L^2-BETTI NUMBERS, ISOMORPHISM CONJECTURES AND NONCOMMUTATIVE LOCALIZATION

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ABSTRACT. In this paper we want to discuss how the question about the rationality of L^2-Betti numbers is related to the Isomorphism Conjecture in algebraic K-theory and why in this context noncommutative localization appears as an important tool.

CONTENTS

L^2-*Betti numbers* are invariants of spaces which are defined analogously to the ordinary Betti-numbers but they take information about the fundamental group into account and are a priori real valued.

The *Isomorphism Conjecture in algebraic K-theory* predicts that $K_0(\mathbb{C}\Gamma)$, the Grothendieck group of finitely generated projective $\mathbb{C}\Gamma$-modules, should be computable from the K-theory of the complex group rings of finite subgroups of Γ.

Given a commutative ring one can always invert the set of all non-zerodivisors. Elements in the resulting ring have a nice description in terms of fractions. For noncommutative rings like group rings this may no longer be the case and other concepts for a *noncommutative localization* can be more suitable for specific problems.

The question whether L^2-Betti numbers are always rational numbers was asked by Atiyah in [1]. The question turns out to be a question about modules over the group

ring of the fundamental group Γ. In [33] Linnell was able to answer the question affirmatively if Γ belongs to a certain class of groups which contains free groups and is stable under extensions by elementary amenable groups (one also needs a bound on the orders of finite subgroups). In fact Linnell proves the stronger result that there exists a semisimple subring in $\mathcal{U}\Gamma$, the algebra of operators affiliated to the group von Neumann algebra, which contains the complex group ring.

The main purpose of this short survey is to give a conceptional framework for Linnell's result, to explain how the question about the rationality of L^2-Betti numbers relates to the Isomorphism conjecture, and why this may involve studying noncommutative localizations of group rings. (The impatient reader should right away take a look at Proposition 3.4, Theorem 6.3 and Addendum 6.4.)

Since probably not every reader is familiar with all three circles of ideas – L^2-Betti numbers – Isomorphism Conjectures – Noncommutative Localization – the paper contains introductions to all of these.

After a brief introduction to group von Neumann algebras and the notion of Γ-dimension we proceed to explain the algebra $\mathcal{U}\Gamma$ of operators affiliated to a group von Neumann algebra and introduce L^2-Betti numbers in a very algebraic fashion. (Once $\mathcal{U}\Gamma$ has been defined there is no more need for Hilbert-spaces.) Section 3 explains the Atiyah Conjecture and contains in particular Proposition 3.4 which is a kind of strategy for its proof. That Proposition says that if one can factorize the inclusion $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ over a ring $\mathcal{S}\Gamma$ with good ring-theoretical properties in such way that a certain K-theoretic condition is satisfied, then the Atiyah conjecture follows. In Section 5 we present a number of candidates for the ring $\mathcal{S}\Gamma$. To do this we first review a number of concepts from the theory of noncommutative localization in Section 4. One of the candidates is the universal localization of $\mathbb{C}\Gamma$ with respect to all matrices that become invertible over $\mathcal{U}\Gamma$. Section 6 contains Linnell's result. We would like to emphasize that the intermediate rings Linnell constructs can also be viewed as universal localizations, see Addendum 6.4 (U). In Section 7 we discuss the Isomorphism Conjecture which seems to be closely related to the K-theoretical condition mentioned above. In the last Section we discuss to what extent the functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$, which plays an important role when one studies L^2-Betti numbers, is exact.

The only new result in this paper is the following.

Theorem. *Let $\mathcal{D}\Gamma$ denote the division closure of $\mathbb{C}\Gamma$ inside $\mathcal{U}\Gamma$. The flat dimension of $\mathcal{D}\Gamma$ over $\mathbb{C}\Gamma$ is ≤ 1 for groups in Linnell's class which have a bound on the orders of finite subgroups.*

For the division closure see Definition 4.8, for Linnell's class of groups see Definition 6.1. The result is proven as Theorem 8.6 below. As an immediate corollary one obtains.

Corollary. *If the infinite group Γ belongs to Linnell's class $\mathcal{C}$ and has a bound on the orders of finite subgroups then the L^2-Euler characteristic (and hence the ordinary one whenever defined) satisfies*

$$\chi^{(2)}(\Gamma) \leq 0.$$

We also would like to mention that the above theorem leads to interesting nontrivial examples of stably flat universal localizations which appear in [45].

A reader who is interested in more information about L^2-Betti numbers and the Atiyah Conjecture should consult the book [39]. Almost all topics discussed here

are also treated there in detail. More information and further results about the Atiyah Conjecture can be found in [34], [24], [55] and [56].

Acknowledgement

1. The von Neumann Dimension

In this section we want to introduce group von Neumann algebras and explain a notion of dimension for finitely generated projective modules over such algebras. For a (discrete) group Γ we denote by $\mathbb{C}\Gamma$ the complex group ring and by $l^2\Gamma$ the complex Hilbert space with orthonormal basis Γ. Each group element operates from the left on $l^2\Gamma$. Linearly extending this action we obtain an inclusion

$$\mathbb{C}\Gamma \to \mathcal{B}(l^2\Gamma)$$

into the algebra $\mathcal{B}(l^2\Gamma)$ of bounded linear operators on the Hilbert space $l^2\Gamma$. The group von Neumann algebra $\mathcal{N}\Gamma$ is defined as the closure of $\mathbb{C}\Gamma$ inside $\mathcal{B}(l^2\Gamma)$ with respect to the weak (or strong, it doesn't matter) operator topology. This algebra is closed under taking the adjoint, i.e. $a^* \in \mathcal{N}\Gamma$ for every operator $a \in \mathcal{N}\Gamma$.

Digression 1.1. A von Neumann algebra is by definition a $*$-closed subalgebra of the algebra of bounded linear operators on some Hilbert-space which is closed with respect to the strong (or weak, it doesn't matter) operator topology. Similarly a C^*-algebra can be defined as a $*$-closed subalgebra of the algebra of bounded operators on some Hilbert space which is closed with respect to the topology given by the operator-norm. Every von Neumann algebra is in particular a C^*-algebra. In the situation described above the operator-norm closure of $\mathbb{C}\Gamma$ inside $\mathcal{B}(l^2\Gamma)$ defines the so called reduced C^*-algebra $C_r^*\Gamma$ and we have a natural inclusion of $C_r^*\Gamma$ in $\mathcal{N}\Gamma$.

The bicommutant theorem of von Neumann (see for example Theorem 5.3.1 in [28]) is a first hint that the definition of $\mathcal{N}\Gamma$ is very natural also from a purely algebraic point of view (at least if we agree to consider $\mathcal{B}(l^2\Gamma)$ as something natural). It says that the von Neumann algebra is the double commutant of $\mathbb{C}\Gamma$, i.e.

$$\mathcal{N}\Gamma = \mathbb{C}\Gamma'',$$

where for a subset $A \subset \mathcal{B}(l^2\Gamma)$ we write $A' = \{b \in \mathcal{B}(l^2\Gamma) | ba = ab \text{ for all } a \in A\}$ for the commutant of A in $\mathcal{B}(l^2\Gamma)$.

The group von Neumann algebra comes equipped with a natural trace. This trace is given as follows:

$$\begin{aligned} \mathrm{tr}_\Gamma : \mathcal{N}\Gamma &\to \mathbb{C} \\ a &\mapsto \langle a(e), e \rangle. \end{aligned}$$

Here $\langle -, - \rangle$ denotes the inner product in $l^2\Gamma$ and e is the unit element of the group considered as a vector in $l^2\Gamma$. Applied to an element $a = \sum a_g g$ in the group ring $\mathbb{C}\Gamma$ the trace yields the coefficient of the identity element a_e. Of course we have the

trace property $\operatorname{tr}_\Gamma(ab) = \operatorname{tr}_\Gamma(ba)$. Once we have such a trace there is a standard procedure to assign a complex number to each finitely generated projective $\mathcal{N}\Gamma$-module: if $p = p^2 = (p_{ij}) \in M_n(\mathcal{N}\Gamma)$ is an idempotent matrix over $\mathcal{N}\Gamma$ which represents P, i.e. such that $P \cong \operatorname{im}(p : \mathcal{N}\Gamma^n \to \mathcal{N}\Gamma^n)$ then we set

$$(1) \qquad \dim_\Gamma P = \Sigma_{i=1}^n \operatorname{tr}_\Gamma(p_{ii})$$

and call it the Γ-dimension of P. We have the following standard facts.

Proposition 1.2. *The Γ-dimension has the following properties.*

(i) $\dim_\Gamma P$ *is a nonnegative real number.*
(ii) $\dim_\Gamma P$ *depends only on the isomorphism class of P.*
(iii) **Normalization.** *We have $\dim_\Gamma \mathcal{N}\Gamma = 1$.*
(iv) **Additivity.** *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated projective modules then*

$$\dim_\Gamma M = \dim_\Gamma L + \dim_\Gamma N.$$

(v) **Faithfulness.** $\dim_\Gamma P = 0$ *if and only if $P = 0$.*

Proof. (i) follows since one can always arrange that the idempotent $p = p^2$ in (1) is a projection, i.e. $p = p^2 = p^*$ (see for example Proposition 4.6.2 on p.23 in [5]). (v) follows from the fact that the trace is faithful, i.e. $\operatorname{tr}(a^*a) = 0$ implies $a = 0$. (ii)-(iv) are straightforward. $\square$

Let $K_0(\mathcal{N}\Gamma)$ denote the Grothendieck-group of finitely generated projective $\mathcal{N}\Gamma$-modules then because of (i)-(iv) above we obtain a homomorphism

$$K_0(\mathcal{N}\Gamma) \xrightarrow{\quad \dim_\Gamma \quad} \mathbb{R}.$$

We recall some terminology, compare page 5 in [57].

Definition 1.3. A projective rank function ρ on a ring R is a homomorphism $\rho : K_0(R) \to \mathbb{R}$ satisfying $\rho([R^1]) = 1$ and $\rho([P]) \geq 0$ for every finitely generated projective R-module P. It is called faithful if moreover $\rho([P]) = 0$ implies $P = 0$.

In this terminology we can summarize the content of the proposition above by saying that $\dim_\Gamma : K_0(\mathcal{N}\Gamma) \to \mathbb{R}$ is a faithful projective rank function.
Other natural examples of faithful projective rank functions occur as follows: Suppose the ring R is embedded in a simple artinian ring $M_n(D)$, where D is a skew field. Then $P \mapsto \frac{1}{n} \dim_D P \otimes_R M_n(D)$ defines a faithful projective rank function on R.
We would like to emphasize the following additional properties of the Γ-dimension for $\mathcal{N}\Gamma$-modules which are not true for arbitrary projective rank functions. They give further justification for the the use of the word "dimension" in this context.

Proposition 1.4. *The Γ-dimension satisfies:*

(v) **Monotony.** *The $\mathcal{N}\Gamma$-dimension is monotone, i.e. $P \subset Q$ implies that $\dim_\Gamma P \leq \dim_\Gamma Q$.*
(vi) **Cofinality.** *If $P = \bigcup_{i \in I} P_i$ is a directed union of submodules then*

$$\dim_\Gamma P = \sup_{i \in I} \dim_\Gamma P_i.$$

Of course cofinality implies monotony. To convince the reader that these properties are not automatic for projective rank functions we would like to treat an example.

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Example 1.5. Let Γ be a free group on two generators x and y. By work of Cohn [11] we know that $\mathbb{C}\Gamma$ is a free ideal ring. In particular every finitely generated projective module is free and taking its rank yields an isomorphism

$$K_0(\mathbb{C}\Gamma) \xrightarrow{\cong} \mathbb{Z}.$$

This is a faithful projective rank function with values in $\mathbb{Z}$. However there is an exact sequence

$$0 \longrightarrow \mathbb{C}\Gamma^2 \xrightarrow{(x-1,y-1)} \mathbb{C}\Gamma \longrightarrow \mathbb{C} \longrightarrow 0$$

which shows that the rank function is not monotone. (Geometrically the above resolution of $\mathbb{C}$ is obtained as the cellular chain complex with complex coefficients of the universal cover $E\Gamma$ of the model for the classifying space $B\Gamma$ given by the wedge of two circles.)

In fact one can always compose $\dim_\Gamma$ with the natural map $K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{N}\Gamma)$. In this way we obtain naturally a faithful projective rank function on $\mathbb{C}\Gamma$ for every group Γ. One rediscovers the example above in the case where Γ is the free group on two generators.

2. The Algebra of Operators affiliated to $\mathcal{N}\Gamma$.

The category of finitely generated projective $\mathcal{N}\Gamma$-modules has one drawback: it is not abelian. In particular if we start out with a complex of finitely generated projective $\mathcal{N}\Gamma$-modules then the homology modules are not necessarily finitely generated projective and hence the $\mathcal{N}\Gamma$-dimension as explained above is a priori not available. But this is exactly what we would like to do in order to define L^2-Betti numbers, i.e. we want to consider

$$C_*^{cell}(\widetilde{X}) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}\Gamma$$

the cellular chain-complex of the universal covering of a CW-complex X of finite type tensored up to $\mathcal{N}\Gamma$ and assign a dimension to the homology modules.

There are several ways to get around this problem. The traditional way to deal with it is to work with certain Hilbert spaces with an isometric Γ-operation instead of modules, e.g. with $l^2\Gamma^n$ instead of $\mathcal{N}\Gamma^n$. These Hilbert spaces have a Γ-dimension and one (re-)defines the homology as the kernel of the differentials modulo the *closure* of their images. This is then again a Hilbert space with an isometric Γ-action and has a well defined Γ-dimension.

A different approach is taken in [36]: *finitely presented* $\mathcal{N}\Gamma$-modules do form an abelian category (because $\mathcal{N}\Gamma$ is a semihereditary ring) and the $\mathcal{N}\Gamma$-dimension can be extended to these modules in such a way that the properties (i)-(vi) still hold. (In fact in [37] the Γ-dimension is even extended to arbitrary $\mathcal{N}\Gamma$-modules.)

A third possible approach is to introduce the algebra $\mathcal{U}\Gamma$ of operators affiliated to $\mathcal{N}\Gamma$. This algebra has better ring-theoretic properties and indeed finitely generated projective $\mathcal{U}\Gamma$-modules do form an abelian category. Moreover the notion of Γ-dimension extends to that algebra. We want to explain this approach in some detail in this section.

Recall that an unbounded operator $a : \text{dom}(a) \to H$ on a Hilbert space H is a linear map which is defined on a linear subspace $\text{dom}(a) \subset H$ called the domain of a. It is called densely defined if $\text{dom}(a)$ is a dense subspace of H and it is called closed if its

graph considered as a subspace of $H \oplus H$ is closed. Each bounded operator is closed and densely defined. For unbounded operators a and b the symbol $a \subset b$ means that restricted to the possibly smaller domain of a the two operators coincide. The following definition goes back to [44].

Definition 2.1 (Affiliated Operators). A closed and densely defined (possibly unbounded) operator $a : \mathrm{dom}(a) \to l^2\Gamma$ is affiliated to $\mathcal{N}\Gamma$ if $ba \subset ab$ for all $b \in \mathcal{N}\Gamma'$. The set

$$\mathcal{U}\Gamma = \{a : \mathrm{dom}(a) \to l^2\Gamma \mid a \text{ is } \begin{array}{l} \text{closed,} \\ \text{densely defined} \\ \text{and affiliated to } \mathcal{N}\Gamma \end{array} \}$$

is called the algebra of operators affiliated to $\mathcal{N}\Gamma$.

Remark 2.2. Each group element $\gamma \in \Gamma$ acts by right multiplication on $l^2\Gamma$. This defines an element $r_\gamma \in \mathcal{N}\Gamma'$ (we had Γ acting from the left when we defined $\mathcal{N}\Gamma$). In order to prove that a closed densely defined operator a is affiliated it suffices to check that its domain $\mathrm{dom}(a)$ is Γ-invariant and that for all vectors $v \in \mathrm{dom}(a)$ we have $r_\gamma a(v) = a r_\gamma(v)$ for all $\gamma \in \Gamma$. In this sense the affiliated operators are precisely the Γ-equivariant unbounded operators.

Observe that the naive composition of two unbounded operators c and d yields an operator dc which is only defined on $c^{-1}(\mathrm{dom}(d))$. Similarly addition is only defined on the intersection of the domains. It is hence not obvious that $\mathcal{U}\Gamma$ is an algebra.

Proposition 2.3. *The set $\mathcal{U}\Gamma$ becomes a $\mathbb{C}$-algebra if we define addition and a product as the closure of the naive addition respectively composition of operators.*

Proof. This is proven in Chapter XVI in [44]. A proof is reproduced in Appendix I in [53] and also in Chapter 8 of [39]. $\square$

The subalgebra of all bounded operators in $\mathcal{U}\Gamma$ is $\mathcal{N}\Gamma$. In contrast to $\mathcal{N}\Gamma$ there seems to be no useful topology on $\mathcal{U}\Gamma$. So we left the realm of C^*-algebras and C^*-algebraic methods. The reason $\mathcal{U}\Gamma$ is nevertheless very useful is that we have gained good ringtheoretical properties. Let us recall the definition of von Neumann regularity.

Definition 2.4. A ring R is called von Neumann regular if one of the following equivalent conditions is satisfied.

(i) Every R-module M is flat, i.e. for every module M the functor $- \otimes_R M$ is exact.

(ii) Every finitely presented R-module is already finitely generated projective.

(iii) The category of finitely generated projective R-modules is abelian.

(iv) For all $x \in R$ there exists a $y \in R$ such that $xyx = x$.

Proof. For (i) $\Leftrightarrow$ (iv) see for example Theorem 4.2.9 in [65]. (i) $\Rightarrow$ (ii) follows since every finitely presented flat R-module is projective, see Theorem 3.2.7 in [65]. Since the tensor product is compatible with colimits, directed colimits are exact and every module is a directed colimit of finitely presented modules we obtain (ii) $\Rightarrow$ (i). For (ii) $\Rightarrow$ (iii) one needs to check that cokernels and kernels between finitely generated projectives are again finitely generated projective. But a cokernel is essentially a finitely presented module. The argument for the kernel and (iii) $\Rightarrow$ (ii) are elementary. $\square$

Note that in particular fields, skew fields, simple artinian rings and semisimple rings are von Neumann regular (every module is projective over such rings). The first condition says that von Neumann regular rings form a very natural class of rings from a homological algebra point of view: they constitute precisely the rings of weak homological dimension 0. The last condition, which seems less conceptional to modern eyes, was von Neumann's original definition [61] and has the advantage that one can explicitly verify it in the case we are interested in. More information about von Neumann regular rings can be found in [21].

Proposition 2.5. *The algebra $\mathcal{U}\Gamma$ is a von Neumann regular ring.*

Proof. Using the polar decomposition and functional calculus one can explicitly construct a y as it is required in the characterization 2.4 (iii) of von Neumann regularity given above. Compare Proposition 2.1 (v) in [54]. $\qquad\square$

In order to define L^2-Betti numbers it remains to establish a notion of dimension for finitely generated projective $\mathcal{U}\Gamma$-modules.

Proposition 2.6. *We have the following facts about the inclusion $\mathcal{N}\Gamma \subset \mathcal{U}\Gamma$.*

(i) *The natural map $K_0(\mathcal{N}\Gamma) \to K_0(\mathcal{U}\Gamma)$ is an isomorphism. In particular there is a Γ-dimension for finitely generated projective $\mathcal{U}\Gamma$-modules which we simply define via the following diagram:*

$$\begin{array}{ccc} K_0(\mathcal{N}\Gamma) & \xrightarrow{\;\cong\;} & K_0(\mathcal{U}\Gamma) \\ & \searrow{\scriptstyle \dim_\Gamma} \quad {\scriptstyle \dim_\Gamma}\swarrow & \\ & \mathbb{R} & \end{array}$$

(ii) *The ring $\mathcal{U}\Gamma$ is the Ore-localization (compare Proposition 4.3) of $\mathcal{N}\Gamma$ with respect to the multiplicative subset of all non-zerodivisors. In particular $- \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma$ is an exact functor.*

Proof. See [54] Proposition 6.1 (i) and Proposition 2.1 (iii). $\qquad\square$

If we now start with a finitely presented (as opposed to finitely generated projective) $\mathbb{C}\Gamma$-module M then because of 2.4 (ii) we know that $M \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$ is a finitely generated projective $\mathcal{U}\Gamma$-module and it makes sense to consider its Γ-dimension.

Remark 2.7. The assignment $M \mapsto \dim_\Gamma(M \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma)$ is a Sylvester module rank function for finitely presented $\mathbb{C}\Gamma$-modules in the sense of Chapter 7 in [57].

We are now prepared to give a definition of L^2-Betti numbers using the Γ-dimension for $\mathcal{U}\Gamma$-modules. Let X be a CW-complex of finite type, i.e. there are only finitely many cells in each dimension. Let $\widetilde{X}$ denote the universal covering. It carries a natural CW-structure and a cellular free $\Gamma = \pi_1(X)$-action. There is one Γ-orbit of cells in $\widetilde{X}$ for each cell in X and in particular the cellular chain complex $C_*^{cell}(\widetilde{X})$ is a complex of finitely generated free $\mathbb{Z}\Gamma$-modules.

Definition 2.8. For a CW-complex X of finite type we define its L^2-Betti numbers as

$$b_p^{(2)}(X) = \dim_{\mathcal{U}\Gamma} H_p(C_*^{cell}(\widetilde{X}) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma).$$

Note that by 2.4 (iii) the homology modules are finitely generated projective $\mathcal{U}\Gamma$-modules and hence have a well defined $\mathcal{U}\Gamma$-dimension.

Remark 2.9. As already mentioned it is possible to extend the notion of Γ-dimension to arbitrary $\mathcal{N}\Gamma$-modules in such a way that one still has "additivity" and "cofinality" [37]. Of course one has to allow the value ∞, and in cases where this value occurs one has to interpret "additivity" and "cofinality" suitably. In [54] it is shown that analogously there is a Γ-dimension for arbitrary $\mathcal{U}\Gamma$-modules which is compatible with the one for $\mathcal{N}\Gamma$-modules in the sense that for an $\mathcal{N}\Gamma$-module M we have

$$(2) \qquad \dim_{\mathcal{U}\Gamma} M \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma = \dim_{\mathcal{N}\Gamma} M.$$

Both notions of extended dimension can be used to define L^2-Betti numbers for arbitrary spaces by working with the singular instead of the cellular chain complex. From 2.6 (ii) we conclude that for a complex C_* of $\mathcal{N}\Gamma$-modules we have

$$H_*(C_* \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma) = H_*(C_*) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma.$$

If we combine this with (2) we see that the two possible definitions of L^2-Betti numbers coincide. In the following we will not deal with L^2-Betti numbers in this generality. We restrict our attention to CW-complexes of finite type and hence to finitely generated projective $\mathcal{U}\Gamma$-modules.

In order to illustrate the notions defined so far we would like to go through two easy examples.

Example 2.10. Suppose Γ is a finite group of order $\#\Gamma$. In this case all the functional analysis is obsolete. We have $\mathbb{C}\Gamma = \mathcal{N}\Gamma = \mathcal{U}\Gamma$ and $l^2\Gamma = \mathbb{C}\Gamma$. A finitely generated projective module P is just a finite dimensional complex Γ-representation. One can check that

$$\dim_\Gamma P = \frac{1}{\#\Gamma} \dim_\mathbb{C} P.$$

Example 2.11. Suppose $\Gamma = C$ is the infinite cyclic group written multiplicatively with generator $z \in C$. In this case (using Fourier transformation) the Hilbert-space $l^2\Gamma$ can be identified with $L^2(S^1)$, the square integrable functions on the unit circle equipped with the standard normalized measure $\mu = \frac{1}{2\pi}dz$. Under this correspondence the group element z corresponds to the function $z \mapsto z$, where we think of S^1 as embedded in the complex plane. The algebras $\mathbb{C}\Gamma$, $\mathcal{N}\Gamma$ and $\mathcal{U}\Gamma$ can be identified as follows:

$$\mathbb{C}\Gamma \quad \leftrightarrow \quad \mathbb{C}[z^{\pm 1}] \quad \begin{matrix} \text{Laurent-polynomials considered} \\ \text{as functions on } S^1 \end{matrix}$$

$$\mathcal{N}\Gamma \quad \leftrightarrow \quad L^\infty(S^1) \quad \begin{matrix} \text{essentially bounded} \\ \text{functions on } S^1 \end{matrix}$$

$$\mathcal{U}\Gamma \quad \leftrightarrow \quad L(S^1) \quad \begin{matrix} \text{measurable functions} \\ \text{on } S^1 \end{matrix}$$

The action on $L^2(S^1)$ in each case is simply given by multiplication of functions. The trace on $\mathcal{N}\Gamma$ becomes the integral $f \mapsto \int_{S^1} f d\mu$. For a measurable subset $A \subset S^1$ let χ_A denote its characteristic function. Then $p = \chi_A$ is a projection and $P_A = pL^\infty(S^1)$ is a typical finitely generated projective $L^\infty(S^1)$-module. We have

$$\dim_\Gamma P_A = \text{tr}_\Gamma(p) = \int_{S^1} \chi_A d\mu = \mu(A).$$

In particular we see that every nonnegative real number can occur as the Γ-dimension of a finitely generated projective $\mathcal{N}\Gamma$- or $\mathcal{U}\Gamma$-module. The module $L^\infty(S^1)/(z-1)L^\infty(S^1)$ is an example of a module which becomes trivial (and hence projective) over $L(S^1)$, because $(z-1)$ becomes invertible. In fact one can show that the there is an isomorphism

$$K_0(L^\infty(S^1)) \cong K_0(L(S^1)) \xrightarrow{\cong} L^\infty(S^1;\mathbb{Z}),$$

where $L^\infty(S^1;\mathbb{Z})$ denotes the space of integer valued measurable bounded functions on S^1, compare Proposition 6.1 (iv) in [54]. Every such function can be written in a unique way as a finite sum $f = \sum_{n=-\infty}^{\infty} n \cdot \chi_{A_n}$ with $A_n = f^{-1}(\{n\}) \subset S^1$ and corresponds to $\sum_{n=-\infty}^{\infty} n[P_{A_n}]$ under the above isomorphism.

Once we have the notion of L^2-Betti numbers it is natural to define

$$\chi^{(2)}(X) = \sum (-1)^i b_i^{(2)}(X).$$

A standard argument shows that for a finite CW-complex this L^2-Euler characteristic coincides with the ordinary Euler-characteristic. But in fact since L^2-Betti numbers tend to vanish more often than the ordinary Betti-numbers the L^2-Euler characteristic is often defined in cases where the ordinary one is not. We also define L^2-Betti numbers and the L^2-Euler characteristic of a group as

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(B\Gamma) \text{ and } \chi^{(2)}(\Gamma) = \chi^{(2)}(B\Gamma).$$

As an example of an application we would like to mention the following result which is due to Cheeger and Gromov [8].

Theorem 2.12. *Let Γ be a group which contains an infinite amenable normal subgroup, then*

$$b_p^{(2)}(\Gamma) = 0 \text{ for all } p, \text{ and hence } \chi^{(2)}(\Gamma) = 0.$$

3. THE ATIYAH CONJECTURE

The question arises which real numbers do actually occur as values of L^2-Betti numbers. This question was asked by Atiyah in [1] where he first introduced the notion of L^2-Betti numbers. (The definition of L^2-Betti numbers at that time only applied to manifolds and was given in terms of the Laplace operator on the universal covering.) It turns out that the question about the values can be phrased as a question about the passage from finitely presented $\mathbb{Z}\Gamma$- or $\mathbb{Q}\Gamma$-modules to $\mathcal{U}\Gamma$-modules.

Proposition 3.1. *Let Λ be an additive subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$. Let Γ be a finitely presented group. The following two statements are equivalent.*

(i) *For all CW-complexes X of finite type with fundamental group Γ and all $p \geq 0$ we have*

$$b_p^{(2)}(X) \in \Lambda.$$

(ii) *For all finitely presented $\mathbb{Z}\Gamma$-modules M we have*

$$\dim_{\mathcal{U}\Gamma}(M \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma) \in \Lambda.$$

Proof. Using the additivity of the dimension and the fact that the finitely generated free modules of the complex $C_*^{cell}(\widetilde{X}) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma$ have integer dimensions (ii) $\Rightarrow$ (i) is straightforward. For the reverse direction one needs to construct a CW-complex X with fundamental group Γ such that the presentation matrix of M appears as the, say 5-th differential in $C_*^{cell}(\widetilde{X})$ whereas the 4-th differential is zero. This is possible by standard techniques. For details see Lemma 10.5 in [39]. $\qquad \square$

More generally one can induce up finitely presented modules over $R\Gamma$ for every coefficient ring R with $\mathbb{Z} \subset R \subset \mathbb{C}$ and ask about the values of the corresponding Γ-dimensions. Let $S \subset R$ be a multiplicatively closed subset. Since each finitely presented $(S^{-1}R)\Gamma$-module is induced from a finitely presented $R\Gamma$-module (clear denominators in a presentation matrix) we can without loss of generality assume that R is a field. In the following we will work for simplicity with the maximal choice $R = \mathbb{C}$.

Let us describe a candidate for Λ. We denote by $\frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z}$ the additive subgroup of $\mathbb{R}$ which is generated by the set of numbers $\{\frac{1}{H} \mid H$ a finite subgroup of $\Gamma\}$. If there is a bound on the orders of finite subgroups then $\frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z} = \frac{1}{l}\mathbb{Z}$ where l is the least common multiple of the orders of finite subgroups. If Γ is torsionfree then $\frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z} = \mathbb{Z}$.

The following Conjecture turned out to be too optimistic in general (compare Remark 3.8 below). But it still has a chance of being true if one additionally assumes a bound on the orders of finite subgroups.

Conjecture 3.2 (Strong Atiyah Conjecture). *Let M be a finitely presented $\mathbb{C}\Gamma$-module then*

$$\dim_{\mathcal{U}\Gamma} M \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma \in \frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z}.$$

We will see below in 5.3 that this Conjecture implies the Zero-Divisor Conjecture.

Remark 3.3. As explained above the conjecture makes sense with any field $\mathbb{F}$ such that $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{C}$ as coefficients for the group ring. With $\mathbb{F} = \mathbb{Q}$ the conjecture is equivalent to the corresponding conjecture about the values of L^2-Betti numbers. The conjecture with $\mathbb{F} = \mathbb{C}$ clearly implies the conjecture formulated with smaller fields.

To get a first idea let us dicuss the Conjecture in the easy case where Γ is the infinite cyclic group. We have already seen in Example 2.11 that in this case the inclusion $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ can be identified with $\mathbb{C}[z^{\pm 1}] \subset L(S^1)$, the Laurent polynomials considered as functions on S^1 inside the algebra of all measurable functions on S^1. Clearly $\mathbb{C}\Gamma$ corresponds to $\mathbb{C}[z^{\pm 1}]$. The crucial observation now is that in this case we find a field in between $\mathbb{C}\Gamma$ and $\mathcal{U}\Gamma$. Let $\mathbb{C}(z)$ denote the field of fractions of the polynomial ring $\mathbb{C}[z]$ then we have

$$\mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z) \subset L(S^1).$$

Now let M be a finitely presented $\mathbb{C}[z^{\pm 1}]$-module then $M \otimes_{\mathbb{C}[z^{\pm 1}]} \mathbb{C}(z)$ is a finitely generated free $\mathbb{C}(z)$-module because $\mathbb{C}(z)$ is a field and hence $M \otimes_{\mathbb{C}[z^{\pm 1}]} L(S^1)$ is a finitely generated free $L(S^1)$-module. In particular its Γ-dimension is an integer as predicted by Conjecture 3.2.

Note that $\mathbb{C}(z)$ is not contained in the group von Neumann algebra $L^\infty(S^1)$ because a rational function like for example $z \mapsto \frac{1}{z-1}$ which has a pole on S^1 can not be

essentially bounded. It hence was crucial for this proof that we had the algebra of affiliated operators $\mathcal{U}\Gamma$, here $L(S^1)$, available.

The following generalizes these simple ideas.

Proposition 3.4. *Suppose the inclusion map $\mathbb{C}\Gamma \to \mathcal{U}\Gamma$ factorizes over a ring $\mathcal{S}\Gamma$ such that the following two conditions are fulfilled.*

(K) *The composite map*

$$\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}\Gamma) \longrightarrow K_0(\mathcal{S}\Gamma)$$

is surjective.

(R) *The ring $\mathcal{S}\Gamma$ is von Neumann regular.*

Then Conjecture 3.2 holds for the group Γ.

In the source of the map in (K) the colimit is taken over the finite subgroups of Γ. The structure maps in the colimit are induced by inclusions $K \subset H$ and conjugation maps $c_g : H \to H^g$, $h \mapsto ghg^{-1}$.

We will see below (compare Theorem 6.3) that there is a reasonably large class of groups for which a factorization of the inclusion $\mathbb{C}\Gamma \to \mathcal{U}\Gamma$ as required above is known to exist. In order to prove Proposition 3.4 we need one more fact about Γ-dimensions.

Proposition 3.5. *The Γ-dimension is compatible with induction, i.e. if G is a subgroup of Γ then there is a natural inclusion $\mathcal{U}G \subset \mathcal{U}\Gamma$ and for a finitely generated projective $\mathcal{U}G$-module P we have*

$$\dim_{\mathcal{U}\Gamma} P \otimes_{\mathcal{U}G} \mathcal{U}\Gamma = \dim_{\mathcal{U}G} P.$$

Proof. There exists a natural inclusion $i : \mathcal{U}G \to \mathcal{U}\Gamma$ which extends the inclusion $i : \mathcal{N}G \to \mathcal{N}\Gamma$ because $\mathcal{U}G$ is the Ore localization of $\mathcal{N}G$. The latter inclusion is compatible with the trace, i.e. $\operatorname{tr}_\Gamma(i(a)) = \operatorname{tr}_G(a)$ for $a \in \mathcal{N}\Gamma$, see Lemma 1.24 in [39]. The claim follows from these facts. $\qquad\square$

Proof of Proposition 3.4. Let M be a finitely presented $\mathbb{C}\Gamma$-module. Then also $M \otimes_{\mathbb{C}\Gamma} \mathcal{S}\Gamma$ is finitely presented and hence finitely generated projective by 2.4 (ii) because we assume that $\mathcal{S}\Gamma$ is von Neumann regular. In particular $M \otimes_{\mathbb{C}\Gamma} \mathcal{S}\Gamma$ defines a class in $K_0(\mathcal{S}\Gamma)$. Our second assumption implies that this class comes from $\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H)$ via the natural map. It remains to check that the composition

$$\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}\Gamma) \longrightarrow K_0(\mathcal{S}\Gamma) \longrightarrow K_0(\mathcal{U}\Gamma) \xrightarrow{\dim_\Gamma} \mathbb{R}$$

lands inside the subgroup $\frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z}$ of $\mathbb{R}$. But from Example 2.10 together with Proposition 3.5 we conclude that for a finite subgroup H and a finitely generated projective $\mathbb{C}H$-module P we have

$$\dim_{\mathcal{U}\Gamma} P \otimes_{\mathbb{C}H} \mathcal{U}\Gamma = \dim_{\mathbb{C}H} P = \frac{1}{\#H} \dim_\mathbb{C} P.$$

$\qquad\square$

Remark 3.6. From 2.4 (iv) it follows that the homomorphic image of a von Neumann regular ring is again von Neumann regular. In particular the image of $\mathcal{S}\Gamma$ in $\mathcal{U}\Gamma$ would be von Neumann regular if $\mathcal{S}\Gamma$ is. (But it is not clear that the induced map for K_0 is surjective, compare Question 7.5.)

Note 3.7. Suppose $\mathcal{S}\Gamma$ is a subring of $\mathcal{U}\Gamma$ which contains $\mathbb{C}\Gamma$. If we assume the properties (K) and (R) and additionally we assume that Γ has a bound on the orders of finite subgroups, then $\mathcal{S}\Gamma$ is semisimple.

Proof. The assumptions imply that the projective rank function $P \mapsto \dim_\Gamma P \otimes_{\mathcal{S}\Gamma} \mathcal{U}\Gamma$ for finitely generated $\mathcal{S}\Gamma$ modules takes values in $\frac{1}{l}\mathbb{Z}$, where l is the least common multiple of the orders of finite subgroups. Since each finitely generated projective $\mathcal{S}\Gamma$-module is a subset of a $\mathcal{U}\Gamma$-module it is easy to see that the projective rank function is faithful. In order to prove that a von Neumann regular ring is semisimple it suffices to show that there are no infinite chains of ideals, see page 21 in [21]. Since each ideal is a direct summand of $\mathcal{S}\Gamma$ and each subideal of a given ideal is a direct summand this can be checked using the faithful projective rank function with values in $\frac{1}{l}\mathbb{Z}$. $\qquad\square$

Remark 3.8. The lamplighter group is the semidirect product of $\mathbb{Z}$ and $\bigoplus_{-\infty}^{\infty} \mathbb{Z}/2$ where $\mathbb{Z}$ acts via shift on $\bigoplus_{-\infty}^{\infty} \mathbb{Z}/2$. The orders of finite subgroups that occur are precisely all powers of 2. Conjecture 3.2 hence predicts $\mathbb{Z}[\frac{1}{2}]$ as the range for the dimensions. However in [24] a finitely presented $\mathbb{Q}\Gamma$-module is constructed whose Γ-dimension is $\frac{1}{3}$.

4. Noncommutative Localization

Our next aim is to present several candidates for the ring $\mathcal{S}\Gamma$ which appears in Proposition 3.4. In order to do this we first want to fix some language and review a couple of concepts from the theory of localization for noncommutative rings. For more on this subject the reader should consult Chapter II in [58], Chapter 7 in [12] and Chapter 4 in [57].

Ore Localization. Classically the starting point for the localization of rings is the wish that certain *elements* in the ring should become invertible. In mathematical terms we have the following universal property.

Definition 4.1. Let $T \subset R$ be a subset which does not contain any zero-divisors. A ring homomorphism $f : R \to S$ is called T-inverting if $f(t)$ is invertible for all $t \in T$. A T-inverting ringhomomorphism $i : R \to R_T$ is called universally T-inverting if it has the following universal property: given any T-inverting ring homomorphism $f : R \to S$ there exists a unique ring homomorphism $\Phi : R_T \to S$ such that

$$
\begin{array}{ccc}
 & & R_T \\
 & \nearrow^{i} & \downarrow{\Phi} \\
R & \xrightarrow{f} & S
\end{array}
$$

commutes.

A generator and relation construction shows that there always exists a universal T-inverting ring and as usual it is unique up to canonical isomorphism. Given a ring homomorphism $R \to S$ let us agree to write

$$T(R \to S)$$

for the set of elements in R which become invertible in S. If one replaces T by $\overline{T} = T(R \to R_T)$ the universal inverting ring does not change. We can hence

always assume that T is multiplicatively closed. A natural maximal choice for T is the set $\mathrm{NZD}(R)$ of all non-zerodivisors of R.

If the ring R is commutative it is well known that there is a model for R_T whose elements are "fractions" or more precisely equivalence classes of pairs $(a,t) \in R \times T$. For noncommutative rings the situation is more complicated. It goes back to Ore that under a suitable assumption such a calculus of fractions still exists.

Definition 4.2. A multiplicatively closed subset $T \subset R$ which does not contain zero-divisors or zero itself satisfies the right Ore-condition if for given $(a,s) \in R \times T$ there always exists a $(b,t) \in R \times T$ such that $at = sb$.

It is clear that this condition is necessary if a calculus of right fractions exists because we need to be able to write a given wrong way (left) fraction $s^{-1}a$ as bt^{-1}. It is a bit surprising that this is the only condition.

Proposition 4.3. *Let $T \subset R$ be a multiplicatively closed subset without zero divisors which satisfies the right Ore condition, then there exists a ring RT^{-1} and a universal T-inverting ring-homomorphism $i : R \to RT^{-1}$ such that every element of RT^{-1} can be written as $i(a)i(t)^{-1}$ with $(a,t) \in R \times T$.*

Proof. Elements in RT^{-1} are equivalence classes of pairs $(a,t) \in R \times T$. The pair (a,t) is equivalent to (b,s) if there exist elements u, $v \in R$ such that $au = bv$, $su = tv$ and $su = tv \in S$. For more details see Chapter II in [58]. $\qquad\square$

Remark 4.4. Ore-localization is an exact functor, i.e. RT^{-1} is a flat R-module, see page 57 in [58].

Example 4.5. Let Γ be the free group on two generators x and y. The group ring $\mathbb{C}\Gamma$ does not satisfy the Ore condition with respect to the set $\mathrm{NZD}(\mathbb{C}\Gamma)$ of all non-zerodivisors. Let $C \subset \Gamma$ be the infinite cyclic subgroup generated by x. Now $x-1$ is a non-zerodivisor since it becomes invertible in $\mathcal{U}C$ (compare Example 2.11) and therefore in the overring $\mathcal{U}\Gamma$. In fact every non-trivial element in $\mathbb{C}\Gamma$ is a non-zerodivisor since one can embed $\mathbb{C}\Gamma$ in a skew field. The Ore condition would imply the existence of $(b,t) \in \mathbb{C}\Gamma \times \mathrm{NZD}(\mathbb{C}\Gamma)$ with $(y-1)t = (x-1)b$ alias

$$(x-1)^{-1}(y-1) = bt^{-1}.$$

This implies that $(-b,t)^{tr}$ is in the kernel of the map $(x-1, y-1) : \mathbb{C}\Gamma^2 \to \mathbb{C}\Gamma$. But this map is injective, compare Example 1.5.

Localizing Matrices. Instead of elements one can try to invert maps. Let Σ be a set of homomorphisms between right R-modules. A ring homomorphism $R \to S$ is called Σ-inverting if for every map $\alpha \in \Sigma$ the induced map $\alpha \otimes_R \mathrm{id}_S$ is an isomorphism.

Definition 4.6. A Σ-inverting ring homomorphism $i : R \to R_\Sigma$ is called universal Σ-inverting if it has the following universal property. Given any Σ-inverting ring homomorphism $f : R \to S$ there exists a unique ring homomorphism $\Psi : R_\Sigma \to S$ such that the following diagram commutes.

From now on let us assume that Σ is a set of matrices over R. For a ring homomorphism $R \to S$ we will write

$$\Sigma(R \to S)$$

for the set of all matrices over R which become invertible over S. One can always replace a given set of matrices Σ by $\overline{\Sigma} = \Sigma(R \to R_\Sigma)$ without changing the universal Σ-inverting ringhomomorphism. There are different constructions which prove the existence of a universal Σ-inverting ring homomorphism. One possibility is a generator and relation construction where one starts with the free ring on a set of symbols $\overline{a}_{i,j}$ where $(a_{i,j})$ runs through the matrices in Σ and imposes the relations which are given in matrix form as $\overline{A}A = A\overline{A} = 1$, compare Theorem 2.1 in [12]. For more information the reader should consult Chapter 7 in [12] and Chapter 4 in [57].

Another construction due to Malcolmson [41] (see also [4]), a kind of calculus of fractions for matrices, allows a certain amount of control over the ring R_Σ.

As an easy example we would like to mention the following: A set of matrices is lower multiplicatively closed if $1 \in \Sigma$ and $a, b \in \Sigma$ implies that

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \in \Sigma$$

for arbitrary matrices c of suitable size. Observe that $\Sigma(R \to S)$ is always lower multiplicatively closed.

Proposition 4.7 (Cramer's rule)**.** *Let R be a ring and Σ be a lower multiplicatively closed set of matrices over R then every matrix a over R_Σ satisfies an equation of the form*

$$s \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = b$$

with $s \in \Sigma$, $x \in M(R_\Sigma)$ and $b \in M(R)$.

Proof. See Theorem 4.3 on page 53 in [57]. $\qquad\qquad\square$

In particular every matrix a over R_Σ is stably associated over R_Σ to a matrix b over R, i.e. there exist invertible matrices $c, d \in GL(R_\Sigma)$ such that

$$c \begin{pmatrix} a & 0 \\ 0 & 1_n \end{pmatrix} d^{-1} = \begin{pmatrix} b & 0 \\ 0 & 1_m \end{pmatrix}$$

with suitable m and n.

Division Closure and Rational Closure. Recall that for a given ring homomorphism $R \to S$ we denoted by $T(R \to S)$ the set of all elements in R which become invertible in S and by $\Sigma(R \to S)$ the set of all matrices over R that become invertible over S. The universal localizations $R_{T(R \to S)}$ and $R_{\Sigma(R \to S)}$ come with a natural map to S. In the case where $R \to S$ is injective one may wonder whether these maps embed the universal localizations into S. The intermediate rings in the following definition serve as potential candidates for such embedded versions of the universal localizations.

Definition 4.8. Let S be a ring.

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(i) A subring $R \subset S$ is called division closed in S if $T(R \subset S) = R^{\times}$, i.e. for every element $r \in R$ which is invertible in S the inverse r^{-1} lies already in R.

(ii) A subring $R \subset S$ is called rationally closed in S if $\Sigma(R \subset S) = \mathrm{GL}(R)$, i.e. for every matrix A over R which is invertible over S the entries of the inverse matrix A^{-1} are all in R.

(iii) Given a subring $R \subset S$ the division closure of R in S denoted

$$\mathcal{D}(R \subset S)$$

is the smallest division closed subring of S which contains R.

(iv) Given a subring $R \subset S$ the rational closure of R in S denoted by

$$\mathcal{R}(R \subset S)$$

is the smallest rationally closed subring of S containing R.

Note that the intersection of division closed intermediate rings is again division closed and similarly for rationally closed rings. This proves the existence of the division and rational closure. Moreover we really have closure-operations, i.e.

$$\mathcal{D}(\mathcal{D}(R \subset S) \subset S) = \mathcal{D}(R \subset S) \text{ and}$$
$$\mathcal{R}(\mathcal{R}(R \subset S) \subset S) = \mathcal{R}(R \subset S).$$

In [12, Chapter 7, Theorem 1.2] it is shown that the set

(3) $\{a_{i,j} \in S \mid (a_{i,j}) \text{ invertible over } S, \ (a_{i,j})^{-1} \text{ matrix over } R\}$

is a subring of S and that it is rationally closed. Since this ring is contained in $\mathcal{R}(R \subset S)$ the two rings coincide. The following observation is very useful in our context.

Proposition 4.9. *A von Neumann regular ring R is division closed and rationally closed in every overring.*

Proof. Suppose $a \in R$ is not invertible in R, then the corresponding multiplication map $l_a : R \to R$ is not an isomorphism. Therefore the kernel or the cokernel is non-trivial. Both split of as direct summands because the ring is von Neumann regular. The corresponding projection onto the kernel or cokernel is given by left multiplication with a suitable idempotent. This idempotent shows that a must be a zerodivisor and hence can not become invertible in any overring. A matrix ring over a von Neumann regular ring is again von Neumann regular and the same reasoning applied to matrix rings over R yields that R is also rationally closed in every overring. $\square$

In particular note that once we know that the division closure $\mathcal{D}(R \subset S)$ is von Neumann regular then it coincides with the rational closure $\mathcal{R}(R \subset S)$. The following proposition relates the division respectively rational closure to the universal localizations $R_{T(R \subset S)}$ and $R_{\Sigma(R \subset S)}$.

Proposition 4.10. *Let $R \subset S$ be a ring extension.*

(i) *The map $R_{T(R \subset S)} \to S$ given by the universal property factorizes over the division closure.*

$$R_{T(R \subset S)}$$
$$\Phi$$
$$R \overset{\subset}{\longrightarrow} \mathcal{D}(R \subset S) \overset{\subset}{\longrightarrow} S$$

(ii) *If the pair $(R, T(R \subset S))$ satisfies the right Ore condition, then Φ is an isomorphism.*

(iii) *The map $R_{\Sigma(R \subset S)} \to S$ given by the universal property factorizes over the rational closure.*

$$R_{\Sigma(R \subset S)}$$
$$\Psi$$
$$R \overset{\subset}{\longrightarrow} \mathcal{R}(R \subset S) \overset{\subset}{\longrightarrow} S$$

The map Ψ is always surjective.

Proof. (i) This follows from the definitions. (ii) Note that $T(R \subset S)$ always consists of non-zerodivisors. Thus we can choose a ring of right fractions as a model for $R_{T(R \subset S)}$. Every element in $\mathrm{im}\Phi$ is of the form at^{-1} with $t \in T(R \subset S)$. Such an element is invertible in S if and only if $a \in T(R \subset S)$. We see that the image of Φ is division closed and hence Φ is surjective. On the other hand the abstract fraction $at^{-1} \in RT(R \subset T)^{-1}$ is zero if and only if $a = 0$ because $T(R \subset S)$ contains no zerodivisors, so Φ is injecitve. (iii) Only the last statement is maybe not obvious. By Cohn's description of the rational closure (compare (3)) we need to find a preimage for $a_{i,j}$, where $(a_{i,j})$ is a matrix invertible over S whose inverse lies over R. The generator and relation construction of the universal localization immediately gives such an element. $\qquad\square$

In general it is not true that the map Ψ is injective.

5. Some Candidates for $\mathcal{S}\Gamma$

We are now prepared to describe the candidates for the ring $\mathcal{S}\Gamma$ which appears in Proposition 3.4. We consider the ring extension $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ and define

$$
\begin{aligned}
\mathcal{D}\Gamma &= \mathcal{D}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma), \\
\mathcal{R}\Gamma &= \mathcal{R}(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma), \\
\mathbb{C}\Gamma_T &= \mathbb{C}\Gamma_{T(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)} \text{ and} \\
\mathbb{C}\Gamma_\Sigma &= \mathbb{C}\Gamma_{\Sigma(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)}.
\end{aligned}
$$

These rings are organized in the following diagram

(4)
$$\mathbb{C}\Gamma_T \longrightarrow \mathbb{C}\Gamma_\Sigma$$
$$\mathbb{C}\Gamma \overset{\subset}{\longrightarrow} \mathcal{D}\Gamma \overset{\subset}{\longrightarrow} \mathcal{R}\Gamma \overset{\subset}{\longrightarrow} \mathcal{U}\Gamma$$

A first hint that the rational or division closure may be a good candidate for $\mathcal{S}\Gamma$ is the following result which is implicit in [33]. At the same time its proof illustrates the usefulness of Cramer's rule 4.7.

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Proposition 5.1. *If* Γ *is a torsionfree group then the Strong Atiyah Conjecture 3.2 implies that* $\mathcal{R}\Gamma$ *is a skew field.*

Proof. For $x \in \mathcal{U}\Gamma$ let $l_x : \mathcal{U}\Gamma \to \mathcal{U}\Gamma$ denote left multiplication with x. From the additivity and faithfulness of the dimension it follows that x is invertible if and only if $\dim \mathrm{im}(l_x) = 1$ or equivalently $\dim \ker(l_x) = 0$ or equivalently $\dim \mathrm{coker}(l_x) = 0$. Now let X be a matrix over $\mathcal{R}\Gamma$ then by 4.10 (iii) we know that we can lift it to a matrix over $\mathbb{C}\Gamma_\Sigma$. Using Cramer's rule 4.7 and projecting down again we see that we can find invertible matrices A and $B \in GL(\mathcal{R}\Gamma)$ such that

$$C = A \begin{pmatrix} 1_n & 0 \\ 0 & X \end{pmatrix} B$$

is a matrix over $\mathbb{C}\Gamma$. In particular if $0 \neq x \in \mathcal{R}\Gamma$ then for $X = (x)$ we know that there exists an n such that

$$\dim \mathrm{im}(l_x) + n = \dim(\mathrm{im} \begin{pmatrix} 1_n & 0 \\ 0 & l_x \end{pmatrix}) = \dim(\mathrm{im} C) \in \mathbb{Z}$$

because we assume for the matrix C over $\mathbb{C}\Gamma$ that the dimension of its image is an integer. It follows that $\dim \mathrm{im}(l_x) = 1$ and hence that x is invertible in $\mathcal{R}\Gamma$. $\qquad\square$

Note 5.2. If one of the rings $\mathcal{D}\Gamma$ or $\mathcal{R}\Gamma$ is a skew field then so is the other and the two coincide.

Proof. If $\mathcal{D}\Gamma$ is a skew field then it is also rationally closed, see 4.9. If $\mathcal{R}\Gamma$ is a skew field then $\mathcal{D}\Gamma$ is a division closed subring of a skew field and hence itself a skew field. $\qquad\square$

Corollary 5.3. *The Atiyah Conjecture 3.2 implies the Zero-Divisor Conjecture, i.e. the conjecture that the complex group ring of a torsionfree group does not contain any zero-divisors.*

Remark 5.4. One can show that for a torsionfree amenable group the Atiyah Conjecture 3.2 is equivalent to the Zero-Divisor Conjecture, see Lemma 10.16 in [39].

Another natural question is in how far the rings discussed above depend functorially on the group. Since an arbitrary group homomorphism $G \to G'$ does not induce a map from $\mathcal{U}G$ to $\mathcal{U}G'$ we can not expect functoriality but at least we have the following.

Note 5.5. An injective group homomorphism induces maps on the rings $\mathcal{D}\Gamma$, $\mathcal{R}\Gamma$, $\mathbb{C}\Gamma_T$ and $\mathbb{C}\Gamma_\Sigma$. These maps are compatible with the maps in diagram 4 above.

Proof. We already know that the inclusion $\mathbb{C}\Gamma \subset \mathcal{U}\Gamma$ is functorial for injective group homomorphisms. Let G be a subgroup of Γ. Since $\mathcal{U}G$ is von Neumann regular it is division closed and rationally closed in every overring, compare 4.9. Therefore $\mathcal{D}\Gamma \cap \mathcal{U}G$ is division-closed in $\mathcal{U}\Gamma$ and $\mathcal{D}G \subset \mathcal{D}\Gamma \cap \mathcal{U}G \subset \mathcal{D}\Gamma$. Analogously one argues for the rational closure. One immediately checks that $T(\mathbb{C}G \subset \mathcal{U}G) \subset T(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$ and $\Sigma(\mathbb{C}G \subset \mathcal{U}G) \subset \Sigma(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$. The universal properties imply the statement for $\mathbb{C}\Gamma_T$ and $\mathbb{C}\Gamma_\Sigma$. $\qquad\square$

6. Linnell's Result

Before we state Linnell's result we would like to introduce the class of groups it applies to.

Definition 6.1 (Linnell's class of groups). Let $\mathcal{C}$ be the smallest class of groups which has the following properties.

(LC1) Free groups are contained in $\mathcal{C}$.

(LC2) If $1 \to G \to \Gamma \to H \to 1$ is an exact sequence of groups such that G lies in $\mathcal{C}$ and H is finite or finitely generated abelian then Γ lies in $\mathcal{C}$.

(LC3) The class $\mathcal{C}$ is closed under directed unions, i.e. if a group $\Gamma = \bigcup_{i \in I} \Gamma_i$ is a directed union of subgroups Γ_i which lie in $\mathcal{C}$ then Γ also lies in $\mathcal{C}$.

To put this definition into perspective we would like to make a couple of remarks.

Remark 6.2. (i) If one replaces (LC1) above by the requirement that the trivial group belongs to $\mathcal{C}$ one obtains the smaller class of elementary amenable groups. Compare [9] and [29]. Elementary amenable groups are in particular amenable (see [16]) but it is not easy to find amenable groups that are not elementary amenable [23]. A group which contains a non-abelian free subgroup is not amenable.
(ii) One can show that if Γ lies in $\mathcal{C}$ and A is an elementary amenable normal subgroup then Γ/A also belongs to $\mathcal{C}$.
(iii) The class $\mathcal{C}$ is closed under free products.

In [33] Linnell proves Conjecture 3.2 for groups in the class $\mathcal{C}$ which additionally have a bound on the orders of finite subgroups. In fact by carefully investigating the proof given there one can obtain the following statements.

Theorem 6.3. *Suppose the group Γ lies in $\mathcal{C}$ and has a bound on the orders of finite subgroups then*

(K) *The composition*

$$\mathrm{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \to K_0(\mathbb{C}\Gamma) \to K_0(\mathcal{D}\Gamma)$$

is surjective.

(R) *The ring $\mathcal{D}\Gamma$ is semi-simple and hence $\mathcal{D}\Gamma = \mathcal{R}\Gamma$ by Proposition 4.9.*

As already mentioned this result is essentially contained in [33]. In the above formulation it is proven in [53]. The proof is published in Chapter 10 in [39]. Below we will only make a couple of remarks about the proof.

Since we formulated the theorem with the division closure $\mathcal{D}\Gamma$ the reader may get the impression that this is the best candidate for an intermediate ring $\mathcal{S}\Gamma$ as in Proposition 3.4. But in fact the situation is not so clear. We already stated that $\mathcal{D}\Gamma = \mathcal{R}\Gamma$ when the theorem applies. Moreover one can show the following.

Addendum 6.4.

(U) *In the situation of Theorem 6.3 the natural map $\mathbb{C}\Gamma_\Sigma \to \mathcal{R}\Gamma$ is an isomorphism and hence $\mathcal{D}\Gamma = \mathcal{R}\Gamma \cong \mathbb{C}\Gamma_\Sigma$.*

(O) *If Γ lies in the smaller class of elementary amenable groups and has a bound on the orders of finite subgroups then $\mathbb{C}\Gamma$ satisfies the the right Ore condition with respect to the set $\mathrm{NZD}(\mathbb{C}\Gamma)$ of all non-zerodivisors and this set coincides with $T(\mathbb{C}\Gamma \subset \mathcal{U}\Gamma)$. Hence $\mathbb{C}\Gamma_T$ can be realized as a ring of fractions and the natural map $\mathbb{C}\Gamma_T \to \mathcal{D}\Gamma$ is an isomorphism, compare 4.10 (iii).*

The statement (O) about the Ore localization appears already in [31]

We will now make some comments about the proof of Theorem 6.3 and the Addendum 6.4. As one might guess from the description of the class of groups to which the Theorem (and the Addendum) applies the proof proceeds via transfinite induction on the class of groups, i.e. one proves the following statements.

(I) (K), (R) and (U) hold for free groups.

(II) If $1 \to G \to \Gamma \to H \to 1$ is an extension of groups where H is finite or infinite cyclic and (K), (R) and (U) hold for G then they hold for Γ. Similar with (O) replacing (U).

(III) If Γ is the directed union of the subgroups Γ_i and (K), (R) and (U) hold for all Γ_i then they hold for Γ if Γ has a bound on the orders of finite subgroups. Similar with (O) replacing (U).

(I) The Kadison Conjecture says that there are no non-trivial idempotents in the group C^*-algebra of a torsionfree group. Linnell observed that Connes conceptional proof of this Conjecture for the free group on two generators given in [14] (see also [27]) can be used to verify the stronger Conjecture 3.2 in this case. Combined with Proposition 5.1 and Note 5.2 one concludes that $\mathcal{D}\Gamma = \mathcal{R}\Gamma$ is a skew field. This yields (R) and also (K) since K_0 of a skew field is $\mathbb{Z}$. Every finitely generated free group is a subgroup of the free group on two generators and every free group is a directed union of finitely generated free subgroups. This is used to pass to arbitrary free groups.

Some rather non-trivial facts about group rings of free groups (see [25] and [30]) are used to verify that $\mathcal{R}\Gamma$ also coincides with the universal localization $\mathbb{C}\Gamma_\Sigma$ and hence to verify (U) in this case. Recall that we saw in Example 4.5 that (O) is false for free groups.

(II) For information about crossed products we refer the reader to [46] and to Digression 6.5 below. If $1 \to G \to \Gamma \to H \to 1$ is an extension of groups then every set-theoretical section μ of the quotient map $\Gamma \to H$ (we can always assume $\mu(e) = e$ and $\mu(g^{-1}) = \mu(g)^{-1}$) allows to describe the group ring $\mathbb{C}\Gamma$ as a crossed product $\mathbb{C}G * H$ of the ring $\mathbb{C}G$ with the group H. Similarly crossed products $\mathcal{D}G * H$, $\mathcal{R}G * H$ and $\mathbb{C}G_\Sigma * H$ exist and can serve as intermediate steps when one tries to prove the statements (R), (K) and (U) for Γ. For example there are natural inclusions

$$\mathbb{C}G * H \to \mathcal{D}G * H \to \mathcal{D}\Gamma.$$

If H is a finite group and $\mathcal{D}G$ is semisimple then $\mathcal{D}G * H$ is semisimple and coincides with $\mathcal{D}\Gamma$. It is relatively easy to verify that $\mathcal{D}G * H$ is noetherian and semiprime if H is an infinite cyclic group and then Goldie's theorem (a criterion for the existence of an Ore-localization, see Section 9.4 in [13]) together with results from [32] are used to verify that $\mathcal{D}\Gamma$ is the Ore-localization of $\mathcal{D}G * H$ with respect to the set of all non-zerodivisors. This is roughly the line of argument in order to verify that condition (R) survives extensions by finite or infinite cyclic groups. Once we know that $\mathcal{D}\Gamma$ is an Ore localization of $\mathcal{D}G * H$ we can combine this with the assumption (which implies that $\mathbb{C}G_\Sigma * H \to \mathcal{D}G * H$ is an isomorphism) in order to verify (U). Similarly iterating Ore localizations one obtains that (O) is stable under extensions with an infinite cyclic group. Moody's Induction Theorem (see Theorem 7.3 below) plays a crucial role in the argument for (K). Moreover one has to assume that the class of groups which appears in the induction hypothesis is already closed under extensions by finite subgroups. Hence one is forced to start the induction

with virtually free groups and in particular one has to prove that (K) holds for such groups. For this purpose results of Waldhausen [62] about generalized free products can be used. Moreover the map induced by $\mathbb{C}\Gamma \to \mathbb{C}\Gamma_\Sigma$ on K_0 needs to be studied, compare Question 7.5. Here it is important to deal with the universal matrix localization.

(III) If Γ is the directed union of the subgroups Γ_i, $i \in I$ then $\mathcal{D}\Gamma$ is the directed union of the subrings $\mathcal{D}\Gamma_i$ and similar for $\mathcal{R}\Gamma$. A directed union of von Neumann regular rings is again von Neumann regular (use Definition 2.4 (iv)) so $\mathcal{D}\Gamma$ is at least von Neumann regular if all the $\mathcal{D}\Gamma_i$ are semisimple. The fact that K-theory is compatible with colimits yields that (K) holds for Γ if it holds for all the Γ_i. Now the assumption on the bound of the orders of finite subgroups implies that $\mathcal{D}\Gamma$ is even semisimple by Note 3.7. That (U) and (O) are stable under directed unions is straightforward.

Digression 6.5. A crossed product $R * G = (S, \mu)$ of the ring R with the group G consists of a ring S which contains R as a subring together with an injective map $\mu : G \to S^\times$ such that the following holds.

  (i) The ring S is a free R-module with basis $\mu(G)$.
  (ii) For every $g \in G$ the conjugation map $c_{\mu(g)} : S \to S$, $\mu(g)s\mu(g)^{-1}$ can be restricted to R.
  (iii) For all $g, g' \in G$ the element $\tau(g, g') = \mu(g)\mu(g')\mu(gg')^{-1}$ lies in $R^\times$.

7. The Isomorphism Conjecture in algebraic K-theory

Condition (K) in Proposition 3.4 requires that the composite map

$$\text{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}\Gamma) \longrightarrow K_0(\mathcal{S}\Gamma)$$

is surjective. About the first map in this composition there is the following conjecture.

Conjecture 7.1 (Isomorphism Conjecture - Special Case). *For every group Γ the map*

$$\text{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}\Gamma)$$

is an isomorphism. In particular for a torsionfree group Γ we expect

$$K_0(\mathbb{C}\Gamma) \cong \mathbb{Z}.$$

To form the colimit we understand $\mathcal{F}in\Gamma$ as the category whose objects are the finite subgroups of Γ and whose morphisms are generated by inclusion maps $K \subset H$ and conjugation maps $c_g : H \to H^g$, $h \mapsto ghg^{-1}$ with $g \in \Gamma$. Observe that in the torsionfree case the colimit reduces to $K_0(\mathbb{C}) \cong \mathbb{Z}$.

In fact Conjecture 7.1 would be a consequence of a much more general conjecture which predicts the whole algebraic K-theory of a group ring $R\Gamma$ in terms of the K-theory of the coefficients and homological data about the group. This more general conjecture is known as the Farrell-Jones Isomorphism Conjecture for algebraic K-theory [18]. A precise formulation would require a certain amount of preparation and we refer the reader to [15], [2] and in particular to [40] for more information. Conjecture 7.1 would have the following consequence.

Consequence 7.2. *For every finitely generated projective $\mathbb{C}\Gamma$-module P we have*

$$\dim_\Gamma(P \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma) \in \frac{1}{\#\mathcal{F}in\Gamma}\mathbb{Z}.$$

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Proof. Use Example 2.10 and Proposition 3.5. □

So if all finitely presented $\mathbb{C}\Gamma$-modules were also finitely generated projective then the Isomorphism Conjecture would imply the Strong Atiyah Conjecture. But of course this is seldom the case.

Conjecture 7.1 is true for infinite cyclic groups and products of such by the Bass-Heller-Swan formula [3]. Cohn's results in [11] imply the Conjecture for free groups. Work of Waldhausen [62] deals with generalized free products and HNN-extensions. (The reader should consult [52] and [45] for a "noncommutative localization"-perspective on these results.) A version of the following result plays also an important role in the proof of 6.3.

Theorem 7.3 (Moody's induction theorem - Special case)**.** *Let Γ be a polycyclic-by-finite group then the map*

$$\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{C}H) \to K_0(\mathbb{C}\Gamma)$$

is surjective.

Proof. See [42], [43], [10] and Chapter 8 in [46]. □

What happens if we replace the complex coefficients in Conjecture 7.1 by integral coefficients? Thinking about the situation for finite groups it is at first glance very surprising that for infinite groups there are a lot of cases where there are results about $K_0(\mathbb{Z}\Gamma)$ whereas nothing is known about $K_0(\mathbb{C}\Gamma)$. See for example [18]. The reason is that the elements of algebraic K-groups of the *integral* group ring have a topological interpretation. They occur as obstruction groups in certain topological problems. Many people put a lot of effort into solving these topological problems and each time this is successful one obtains a result about the algebraic K-groups of $\mathbb{Z}\Gamma$.

However with integral coefficients one does not expect an isomorphism as in Conjecture 7.1. In the case where Γ is torsionfree one would still expect $K_0(\mathbb{Z}\Gamma) \cong \mathbb{Z}$, but in general so called Nil-groups and also negative K-groups should enter in a "computation" of $K_0(\mathbb{Z}\Gamma)$. Moreover by a result of Swan (see Theorem 8.1 in [59]) the map $K_0(\mathbb{Z}H) \to K_0(\mathbb{Q}H)$ is almost the trivial map for a finite group H, i.e. the map on reduced K-groups $\tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathbb{Q}H)$ is trivial. Summarizing: In general in the square

$$
\begin{array}{ccc}
\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{Z}H) & \longrightarrow & K_0(\mathbb{Z}\Gamma) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{H \in \mathcal{F}in\Gamma} K_0(\mathbb{Q}H) & \longrightarrow & K_0(\mathbb{Q}\Gamma).
\end{array}
$$

neither the upper horizontal arrow nor the vertical arrows are surjective. We see that the comparison to the integral group ring is not very useful for the question we are interested in.

The main techniques to prove results about the K-theory of $\mathbb{Z}\Gamma$ stems from "controlled topology". See [51], [49], [50], [17], [19] and [26]. The set-up has been adapted to a more algebraic setting [48] and this "controlled algebra" (see [48],[6] and [47]) was used successfully to obtain "lower bounds" for the K-theory of group rings with arbitrary coefficients under certain curvature conditions on the group [7].

A result about Conjecture 7.1 which uses this "controlled algebra" is the following result from [2]. Recall that a ring is called (right)-regular if it is right noetherian and every finitely generated right R-module admits finite dimensional projective resolution.

Theorem 7.4. *Let Γ be the fundamental group of a closed riemannian manifold with strictly negative sectional curvature. Let R be a regular ring, e.g. $R = \mathbb{C}$ then*

$$K_0(R) \cong K_0(R\Gamma).$$

Moreover $K_{-n}(R\Gamma) = 0$ and $K_1(R\Gamma) = \Gamma_{ab} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R)$, where Γ_{ab} denotes the abelianized group.

The assumption about Γ implies that Γ is torsionfree so the above verifies Conjecture 7.1.

The author is optimistic that in the near future techniques similar to the ones used in [2] will lead to further results about Conjecture 7.1. In view of condition (K) in Proposition 3.4 the following seems to be an important question.

Question 7.5. *Are the maps*

$$
\begin{aligned}
K_0(\mathbb{C}\Gamma) &\rightarrow K_0(\mathbb{C}\Gamma_\Sigma), \\
K_0(\mathbb{C}\Gamma) &\rightarrow K_0(\mathcal{R}\Gamma) \\
\text{or} \quad K_0(\mathbb{C}\Gamma) &\rightarrow K_0(\mathcal{D}\Gamma)
\end{aligned}
$$

surjective?

Note that this is true for groups in Linnell's class $\mathcal{C}$ with a bound on the orders of finite subgroups by Theorem 6.3 (K).

8. EXACTNESS PROPERTIES

In this section we want to investigate to what extent the functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$ and related functors are exact. Recall that this functor is crucial for the definition of L^2-Betti numbers, compare Definition 2.8.

Note 8.1. If Γ is elementary amenable and there is a bound on the orders of finite subgroups then $- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$ is exact.

Proof. From Addendum 6.4 (O) we know that for these groups $\mathcal{D}\Gamma$ is an Ore-localization of $\mathbb{C}\Gamma$. In particular in this case $- \otimes_{\mathbb{C}\Gamma} \mathcal{D}\Gamma$ is exact. Since by Theorem 6.3 (R) $\mathcal{D}\Gamma$ is also semisimple (and hence von Neumann regular) we know that every module is flat over $\mathcal{D}\Gamma$. $\square$

The following tells us that we cannot always have exactness.

Note 8.2. Suppose for the infinite group Γ the functor $- \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma$ is exact, then all L^2-Betti numbers and also the Euler-characteristic $\chi^{(2)}(\Gamma)$ of the group Γ vanishes.

Proof. Flatness implies

$$H_p(C_*(E\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma) = Tor_p^{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathcal{U}\Gamma) = Tor_p^{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma = 0$$

for $p > 0$. Moreover $b_0^{(2)}(\Gamma) = 0$ for every infinite group (see Theorem 6.54 (8) (b) in [39]). $\square$

In particular we see that for the free group on two generators we cannot have exactness. We saw this phenomenon already in Example 1.5 because exactness of $-\otimes_{\mathbb{C}\Gamma}\mathcal{N}\Gamma$ would contradict the monotony of the dimension. (Recall from Proposition 2.6 (ii) that $-\otimes_{\mathcal{N}\Gamma}\mathcal{U}\Gamma$ is always exact.)
More generally we have.

Note 8.3. If Γ contains a nonabelian free group, then neither $\mathcal{D}\Gamma$ nor $\mathcal{R}\Gamma$, $\mathbb{C}\Gamma_\Sigma$ or $\mathcal{U}\Gamma$ can be flat over $\mathbb{C}\Gamma$.

Proof. Every free group contains a free group on two generators. Let $G \subset \Gamma$ be a free subgroup on two generators. Let $\mathbb{C}G^2 \to \mathbb{C}G$ be the injective homomorphism from Example 1.5. Since $\mathbb{C}\Gamma$ is flat over $\mathbb{C}G$ we obtain an injective map $\mathbb{C}\Gamma^2 \to \mathbb{C}\Gamma$. On the other hand since $\mathcal{D}G$ is a skew-field we know that the non-trivial kernel of the corresponding map $\mathcal{D}G^2 \to \mathcal{D}G$ (which must appear for dimension reasons since $-\otimes_{\mathcal{D}G}\mathcal{U}G$ is exact and the Γ-dimension is faithful) is a one-dimensional free module which splits off $\mathcal{D}G^2$ as a direct summand. The same remains true for every overring of $\mathcal{D}G$. In particular for $\mathcal{D}\Gamma$, $\mathcal{R}\Gamma$ and $\mathcal{U}\Gamma$. But also for $\mathbb{C}\Gamma_\Sigma$ because $\mathcal{D}G = \mathcal{R}G \cong \mathbb{C}G_\Sigma$ by Addendum 6.4 (O) and since there is a natural map $\mathbb{C}G_\Sigma \to \mathbb{C}\Gamma_\Sigma$. $\square$

In this context we would also like to mention the following result from [38].

Theorem 8.4. *If Γ is amenable then $\mathcal{N}\Gamma$ (and hence $\mathcal{U}\Gamma$) is dimension-flat over $\mathbb{C}\Gamma$, i.e. for $p > 0$ and every $\mathbb{C}\Gamma$-module M we have*

$$\dim_{\mathcal{N}\Gamma} Tor_p^{\mathbb{C}\Gamma}(M,\mathcal{N}\Gamma) = \dim_{\mathcal{U}\Gamma} Tor_p^{\mathbb{C}\Gamma}(M,\mathcal{U}\Gamma) = 0.$$

Proof. See [38] or Theorem 6.37 on page 259 in [39] and recall that $\mathcal{U}\Gamma$ is flat over $\mathcal{N}\Gamma$ and the $\mathcal{U}\Gamma$-dimension and the $\mathcal{N}\Gamma$-dimension are compatible. $\square$

Given these facts it is tempting to conjecture that $-\otimes_{\mathbb{C}\Gamma}\mathcal{U}\Gamma$ is exact if and only if Γ is amenable. However in [35] it is shown that the condition about the bound on the orders of finite subgroups in Note 8.1 is necessary.

Example 8.5. Let H be a nontrivial finite group and let $H \wr \mathbb{Z}$ denote the semidirect product $\bigoplus_{-\infty}^{\infty} H \rtimes \mathbb{Z}$, where $\mathbb{Z}$ is acting via shift on the $\bigoplus_{-\infty}^{\infty} H$. Then neither $\mathcal{D}\Gamma$ nor $\mathcal{U}\Gamma$ is flat over $\mathbb{C}\Gamma$ (see Theorem 1 in [35]).

The main purpose of this section is to prove the following result which measures the deviation from exactness for groups in Linnell's class.

Theorem 8.6. *Let Γ be in the class $\mathcal{C}$ with a bound on the orders of finite subgroups, then*

$$\mathrm{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{D}\Gamma) = 0 \qquad \text{for all } p \geq 2.$$

Note that for these groups $\mathcal{D}\Gamma = \mathcal{R}\Gamma \cong \mathbb{C}\Gamma_\Sigma$ is semisimple and therefore the functor $-\otimes_{\mathcal{D}\Gamma}\mathcal{U}\Gamma$ is exact. The functor $-\otimes_{\mathbb{Z}\Gamma}\mathbb{C}\Gamma$ is always exact. Therefore we obtain the corresponding statements for $\mathrm{Tor}_p^{\mathbb{C}\Gamma}(-;\mathcal{U}\Gamma)$, $\mathrm{Tor}_p^{\mathbb{Z}\Gamma}(-;\mathcal{D}\Gamma)$ and $\mathrm{Tor}_p^{\mathbb{Z}\Gamma}(-;\mathcal{U}\Gamma)$. As an immediate consequence we obtain interesting examples of stably flat universal localizations.

Corollary 8.7. *If Γ lies in Linnell's class $\mathcal{C}$ and has a bound on the orders of finite subgroups then $\mathcal{D}\Gamma \cong \mathbb{C}\Gamma_\Sigma$ is stably flat over $\mathbb{C}\Gamma$, i.e. we have*

$$Tor_p^{\mathbb{C}\Gamma}(\mathcal{D}\Gamma,\mathcal{D}\Gamma) = 0 \qquad \text{for all } p \geq 1.$$

Proof. We know that $\mathcal{D}\Gamma \cong \mathbb{C}\Gamma_\Sigma$ is a universal localization of $\mathbb{C}\Gamma$ and hence $\mathbb{C}\Gamma \to \mathbb{C}\Gamma_\Sigma$ is an epimorphism in the category of rings, see page 56 in [57]. By Theorem 4.8 b) in [57] we know that $Tor_1^{\mathbb{C}\Gamma}(\mathcal{D}\Gamma, \mathcal{D}\Gamma) = 0$. For $p \geq 2$ the result follows from Theorem 8.6. $\square$

Recent work of Neeman and Ranicki [45] shows that for universal localizations which are stably flat there exists a long exact localization sequence which extends Schofield's localization sequence for universal localizations (see Theorem 4.12 in [57]) to the left. In the case of Ore-localizations the corresponding sequence was known for a long time, see [20], [22], [64] and [60]. Observe that because of Note 8.3 we know that whenever Γ contains a free group $\mathbb{C}\Gamma_\Sigma$ cannot be an Ore-localization. Here is another consequence of Theorem 8.6.

Corollary 8.8. *If the infinite group Γ belongs to $\mathcal{C}$ and has a bound on the orders of finite subgroups, then*

$$\chi^{(2)}(\Gamma) \leq 0.$$

Proof. Since the group is infinite we have $b_0^{(2)}(\Gamma) = 0$. Because of

$$H_p(\Gamma; \mathcal{U}\Gamma) = \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathbb{C}; \mathcal{U}\Gamma) = 0 \qquad \text{for all } p \geq 2$$

we know that $b_1^{(2)}(\Gamma)$ is the only L^2-Betti number which could possibly be nonzero. $\square$

The $L^{(2)}$-Euler characteristic coincides with the usual Euler-characteristic and the rational Euler-Characteristic of [63] whenever these are defined.
Before we proceed to the proof of Theorem 8.6 we would also like to mention the following consequences for L^2-homology.

Corollary 8.9 (Universal Coefficient Theorem)**.** *Let Γ be in $\mathcal{C}$ with a bound on the orders of finite subgroups. Then there is a universal coefficient theorem for L^2-homology: Let X be a Γ-space whose isotropy groups are all finite, then there is an exact sequence*

$$0 \to H_n(X; \mathbb{C}) \otimes_{\mathbb{C}\Gamma} \mathcal{U}\Gamma \to H_n^\Gamma(X; \mathcal{U}\Gamma) \to \mathrm{Tor}_1^{\mathbb{C}\Gamma}(H_{n-1}(X; \mathbb{C}); \mathcal{U}\Gamma) \to 0.$$

Proof. We freely use the dimension theory for arbitrary $\mathcal{U}\Gamma$-modules, compare Remark 2.9. If X has finite isotropy, then the set of singular simplices also has only finite isotropy groups. If H is a finite subgroup of Γ, then $\mathbb{C}[\Gamma/H] \cong \mathbb{C}\Gamma \otimes_{\mathbb{C}H} \mathbb{C}$ is induced from the projective $\mathbb{C}H$-module $\mathbb{C}$ and therefore projective. We see that the singular chain complex with complex coefficients $C_* = C_*^{sing}(X; \mathbb{C})$ is a complex of projective $\mathbb{C}\Gamma$-modules. The E^2-term of the Künneth spectral sequence (compare Theorem 5.6.4 on page 143 in [65])

$$E_{pq}^2 = \mathrm{Tor}_p^{\mathbb{C}\Gamma}(H_q(C_*); \mathcal{D}\Gamma) \Rightarrow H_{p+q}(C_* \otimes \mathcal{D}\Gamma) = H_{p+q}(X; \mathcal{D}\Gamma)$$

is concentrated in two columns. The spectral sequence collapses, and we get exact sequences

$$0 \to H_n(X; \mathbb{C}) \otimes_{\mathbb{C}\Gamma} \mathcal{D}\Gamma \to H_n(X; \mathcal{D}\Gamma) \to \mathrm{Tor}_1^{\mathbb{C}\Gamma}(H_{n-1}(X; \mathbb{C}); \mathcal{D}\Gamma) \to 0.$$

Applying the exact functor $- \otimes_{\mathcal{D}\Gamma} \mathcal{U}\Gamma$ yields the result. $\square$

The proof of Theorem 8.6 depends on the following Lemma.

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Lemma 8.10. (i) *Let $R * G \subset S * G$ be compatible with the crossed product structure. Let M be an $R * G$-module. There is a natural isomorphism of right S-modules*

$$\operatorname{Tor}_p^{R*G}(M; S * G) \cong \operatorname{Tor}_p^R(\operatorname{res}_R^{R*G} M; S)$$

for all $p \geq 0$.

(ii) *Suppose $R \subset S$ is a ring extension and $R = \bigcup_{i \in I} R_i$ is the directed union of the subrings R_i. Let M be an R-module. Then there is a natural isomorphism of right S-modules*

$$\operatorname{Tor}_p^R(M; S) \cong \operatorname{colim}_{i \in I} \operatorname{Tor}_p^{R_i}(\operatorname{res}_{R_i}^R M; S_i) \otimes_{S_i} S$$

for all $p \geq 0$.

Proof. (i) We start with the case $p = 0$. We denote the crossed product structure map by μ, compare Digression 6.5. Define a map

$$h_M : \operatorname{res}_R^{R*G} M \otimes_R S \to M \otimes_{R*G} S * G$$

by $m \otimes s \mapsto m \otimes s$. Obviously h is a natural transformation from the functor $\operatorname{res}_R^{R*G}(-) \otimes_R S$ to $- \otimes_{R*G} S * G$. If $M = R * G$ the map $h_{R*G}^{-1} : R * G \otimes_{R*G} S * G \cong S * G \to \operatorname{res}_R^{R*G} R * G \otimes_R S$ given by $s\mu(g) \mapsto g \otimes c_g^{-1}(s)$ is a well-defined inverse. Since h is compatible with direct sums we see that h_F is an isomorphism for all free modules F. Now if M is an arbitrary module choose a free resolution $F_* \to M$ of M and apply both functors to

$$F_1 \to F_0 \to M \to 0 \to 0.$$

Both functors are right exact, therefore an application of the five lemma yields the result for $p = 0$. Now let $P_* \to M$ be a projective resolution of M, then

$$
\begin{aligned}
\operatorname{Tor}_p^R(\operatorname{res}_R^{R*G} M; S) &= H_p(\operatorname{res}_R^{R*G} P_* \otimes_R S) \\
&\overset{\cong}{\to} H_p(P_* \otimes_{R*G} S * G) \\
&= \operatorname{Tor}_p^R(M; S * G).
\end{aligned}
$$

(ii) Again we start with the case $p = 0$. The natural surjections $\operatorname{res}_{R_i}^R M \otimes_{R_i} S \to M \otimes_R S$ induce a surjective map

$$h_M : \operatorname{colim}_{i \in I} \operatorname{res}_{R_i}^R M \otimes_{R_i} S \to M \otimes_R S$$

which is natural in M. Suppose the element of the colimit represented by $\sum_k m_k \otimes s_k \in \operatorname{res}_{R_i}^R M \otimes_{R_i} S$ is mapped to zero in $M \otimes_R S$. By construction the tensor product $M \otimes_R S$ is the quotient of the free module on the set $M \times S$ by a relation submodule. But every relation involves only finitely many elements of R, so we can find a $j \in I$ such that $\sum_k m_k \otimes s_k = 0$ already in $\operatorname{res}_{R_j}^R M \otimes_{R_j} S$. We see that h_M is an isomorphism. Now let $P_* \to M$ be a projective resolution. Since the colimit is an exact functor it commutes with homology and we get

$$
\begin{aligned}
\operatorname{colim}_{i \in I} \operatorname{Tor}_p^{R_i}(M; S) &= \operatorname{colim}_{i \in I} H_p(\operatorname{res}_{R_i}^R P_* \otimes_{R_i} S) \\
&= H_p(\operatorname{colim}_{i \in I}(\operatorname{res}_{R_i}^R P_* \otimes_{R_i} S)) \\
&\overset{\cong}{\to} H_p(P_* \otimes_R S) \\
&= \operatorname{Tor}_p^R(M; S).
\end{aligned}
$$

$\square$

Proof of Theorem 8.6. The proof works via transfinite induction over the group as for the proof of Linnell's Theorem 6.3 itself, compare (I), (II) and (III) on page 19.
(I) The statement for free groups is well known: let Γ be the free group generated by the set S. The cellular chain complex of the universal covering of the obvious 1-dimensional classifying space gives a projective resolution of the trivial module of length one

$$0 \to \bigoplus_S \mathbb{C}\Gamma \to \mathbb{C}\Gamma \to \mathbb{C} \to 0.$$

Now if M is an arbitrary $\mathbb{C}\Gamma$-module we apply $- \otimes_\mathbb{C} M$ to the above complex and get a projective resolution of length 1 for M (diagonal action). (Use that for P a projective $\mathbb{C}\Gamma$-module $P \otimes_\mathbb{C} M$ with the diagonal respectively the left Γ-action are noncanonically isomorphic $\mathbb{C}\Gamma$-modules.)
(II) The next step is to prove that the statement remains true under extensions by finite groups. So let $1 \to G \to \Gamma \to H \to 1$ be an exact sequence with H finite. We know that $\mathcal{D}\Gamma = \mathcal{D}G*H$, see Lemma 10.59 on page 399 in [39] or Proposition 8.13 in [53]. Let M be a $\mathbb{C}\Gamma$-module, then with Lemma 8.10 and the induction hypothesis we conclude

$$
\begin{aligned}
\operatorname{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &= \operatorname{Tor}_p^{\mathbb{C}G*H}(M; \mathcal{D}G * H) \\
&\cong \operatorname{Tor}_p^{\mathbb{C}G}(\operatorname{res}_{\mathbb{C}G}^{\mathbb{C}G*H} M; \mathcal{D}G) \\
&= 0 \quad \text{for } p > 1.
\end{aligned}
$$

The case H infinite cyclic is only slightly more complicated. This time we know from Lemma 10.69 in [39] or Proposition 8.18 in [53] that $\mathcal{D}\Gamma = (\mathcal{D}G * H)T^{-1}$ is an Ore localization, where $T = T(\mathcal{D}G * H \subset \mathcal{U}\Gamma)$, i.e. the set of all elements in $\mathcal{D}G * H$ which become invertible in $\mathcal{U}\Gamma$. Since Ore localization is an exact functor we get

$$
\begin{aligned}
\operatorname{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &= \operatorname{Tor}_p^{\mathbb{C}G*H}(M; (\mathcal{D}G * H)T^{-1}) \\
&\cong \operatorname{Tor}_p^{\mathbb{C}G*H}(M; \mathcal{D}G * H) \otimes_{\mathcal{D}G*H} \mathcal{D}\Gamma
\end{aligned}
$$

and conclude again with Lemma 8.10 that this module vanishes if $p > 1$.
(III) The behaviour under directed unions remains to be checked. Let $\Gamma = \bigcup_{i \in I} \Gamma_i$ be a directed union, then using Definition 2.4 we see that $\bigcup_{i \in I} \mathcal{D}\Gamma_i$ is von Neumann regular and it is easy to check that it coincides with the division closure $\mathcal{D}\Gamma$. Now Lemma 8.10 gives

$$
\begin{aligned}
\operatorname{Tor}_p^{\mathbb{C}\Gamma}(M; \mathcal{D}\Gamma) &\cong \operatorname{colim}_{i \in I} \operatorname{Tor}_p^{\mathbb{C}\Gamma_i}(\operatorname{res}_{\mathbb{C}\Gamma_i}^{\mathbb{C}\Gamma} M; \mathcal{D}\Gamma) \\
&= \operatorname{colim}_{i \in I} \operatorname{Tor}_p^{\mathbb{C}\Gamma_i}(\operatorname{res}_{\mathbb{C}\Gamma_i}^{\mathbb{C}\Gamma} M; \mathcal{D}\Gamma_i) \otimes_{\mathcal{D}\Gamma_i} \mathcal{D}\Gamma \\
&= 0 \quad \text{for } p > 1.
\end{aligned}
$$

$\square$

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