

A FOLIATED SQUEEZING THEOREM FOR GEOMETRIC MODULES

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ABSTRACT. We prove a foliated control theorem for automorphisms of geometric modules. This is the analogue of a result for h-cobordism from [3].

1. INTRODUCTION

For a compact connected Riemannian manifold M it makes sense to talk of the size of an h-cobordism over M . A typical and well known theorem in controlled topology says that for manifolds of dimension ≥ 5 we have (compare [2]):

A sufficiently small h-cobordism over M is trivial.

Let Γ denote the fundamental group of M , then translated to the language of geometric modules (compare [7]) the statement reads:

A sufficiently small automorphism of a geometric module over M represents the trivial class in the Whitehead group $Wh(\Gamma)$.

In connection with the Isomorphism Conjecture in algebraic K -theory we shifted our attention from vanishing results towards computations of K -theory groups of group rings $R\Gamma$, compare [5] and [1]. In general, i.e. for groups containing torsion or coefficient rings R other than \mathbb{Z} , the Whitehead group does not vanish and one should work directly with the K -theory groups. The following statement is equivalent to the above but more convenient for generalizations.

A sufficiently small automorphism of a geometric module over M can be deformed (via a small deformation) to an arbitrarily small automorphism (which represents the same class in $K_1(\mathbb{Z}\Gamma)$).

In the following we call such a statement a squeezing theorem. Our aim in this paper is to prove a foliated squeezing theorem for automorphisms of geometric modules over a foliated Riemannian manifold. Before we formulate the result let us discuss its relevance.

In [1] we prove certain cases of the Isomorphism Conjecture in algebraic K -theory. There the assembly map which is conjectured to be an isomorphism is reinterpreted as a “forget-control map”. In particular the problem of proving surjectivity becomes the problem of regaining control. We there follow the ideas of [3] and use the geodesic flow on a negatively curved Riemannian manifold as a tool to regain control. But the control that is obtained this way is only foliated control instead of ordinary control, see Section 7. In this paper we prove the following theorem which is responsible for the remaining step from foliated to ordinary control.

Theorem 1.1. *Let N be a Riemannian manifold which is equipped with a 1-dimensional smooth foliation. Let $\alpha > 0$ be an arbitrarily large and $\epsilon > 0$ be an arbitrarily small number. Fix a compact subset N_2 of N which does not meet the short closed leaves of the foliation (what “short” means depends on α).*

There exists a constant $C \geq 1$ such that the following holds:

Suppose we are given an automorphism of a (geometric) module over N which together with its inverse is (α, δ) -foliated controlled. If δ is sufficiently small then one can find a deformation such that the deformed automorphism and its inverse are both (non-foliated) ϵ -controlled over N_2 and still $(C\alpha, C\delta)$ -controlled everywhere else.

In fact we need an equivariant version and we also need control over the constructed deformation. For a more detailed (and more technical) statement see Theorem 9.5. Along the way we also reprove the ordinary squeezing statement for compact connected Riemannian manifolds mentioned above, see Proposition 6.6 and Theorem 9.4.

For the reader familiar with [3] the proof of Theorem 1.1 contains no surprise. It relies on two main ingredients: A relative version of an ordinary squeezing theorem for simple geometric standard situations (like finite polyhedra see 6.1 and its Corollaries) which goes back to [8] and the long and thin cell structure from Section 7 in [3]. Apart from these two results the technical difficulty is the bookkeeping for the control data.

In order to make the proof more transparent we develop a uniform language for squeezing theorems and formulate a couple of easy general principles which are then used throughout the paper. In particular we found it helpful to organize the control data in tables which contain all the essential information about a squeezing. In Section 4 we define a “relative squeezing situation” and what it means that “there exists a relative squeezing” for a given such situation (compare Definition 4.3). Then in Section 5 we describe abstractly the induction process that is used later to produce squeezing statements on a large scale out of local relative squeezing statements (see 5.4).

In Section 6 we review the well known relative squeezings for some geometric standard situations. These results go back to [8]. Recently a new proof (which avoids the use of the torus trick) of a squeezing result for finite polyhedra appeared in [6]. We briefly discuss the trivial modifications of that proof necessary to obtain the relative version we need.

Section 7 discusses foliated versions of the general principles developed in Section 5. Once these are established the proof of the foliated squeezing theorem in Section 8 is a straightforward consequence of the existence of a suitable covering as described in Section 7 and 8 of [3].

The first two sections briefly review the definitions of geometric modules and deformations and introduce the different notions of control. Note that finally we will work with equivariant modules and morphisms on the universal covering where short paths are essentially determined by their endpoints. For this reason our morphisms do not involve paths.

2. GEOMETRIC MODULES

In this section we briefly recall the notions of modules and morphisms over a space. For more details the reader can consult Section 2 and 11 of [1].

Throughout the whole paper we fix an associative ring with unit R . Let X be a topological space. A locally finite R -module M over X is a family of free R -modules $M = (M_x)_{x \in X}$ such that for every compact subset $K \subset X$ the module $\bigoplus_{x \in K} M_x$ is finitely generated. A morphism $\phi = (\phi_{y,x})$ from M to N is a family of R -linear maps $\phi_{y,x} : M_x \rightarrow N_y$ such that for fixed x the set of y with $\phi_{y,x} \neq 0$ is finite and for fixed y the set of x with $\phi_{y,x} \neq 0$ is finite. The composition of morphisms is given by matrix multiplication, i.e.

$$(\phi \circ \psi)_{z,x} = \sum_y \phi_{z,y} \circ \psi_{y,x}.$$

The category of all locally finite modules and morphisms over X is denoted $\mathcal{C}(X)$. There is an obvious notion of direct sum of modules and in fact $\mathcal{C}(X)$ is an additive category. As in every additive category there is a notion of elementary morphisms and deformations. An automorphism $e : M \rightarrow M$ is called elementary if there is a decomposition $M = M_0 \oplus \cdots \oplus M_k$ such that with respect to this decomposition e has only one nonzero off-diagonal entry and identities on the diagonal. A deformation $\eta = (e_1, \dots, e_l)$ is a finite sequence of elementary automorphisms with respect to a fixed decomposition of a fixed module M . Every deformation η has an underlying automorphism namely the product $\bar{\eta} = e_1 \circ \cdots \circ e_l$.

An automorphism ϕ represents an element $[\phi]$ in $K_1(\mathcal{C}(X))$. A deformation η represents an element in $K_2(\mathcal{C}(X))$. Note that composing an automorphism with the underlying automorphism of a deformation does not change its class in $K_1(\mathcal{C}(X))$.

Given a locally finite R -module M we define the support of M as

$$\text{supp}(M) = \{x \in X \mid M_x \neq 0\} \subset X$$

and the support of a morphism ϕ as

$$\text{supp}(\phi) = \{(x, y) \mid \phi_{y,x} \neq 0\} \subset X \times X.$$

For deformations we are mostly interested in the support of the off diagonal entries. We define the support of a deformation $\eta = (e_1, \dots, e_l)$ as the union of the sets $\text{supp}(\phi) - \Delta$, where ϕ varies over all partial products of the e_i and Δ denotes the diagonal in $X \times X$. Note that the support of a deformation η does not coincide with the support of the underlying automorphism $\bar{\eta}$. If X is a metric space, then a morphism is said to be bounded if

$$|\phi| = \sup\{d(x, y) \mid x, y \in \text{supp}\} < \infty.$$

If $A \subset X$ and $\delta > 0$ we denote by A^δ the δ -thickening of A , i.e. the set of all points with distance less than δ to A . More general, if E is a symmetric neighborhood of the diagonal in $X \times X$ (where X is no longer necessary a metric space), we set

$$A^E = \{x \in X \mid (x, a) \in E \text{ for some } a \in A\}.$$

In a metric space we clearly have $A^\delta = A^{E_\delta}$ where E_δ consists of all pairs of points of distance less than δ . We will also use the notations $A^{-\delta}$ for the set of point with distance greater than δ to the complement of A , i.e. $A^{-\delta} = ((A^c)^\delta)^c$ if A^c denotes the complement of A in X . Observe that $A^\delta \subset B$ is equivalent to $A \subset B^{-\delta}$.

3. CONTROL

In the following it will be important to use the right notion of ‘‘control over a subset’’. For a module M over X and a subset $B \subset X$ we denote by $M|_B$ the largest submodule of M with support in B . Let i_B and p_B be the inclusion of respectively the projection onto $M|_B$. For a morphism ϕ we then have the restriction $\phi|_B = p_B \circ \phi \circ i_B$.

Definition 3.1 (Control). Let X be a metric space or let $E \subset X \times X$ be a symmetric neighbourhood of the diagonal. Let $B \subset X$. Let ϕ and ψ be morphisms in $\mathcal{C}(X)$.

- (i) A morphism ϕ is α -controlled if $|\phi| \leq \alpha$. A morphism ϕ is E -controlled if $\text{supp}(\phi) \subset E$.
- (ii) An automorphism ϕ in $\mathcal{C}(X)$ is called an α -automorphism if ϕ and ϕ^{-1} are α -controlled, it is called an E -automorphism if ϕ and ϕ^{-1} are E -controlled.
- (iii) A morphism ϕ is α -controlled over $B \subset X$ if for every $x \in X$ and $y \in B$ with $d(x, y) > \alpha$ we have $\phi_{x,y} = \phi_{y,x} = 0$. A morphism ϕ is said to be E -controlled over B if

$$\text{supp}(\phi \circ i_B) \subset E \quad \text{and} \quad \text{supp}(p_B \circ \phi) \subset E$$

or equivalently $\phi \circ i_B = i_{B^E} \circ p_{B^E} \circ \phi \circ i_B$ and $p_B \circ \phi = p_B \circ \phi \circ i_{B^E} \circ p_{B^E}$.

Warning 3.2. Control over B is not a local notion, i.e. we can not compute the control of ϕ over B from the knowledge of $\phi|_B$.

However we have the following two lemmata which we will use in the sequel. Both are easy consequences of the definitions.

Lemma 3.3. *Let \mathcal{U} be a set of subsets of X and let ϕ be a morphism in $\mathcal{C}(X)$, then ϕ is E -controlled over $\bigcup_{U \in \mathcal{U}} U$ if and only if for each $U \in \mathcal{U}$ it is E -controlled over U .*

Lemma 3.4. *Let ϕ and ψ in $\mathcal{C}(X)$ be E -controlled and let A and B be subsets of X with $B \supset A^E$, then $(\phi|_B \circ \psi|_B)|_A = (\phi \circ \psi)|_A$. Moreover ϕ and $\phi|_B$ have the same control over A .*

We also need control notions for deformations. Note that they are stronger than the corresponding notions for the underlying automorphism.

Definition 3.5. Let X be a metric space or let $E \subset X \times X$ be a symmetric neighborhood of the diagonal. A deformation $\eta = (e_1, \dots, e_l)$ is α -controlled if $\sum |e_i| \leq \alpha$. The deformation is E -controlled if all partial products of the e_i are E -controlled.

4. RELATIVE SQUEEZINGS

Squeezing theorems come as a patchwork of local relative squeezing theorems (or briefly local relative squeezings) which are put together according to the principles we will develop in Section 5. Here *local* refers to the fact that we are dealing with a small part of a larger ambient space and *relative* indicates that over a certain part of this space we already achieved very good control and our squeezing should not destroy this.

In this section we formalize the notion of a relative squeezing (see Definition 4.3). A special case is an absolute squeezing, the type of squeezing we are really after.

Let U be a metric space. Let U_1, U_2, U_4 and U_3 be subsets of U with

$$U_2 \subset U_4 \subset U_3.$$

The slightly odd indexing is chosen to better fit with the notation for the “models” in 6.4 in [3]. We call such a 5-tuple (U, U_1, U_2, U_4, U_3) a *relative squeezing situation*. If $U_1 = \emptyset$ we talk of an *absolute squeezing situation*. In the following one should think of these subsets as follows:

- U_1 = region where we already have very good control
- U_2 = region where we want to improve control
- U_4 = the working zone, i.e. here the deformation will take place

Recall that control is not a local notion (compare Warning 3.2) and hence even though the deformation takes place over U_4 the control may change over a larger region. Hence

$$U_3 = \text{region where the control changes at all.}$$

In practice squeezing situations will only be useful if we have certain security-zones:

Definition 4.1 (Security). Let $\epsilon_s > 0$. A relative squeezing situation (U, U_i) has ϵ_s -security if $U_2^{\epsilon_s} \subset U_4$ and $U_4^{\epsilon_s} \subset U_3$. More generally let $E \subset U \times U$ be a symmetric neighborhood of the diagonal. We say that (U, U_i) has E -security if $U_2^E \subset U_4$ and $U_4^E \subset U_3$.

Usually U is embedded in some larger metric space. In Section 5 we will discuss the relation to this ambient space and it will be convenient to have the following definition.

Definition 4.2 (Security in Y). Let $\epsilon_s > 0$. Let U be a subspace of the metric space Y . A relative squeezing situation (U, U_i) has ϵ_s -security in Y if $U_2^{\epsilon_s} \subset U_4$, $U_4^{\epsilon_s} \subset U_3$ and $U_3^{\epsilon_s} \subset U$. Here the thickenings are taken in Y . Let $E \subset Y \times Y$ be a symmetric neighborhood of the diagonal. We say that (U, U_i) has E -security in Y if $U_2^E \subset U_4$, $U_4^E \subset U_3$ and $U_3^E \subset U$.

Note that in the case where $U = Y$ we get back Definition 4.1 because the condition $U_3^{\epsilon_s} \subset U$ is trivially satisfied. The reader who wants to see typical relative squeezing situations should consult Example 5.5.

The following definition is crucial for the whole paper. Note that it makes sense for an arbitrary squeezing situation.

Definition 4.3 (Relative squeezing). We say that there exists a *relative squeezing for the situation* (U, U_1, U_2, U_4, U_3) if the Statement 4.4 below is true. The constant ϵ_0 is called the *a priori control*, the function r is called the *response function*. In the special case where $U_1 = \emptyset$ we talk of an *absolute squeezing*.

Statement 4.4. There exists an $\epsilon_0 > 0$ and a homeomorphism

$$r : [0, \infty) \rightarrow [0, \infty)$$

with $r(\epsilon) \geq \epsilon$ such that for every ϵ_{vg} and ϵ_g with $0 < \epsilon_{vg} \leq \epsilon_g \leq \epsilon_0$ the following holds:

Let ϕ and ψ be endomorphisms of a module M over U which are inverses of one another over U_3 , i.e.

$$(\phi \circ \psi)|_{U_3} = (\psi \circ \phi)|_{U_3} = \text{id}_{M|_{U_3}}.$$

Suppose that both endomorphisms are ϵ_g -controlled over U_3 and both are ϵ_{vg} -controlled over U_1 , then we can find a stabilizing module L and a deformation η on $M \oplus L$ such that

- (i) $\text{supp}(L) \subset U_4$ and $\text{supp}(\eta) \subset U_4 \times U_4$.
- (ii) η (and hence $\bar{\eta}$) is $r(\epsilon_g)$ -controlled.

Moreover the deformed endomorphisms

$$\phi_{new} = \bar{\eta} \circ \phi \quad \text{and} \quad \psi_{new} = \psi \circ \bar{\eta}^{-1}$$

are both $r(\epsilon_g)$ -controlled over U_3 and they are both $r(\epsilon_{vg})$ -controlled over $U_1 \cup U_2$.

Briefly: There exists an $\epsilon_0 > 0$ and a response function $r : [0, \infty) \rightarrow [0, \infty)$ such that given any ϵ_{vg} and ϵ_g with $0 < \epsilon_{vg} \leq \epsilon_g \leq \epsilon_0$ we can achieve the following control improvement by working only on U_4 :

control over	U_1	U_2	U_3	\parallel	$U - U_3$
before	ϵ_{vg}		$\epsilon_g \leq \epsilon_0$	\parallel	
after	$r(\epsilon_{vg})$	$r(\epsilon_{vg})$	$r(\epsilon_g)$	\parallel	no change

Note that in the absolute case, where U_1 is empty we get the following statement because r is a homeomorphism:

There exists an $\epsilon_0 > 0$ and a response function r such that given any $0 < \epsilon_{vg} \leq \epsilon_g \leq \epsilon_0$ we can achieve the following control improvement working on U_4 :

control over	U_2	U_3	\parallel	$U - U_3$
before		$\epsilon_g \leq \epsilon_0$	\parallel	
after	ϵ_{vg}	$r(\epsilon_g)$	\parallel	no change

In particular we can achieve arbitrary good control over U_2 .

Remark 4.5. Suppose there exists a relative squeezing for the situation (U, U_i) . Then there exists a relative squeezing with the same response function and the same a priori control for every squeezing situation (U, U'_i) obtained by enlarging U_4 and U_3 , making U_2 smaller and keeping U_1 . The only not completely trivial observation here is that U'_3 is the disjoint union of U_3 (where we have $r(\epsilon_g)$ control afterwards) and $U'_3 \cap (U - U_3)$ (where nothing changes and we had ϵ_g and hence $r(\epsilon_g)$ -control before).

5. SOME GENERAL PRINCIPLES

In this section we discuss a couple of general principles which hold for squeezings.

Let (U, U_i) be a relative squeezing situation where U is embedded in some larger space Y . Suppose there exists a squeezing for (U, U_i) . The following lengthy Remark basically says that if (U, U_i) has E -security in Y and we are given an E -automorphism then we can cut down the automorphisms to U , apply a relative squeezing for U (if the a priori control-condition is

fulfilled) and glue the result back to Y . The second remark says that we can work simultaneously over different pieces of the big space if the zones where we change control (the U_3 's) are pairwise disjoint.

Remark 5.1. Global to local and back.

Let U be a subset of Y and let (U, U_i) be a relative squeezing situation. Let E be a symmetric neighborhood of the diagonal in $Y \times Y$. Let Φ and Ψ be endomorphisms of the module M in $\mathcal{C}(Y)$ which are E -controlled over U_3 .

Global to local.

Suppose Φ and Ψ are inverse to one another over U_3 , i.e.

$$(\Phi \circ \Psi)|_{U_3} = (\Psi \circ \Phi)|_{U_3} = \text{id}_{M|_{U_3}}$$

If $U_3^E \subset U$, i.e. there is a security zone between U_3 and $Y - U$ which is at least E -thick. Then the endomorphisms $\phi = \Phi|_U$ and $\psi = \Psi|_U$ are inverse to one another over U_3 (cf. Lemma 3.4). Moreover ϕ and ψ respectively Φ and Ψ have the same control over U_3 .

Local to global.

Suppose L_U is a stabilizing module over U and η_U is a deformation on $M|_U \oplus L_U$ such that

- (a) $\text{supp}(L_U) \subset U_4$ and $\text{supp}(\eta_U) \subset U_4 \times U_4$.

Let L be the module L_U considered as a module over Y . Extend η_U trivially to a deformation $\eta = \eta_U \oplus \text{id}_{M|_{Y-U}}$ on $M \oplus L$. Then L and η again satisfy

- (a) above. Let $\Phi_{new} = \bar{\eta} \circ (\Phi \oplus \text{id}_L)$ and $\Psi_{new} = (\Psi \oplus \text{id}_L) \circ \bar{\eta}^{-1}$.

Suppose (U, U_i) has E -security in Y then the support of Φ_{new} and Ψ_{new} can be computed as follows (cf. Lemma 3.3).

- (i) The control does not change outside of U_3 , i.e. Φ and Φ_{new} have the same control over $Y - U_3$. Similar for Ψ and Ψ_{new} .
- (ii) The control change over U_3 is determined by the local control change, i.e. Φ_{new} and $\phi_{new} = \eta_U \circ (\Phi|_U \oplus \text{id}_L)$ have the same control over U_3 . Similar for Ψ_{new} .

Note that if $\Phi = \Psi^{-1}$ is an automorphism then $\Phi_{new} = \Psi_{new}^{-1}$ is also an automorphism and Φ_{new} and Φ represent the same class in $K_1(\mathcal{C}(Y))$.

Even though it is convenient to state part of the assumption above as “ (U, U_i) has E -security in Y ” we did not really use all conditions of Definition 4.2. Namely $U_2^E \subset U_4$ was not used above. This condition will only be important later in 6.1.

Remark 5.2. Working simultaneously over disjoint pieces.

Suppose we are in the situation of the previous remark but instead of a single subset U we have a covering \mathcal{U} of a subset of Y , such that each $U \in \mathcal{U}$ comes equipped with a relative squeezing situation (U, U_i) . Let Φ and Ψ be E -controlled over $\bigcup_{U \in \mathcal{U}} U_3$. Suppose each (U, U_i) has E -security in Y . Suppose there exists for each $U \in \mathcal{U}$ an L_U and an η_U satisfying (a) as in Remark 5.1 above.

If the U_3 are pairwise disjoint, then there exists a module L and a deformation η satisfying

$$(a') \text{ supp}(L) \subset \bigcup_{U \in \mathcal{U}} U_4 \text{ and } \text{supp}(\eta) \subset \bigcup_{U \in \mathcal{U}} U_4 \times U_4.$$

The deformed endomorphisms $\Phi_{new} = \bar{\eta} \circ (\Phi \oplus \text{id}_L)$ and $\Psi_{new} = (\Psi \oplus \text{id}_L) \circ \bar{\eta}^{-1}$ have no control change outside of $\bigcup_{U \in \mathcal{U}} U_3$ (compare (i)) and the control change over each U_3 is determined by the local control change as in (ii) above.

Warning 5.3. Let the situation be as in the previous remark. In particular suppose we have disjoint control change-zones. Suppose each (U, U_i) admits a relative squeezing with response function r and a priori control ϵ_0 . It may happen that while applying a squeezing simultaneously over the disjoint pieces as above we destroy control which was already very good. This can happen if $U_3 - (U_1 \cup U_2)$ meets U'_1 for different $U, U' \in \mathcal{U}$. Abbreviate $U_p = U_3 - (U_1 \cup U_2)$ (“p” for possibly problematic). Let \mathcal{U}_i denote the collection of sets $\{U_i | U \in \mathcal{U}\}$ and write

$$|\mathcal{U}_i| = \bigcup_{U_i \in \mathcal{U}_i} U_i.$$

The best squeezing statement we can obtain is the following: Working over $|\mathcal{U}_4|$ we can achieve:

control over	$ \mathcal{U}_1 $	$ \mathcal{U}_2 $	$ \mathcal{U}_3 $	$ $	$ \mathcal{U} - \mathcal{U}_3 $
before	ϵ_{vg}		$\epsilon_g \leq \epsilon_0$	$ $	
control over	$ \mathcal{U}_1 - \mathcal{U}_p $	$ \mathcal{U}_2 $	$ \mathcal{U}_3 $	$ $	$ \mathcal{U} - \mathcal{U}_3 $
after	$r(\epsilon_{vg})$	$r(\epsilon_{vg})$	$r(\epsilon_g)$	$ $	no change

The Induction Principle

We are now prepared to describe the formal induction process which out of local relative squeezings for small pieces of a space produces a squeezing on a larger scale. Let Y be a space. Let \mathcal{U} be a covering of some subset of Y which comes as a disjoint union

$$\mathcal{U} = \mathcal{U}^{(0)} \cup \mathcal{U}^{(1)} \cup \dots \cup \mathcal{U}^{(n)}.$$

Suppose every covering set U is equipped with subsets U_i , such that each (U, U_i) is a relative squeezing situation. Let $Y_1 \subset Y$ be given and define inductively the region $R^{(q)}$ where we want to achieve very good control after the q -th induction step:

$$\begin{aligned} R^{(-1)} &= Y_1 \\ R^{(q)} &= \left(R^{(q-1)} \cup |\mathcal{U}_2^{(q)}| \right) - \bigcup_{U \in \mathcal{U}^{(q)}} (U_3 - (U_1 \cup U_2)) \end{aligned}$$

This definition reflects the fact that on those parts of the working zone which do not lie in U_1 or U_2 we may destroy control we obtained in previous induction steps. Compare Warning 5.3. Suppose that

- (i) For each $U \in \mathcal{U}^{(q)}$ we have $U_1 \subset R^{(q-1)}$.

This will ensure the hypothesis for the q -th induction step. Note that in particular $U_1 = \emptyset$ is allowed.

- (ii) For fixed q we have disjoint security zones, i.e. the U_3 with $U \in \mathcal{U}^{(q)}$ are pairwise disjoint.

Moreover set $Y_2 = |\mathcal{U}_2| \cap R^{(n)}$, $Y_4 = |\mathcal{U}_4|$ and $Y_3 = |\mathcal{U}_3|$. Assume that

- (iii) $Y_1 \subset R^{(n)}$.

This last condition is only necessary to simplify the statement. It follows for example if for every $U \in \mathcal{U}$ we have $(U_3 - U_1 \cup U_2) \cap Y_1 = \emptyset$. In applications that produce absolute squeezings, we have of course $Y_1 = \emptyset$ and condition (iii) is empty.

Proposition 5.4 (Induction Principle). *Let $\epsilon_0 > 0$ and let $r : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. Let \mathcal{U} satisfy the conditions above. Suppose*

- (i) *For each $U \in \mathcal{U}$ there exists a relative squeezing for (U, U_i) with a priori control ϵ_0 and response function r .*
(ii) *Each (U, U_i) has ϵ_s -security in Y with $\epsilon_s \geq \epsilon_0$.*

Then there exists a squeezing for (Y, Y_i) with a priori control $(r^n)^{-1}(\epsilon_0)$ and response function $r^{n+1} = r \circ r \circ \dots \circ r$. Moreover (Y, Y_i) has ϵ_s -security.

Proof. We again abbreviate $U_p = U_3 - U_1 \cup U_2$. One can use the statement in 5.3 with $\mathcal{U}^{(q)}$ instead of \mathcal{U} to see that in the q -th step one can obtain the following by working over $|\mathcal{U}_4^{(q)}|$:

control over	$R^{(q-1)}$	$ \mathcal{U}_2^{(q)} $	$ \mathcal{U}_3^{(q)} $	$Y - \mathcal{U}_3^{(q)} $
before	$r^q(\epsilon_{vg})$		$r^q(\epsilon_g) \leq \epsilon_0$	
control over	$R^{(q-1)} - \mathcal{U}_p^{(q)} $	$ \mathcal{U}_2^{(q)} $	$ \mathcal{U}_3^{(q)} $	$Y - \mathcal{U}_3^{(q)} $
after	$r^{q+1}(\epsilon_{vg})$	$r^{q+1}(\epsilon_{vg})$	$r^{q+1}(\epsilon_g)$	no change

Note that all sets in $\{U_p, U_2 | U \in \mathcal{U}^{(q)}\}$ are pairwise disjoint and hence

$$R^{(q)} = \left(R^{(q-1)} - |\mathcal{U}_p^{(q)}| \right) \cup |\mathcal{U}_2^{(q)}|.$$

□

Example 5.5. For illustration we give a typical example of a covering \mathcal{U} as described above. Consider \mathbb{R}^n equipped with the maximum-metric. Equip \mathbb{R}^n with the standard cellulation by unit cubes. Let X and $X_1, X_2 \subset X$ be

subcomplexes of \mathbb{R}^n . Let $X^{(q)}$ denote the q -skeleton of X and let $X^{(-1)} = \emptyset$. Define the following subsets of X (Thickenings in X not in \mathbb{R}^n !)

$$V^{(q)} = (X^{(q)})^{\frac{1}{2q+2}} \text{ and } W^{(q)} = \overline{V^{(q)} - V^{(q-1)}}.$$

Choose $\epsilon_s > 0$ so small that $4\epsilon_s \leq \frac{1}{2^{n+2}}$. Then $(W^{(q)})^{4\epsilon_s}$ decomposes into path components (which are in one-to-one correspondence with the q -cells of X). Let $\mathcal{U}^{(q)}$ be the set of those path-components which meet X_2 . For each $U \in \mathcal{U}^{(q)}$ define

$$U_2 = U \cap W^{(q)}, U_4 = U \cap (W^{(q)})^{\epsilon_s} \text{ and } U_3 = U \cap (W^{(q)})^{2\epsilon_s}$$

Let $R^{(-1)} = X_1$, $R^{(q)} = X_1 \cup \bigcup_{U \in \mathcal{U}^{(0)} \cup \dots \cup \mathcal{U}^{(q)}} U_2$ and set $U_1 = U \cap R^{(q-1)}$. Note that by construction for each $U \in \mathcal{U}^{(q)}$ we have $(U_3 - (U_1 \cup U_2)) \cap R^{(q-1)} = \emptyset$ (compare 5.3). Note also that in the case where X is a codimension 0-submanifold and $X_1 = \emptyset$ the local squeezing situations (U, U_i) have the shape of the models from 6.4 in [3].

Under mild conditions one can transport a relative squeezing via a homeomorphism. In special situations one can compute a new a priori control and a new response function. In particular we will see that it is very useful to have linear response functions.

Proposition 5.6 (Transporting a squeezing). *Let $f : X \rightarrow U$ be a homeomorphism. Suppose there exists a relative squeezing for (X, X_i) with a priori control ϵ_X and response function r_X . Set $U_i = f(X_i)$. Suppose moreover that $r_X(\epsilon) = \lambda\epsilon$ is linear.*

- (i) *If $X \subset \mathbb{R}^n$ and $f(x) = Cx$ is a rescaling with a constant $C > 0$, then $\epsilon_U = C\epsilon_X$ and the response function does not change, i.e. $r_X = r_U$.*
- (ii) *Suppose f is a bi-Lipschitz homeomorphism with bi-Lipschitz constant $\eta > 1$, i.e. it satisfies*

$$\eta^{-1} \cdot d(x, y) \leq d(f(x), f(y)) \leq \eta \cdot d(x, y),$$

then $\epsilon_U = \eta^{-1}\epsilon_X$ and $r_U = 2\eta r_X$. In particular the new response function is again linear.

If (X, X_i) has ϵ_s -security then (U, U_i) has $C\epsilon_s$ respectively $\eta^{-1}\epsilon_s$ -security.

Proof. Everything follows immediately from the definitions. □

6. RELATIVE SQUEEZINGS IN SOME STANDARD SITUATIONS

So far we have only discussed how to produce new squeezing results out of given ones. To get the process started we need the following theorem which is the main building block for all the squeezing theorems below. This is implicit in Section 4 of [8]. Compare in particular Theorem 4.5 there. Recently a new proof of such a squeezing statement appeared in [6]. We briefly discuss how the argument in [6] needs to be modified to obtain a

relative squeezing instead of the absolute squeezing formulated there. A cubical subcomplex of the boundary of the unit cube in \mathbb{R}^n is a complex consisting entirely of faces of I^n .

Theorem 6.1. *Suppose (X, X_1, X_2, X_4, X_3) is a relative squeezing situation where X and the X_i are cubical subcomplexes of the boundary of the unit cube $I^n \subset \mathbb{R}^n$. Suppose (X, X_i) has positive and hence $\frac{1}{2}$ -security. Then there exists a relative squeezing with a linear response function $r(\epsilon) = \lambda \cdot \epsilon$. The a priori control ϵ_0 and the constant λ both depend only on the dimension of X .*

Proof. We indicate how a minor modification of Theorem 3.6 and Theorem 3.7 in [6] gives the relative result: Identify $I^n \times [1, \infty)$ with the outer part of the open cone over I^n . Let $\epsilon_0 = 2/(3 \cdot 6^{\dim X + 1})$. Let $\phi, \psi, \epsilon_{vg}$ and ϵ_g be given as in Statement 4.4. Construct a module L' and a deformation η' on $X \times [1, \infty)$ as in the proof of Theorem 3.6 in [6] but work only over $X_3 \times [1, \infty)$ so that $\text{supp}(L') \subset X_3 \times [1, \infty)$ and $\text{supp}(\eta') \subset (X_3 \times [1, \infty))^{\times 2}$. Inspecting the construction one observes that η' is $C\epsilon_{vg}$ -controlled over $X_1 \times [1, \infty)$ and $C\epsilon_g$ -controlled over $X_3 \times [1, \infty)$. Here C is an explicit constant only depending on the dimension of X . Now in the proof of Theorem 3.7 instead of cutting down the off-diagonal entries of η' to $X \times [1, R]$ for some large R and then restricting the deformation one obtains to $X \times [1, R + C\epsilon_0]$ do the following: Cut the off-diagonal entries of η' down to $X_4 \times [1, R]$ and then restrict to $X_4 \times [1, R + C\epsilon_0]$. Finally project the result down to X . This gives the desired deformation η . The point is that $\phi \cdot (\eta'|_{X_4 \times [1, R]})$ is the identity over $X_2 \times [1, R - C\epsilon_0]$ so that by projecting down we really improve the control over X_2 , whereas over $X_3 - X_4$ things do not get worse. \square

Corollary 6.2. *Let $X \subset \mathbb{R}^n$ be a finite polyhedron. Let X be equipped with the subspace metric it inherits from \mathbb{R}^n equipped with the Euclidean or maximum-metric. Suppose we are given subcomplexes X_i such that (X, X_i) is a relative squeezing situation with positive security. Then there exists a relative squeezing for (X, X_i) with a linear response function.*

Remark 6.3. It is not hard to check that the metric on X is up to Lipschitz equivalence independent of the given embedding in \mathbb{R}^n since X is finite.

Proof of 6.2. Every finite polyhedron is PL homeomorphic to a cubical subcomplex of the boundary of some unit cube, see the discussion after Lemma 3.4 in [6]. Hence the models are bi-Lipschitz homeomorphic to some (X, X_i) as in Theorem 6.1. Now use Proposition 5.6. \square

In particular we immediately obtain a relative squeezing for the models M_j and M'_j on page 558 in [3].

Corollary 6.4. *Let (X, X_1, X_2, X_3) be one of the models M_j or M'_j from [3]. Set $X_4 = X_2^{\frac{1}{16}}$. There exists a relative squeezing for (X, X_i) with a linear response function.*

Now that we have the basic building block available we can use the principles developed in the last subsection to prove more sophisticated squeezing theorems. To illustrate the general method we prove an absolute squeezing theorem for (not necessarily compact) subcomplexes of the standard cellulation of \mathbb{R}^n .

Proposition 6.5. *Let X and the X_i be (not necessarily compact) subcomplexes of the standard cellulation of \mathbb{R}^n by unit cubes. Suppose (X, X_i) is a relative squeezing situation with positive and hence $\frac{1}{2}$ -security, then there exists a relative squeezing for (X, X_i) with a linear response function.*

Proof. Of course we want to apply the Induction Principle 5.4 with $Y_1 = X_1$ and $Y_2 = X_2$. Let \mathcal{U} be the covering constructed in Example 5.5. The conditions (i) to (iii) before Proposition 5.4 are satisfied. Each local squeezing situation (U, U_i) is a finite polyhedron and has positive security. There are only finitely many different isometry types of such local situations. By Corollary 6.2 there exists a squeezing for each of these with the same linear response function and the same a priori control. The Induction Principle 5.4 applies. Note that iterating the linear response function again gives a linear response function. It remains to observe that by construction of the covering the Y_3 and Y_4 which appear in the Induction Principle are contained in X_3 respectively X_4 because we assume $\frac{1}{2}$ -security. The result follows by Remark 4.5. \square

Similarly we immediately obtain the following squeezing result for compact Riemannian manifolds.

Proposition 6.6. *Let Y be a compact connected Riemannian manifold and let (Y, Y_i) be an absolute squeezing situation with positive security $\epsilon_s > 0$. Then there exists an absolute squeezing for (Y, Y_i) with linear response function.*

Proof. For some large n embed Y into the half space \mathbb{R}_+^n such that the boundary of Y embeds into the boundary of the half space and some standard collar of the half space restricts to a collar for Y . Now note that, since Y is compact, the path length metric induced from the Riemannian metric on Y and the metric Y inherits as a subspace of the half space (or \mathbb{R}^n) are Lipschitz equivalent. We can therefore work with the subspace metric. Let β be a positive number with $\beta < \frac{\epsilon_s}{4}$. Using the normal bundle νY of the embedding we can construct an open neighborhood $\nu_\beta Y$ of Y in \mathbb{R}_+^n together with a Lipschitz retraction

$$p : \nu_\beta Y \rightarrow Y \quad \text{which satisfies} \quad d(p(x), x) < \beta \text{ for all } x \in \nu_\beta Y.$$

Note that this implies:

- (1) $p^{-1}(A) \subset A^\beta$ for any $A \subset Y$ and
- (2) $p(B) \subset B^\beta \cap Y$ for any $B \subset \nu_\beta Y$.

Since Y is compact there clearly exists a $\delta > 0$ such that $\delta < \frac{\epsilon_s}{4}$ and $Y^\delta \subset \nu_\beta Y^{-\delta} \subset \nu_\beta Y$.

For this given $\delta > 0$ let \mathbb{R}_δ^n be \mathbb{R}^n considered as a cellular complex with the rescaled cubical standard cellulation in which the diameter of a cube is δ . For $X \subset \mathbb{R}_\delta^n$ define X^\square as the union of all n -dimensional closed cubes meeting X . Define $X^{-\square}$ as the closure of $((X^c)^\square)^c$ and for any ϵ set $X^{-\epsilon} = ((X^c)^\epsilon)^c$, where X^c denotes the complement in \mathbb{R}^n . Observe that $X^{-\square}$ and X^\square are cubical subcomplexes of \mathbb{R}_δ^n and that we have inclusions

$$X^{-\delta} \subset X^{-\square} \subset X \subset X^\square \subset X^\delta.$$

Set

$$\begin{aligned} K &= \nu_\beta Y^{-\square} \\ K_2 &= p^{-1}(Y_2)^\square \cap K \\ K_4 &= p^{-1}(Y_4)^{-\square} \\ K_3 &= p^{-1}(Y_3)^{-\square}. \end{aligned}$$

Set $\epsilon = \epsilon_s - 2\beta - 2\delta$. Then

$$\left(p^{-1}(Y_2)^\square\right)^\epsilon \subset p^{-1}(Y_2)^{\delta+\epsilon} \subset Y_2^{\delta+\epsilon+\beta} \subset p^{-1}(Y_4)^{-\delta} \subset p^{-1}(Y_4)^{-\square}$$

and

$$\left(p^{-1}(Y_4)^{-\square}\right)^\epsilon \subset p^{-1}(Y_4)^\epsilon \subset Y_4^{\epsilon+\beta} \subset p^{-1}(Y_3)^{-\delta} \subset p^{-1}(Y_3)^{-\square}.$$

Here the third inclusion is equivalent to $Y_2^{2\delta+\epsilon+\beta} \subset p^{-1}(Y_4)$ and hence to $p(Y_2^{2\delta+\epsilon+\beta}) \subset Y_4$. But by (2) and the security assumption we have

$$p(Y_2^{2\delta+\epsilon+\beta}) \subset Y_2^{2\delta+\epsilon+2\beta} \cap Y = Y_2^{\epsilon_s} \cap Y \subset Y_4.$$

The argument for the third inclusion in the second line is similar. One concludes that (K, K_i) is an absolute squeezing situation with security $\epsilon > 0$. By rescaling as in 5.6 and the squeezing Theorem 6.5 for cubical subcomplexes of \mathbb{R}^n or directly from 6.2 we know that there exists a squeezing with a linear response function for the situation (K, K_i) .

Now note that

$$Y \subset \nu_\beta Y^{-\delta} \subset \nu_\beta Y^{-\square} = K \subset \nu_\beta Y$$

and define

$$q : K \rightarrow Y$$

as the restriction of p . Note that q is again Lipschitz with the same Lipschitz constant and is still a retraction onto Y . Via the inclusion we can now consider morphisms over Y as morphisms over K apply the squeezing there

and then map them back to Y via q . Since $K_3 \cap Y \subset Y_3$ the before-control data for Y implies the “same” before-control data for K . The working zone is mapped under q to the working zone, i.e. $q(K_4) \subset Y_4$. The after-control data for Y depends via the Lipschitz constant of q on the after-control data for K because $q^{-1}(Y_2) \subset K_2$, $q^{-1}(Y_3) \subset K$ and $q^{-1}(Y - Y_3) \subset K - K_3$. Note that we do not have $q^{-1}(Y_3) \subset K_3$ but K is the disjoint union of K_3 (where we have $r(\epsilon_g)$ -control afterwards) and $K - K_3$ (here nothing changes so bad control stems from $Y - Y_3$ which does not meet $q^{-1}(Y_3)$). This yields the desired squeezing with linear response function for (Y, Y_i) . \square

7. FOLIATED VERSIONS

For the foliated control theorem we need foliated versions of the notions developed in Sections 4 and 5.

Let Y be a Riemannian manifold equipped with a smooth 1-dimensional foliation. Let $\alpha, \delta > 0$ be given. A pair $(x, y) \in Y \times Y$ is said to be (α, δ) -controlled if there exist points x' and y' in Y and a piecewise smooth path γ which connects x' to y' such that

- (i) γ lies inside one leaf of the foliation.
- (ii) γ is shorter than α .
- (iii) $d(x, x') \leq \delta$ and $d(y, y') \leq \delta$.

Let $E_{(\alpha, \delta)}$ be the set of all pairs $(x, y) \in Y \times Y$ which are (α, δ) -controlled. This is a neighborhood of the diagonal and the E -control notions from Definition 3.1 and Definition 3.5 apply. Usually we talk of (α, δ) -control instead of $E_{(\alpha, \delta)}$ -control.

A *foliated relative squeezing situation* (U, U_i) is a relative squeezing situation where U is a subset of a Riemannian manifold Y which is equipped with a 1-dimensional foliation. Note that Definition 4.2 applies and we can hence talk about (α, δ) -*security in Y* for a foliated relative squeezing situation (U, U_i) with ambient foliated Riemannian manifold Y . In the foliated context a response function

$$r = r_1 \times r_2 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \times [0, \infty)$$

is simply a product of response functions, i.e. r_1 and r_2 are homeomorphisms of $[0, \infty)$ with $r_i(\epsilon) \geq \epsilon$. A response function r is called linear if r_1 and r_2 are linear.

In the following we write

$$(\alpha, \delta) \leq (\alpha', \delta') \text{ if } \alpha \leq \alpha' \text{ and } \delta \leq \delta'.$$

There is an obvious analogue of the Statement 4.4 which in the short version reads as follows:

Statement 7.1. There exists an $(\alpha_0, \epsilon_0) > (0, 0)$ such that given any (α_g, ϵ_g) and $(\alpha_{vg}, \epsilon_{vg})$ with

$$(0, 0) < (\alpha_{vg}, \epsilon_{vg}) \leq (\alpha_g, \epsilon_g) \leq (\alpha_0, \epsilon_0)$$

we can achieve the following control improvement by working only on U_4 :

control over	U_1	U_2	U_3	$U - U_3$
before	$(\alpha_{vg}, \delta_{vg})$		$(\alpha_g, \delta_g) \leq (\alpha_0, \delta_0)$	
after	$r(\alpha_{vg}, \delta_{vg})$	$r(\alpha_{vg}, \delta_{vg})$	$r(\alpha_g, \delta_g)$	no change

Definition 7.2 (Foliated relative squeezing). Let (U, U_i) be a foliated relative squeezing situation. We say that there exists a *foliated relative squeezing* with a *a priori control* (α_0, δ_0) and *response function* $r = r_1 \times r_2$ for (U, U_i) if Statement 7.1 above holds for (U, U_i) .

Note that Remark 5.1 about the global-local passage and Remark 5.2 apply in the context of foliated control. Consequently there is an analogue of the Induction Principle 5.4 whose proof only used Remarks 5.1 and 5.2. In the statement one simply replace ϵ_0 by (α_0, δ_0) and ϵ_s by (α_s, δ_s) . We also need an analogue of Proposition 5.6. The following Proposition describes the behavior of a foliated squeezing with respect to a map which preserves the foliation and stretches by a factor γ in the foliation direction. Its proof is straightforward from the definitions.

Proposition 7.3 (Transporting a foliated squeezing). *Let (X, X_i) be a foliated relative squeezing situation. Let U be an open subset of a foliated Riemannian manifold. Let $f : X \rightarrow U$ be a diffeomorphism which respects the foliations, i.e. maps leaves to leaves. Set $U_i = f(X_i)$. Suppose there exists a foliated relative squeezing with a priori control (α_0, δ_0) and linear response function $r_X(\alpha, \delta) = (\lambda_1 \alpha, \lambda_2 \delta)$ for (X, X_i) . Suppose that the differential of f satisfies the following two Lipschitz conditions.*

- (i) *There is $\eta \geq 1$ such that for all tangent vectors v to X*

$$\eta^{-1}|v| \leq |df(v)| \leq \eta|v|.$$

- (ii) *There are $\beta \geq 1, \gamma > 0$ such that for all vectors v tangent to the foliation on X*

$$\beta^{-1}\gamma|v| \leq |df(v)| \leq \beta\gamma|v|.$$

Then there exists a foliated relative squeezing for (U, U_i) with a priori control $(\gamma\alpha_0\beta^{-1}, \delta_0\eta^{-1})$ and linear response function $r_U(\alpha, \delta) = (\beta^2\lambda_1\alpha, \eta^2\lambda_2\delta)$. If (X, X_i) has (α_s, δ_s) -security then (U, U_i) has security $(\gamma\alpha_s\beta^{-1}, \delta_s\eta^{-1})$.

Here are some trivial remarks concerning the relation between ϵ -control and (α, δ) -control: Let as above $E_{\alpha, \delta}$ be the set of all pairs of points which are (α, δ) apart. Then

$$E_\epsilon \subset E_{\epsilon, \epsilon} \quad \text{and} \quad E_{\alpha, \delta} \subset E_{\alpha+2\delta}.$$

In words: ϵ -control implies (ϵ, ϵ) -foliated control and (α, δ) -control implies $\alpha + 2\delta$ -control. Using this every ordinary squeezing gives rise to a foliated squeezing. In particular there exists a foliated squeezing for the models.

Proposition 7.4. *Equip $\mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1}$ with the foliation by lines parallel to $\mathbb{R} \times \{0\}$. Let (X, X_i) be one of the models M_j or M'_j from Section 6 of [3]. There exists a foliated relative squeezing for (X, X_i) with a priori control (ϵ_0, ϵ_0) and a linear response function $r(\alpha, \delta) = (\lambda \cdot \alpha, \lambda \cdot \delta)$. The constants ϵ_0 and λ depend only on the dimension n .*

Proof. By the remarks above this follows from Corollary 6.4. Note that the foliation breaks the symmetry and we really have to distinguish between the models M_j and M'_j . \square

8. A FOLIATED SQUEEZING THEOREM

In this section we prove the foliated squeezing theorem. Roughly such a theorem states that once one has sufficient foliated control one can achieve arbitrarily good control at least away from short compact leaves. More precisely we have the following:

Let N be a not necessarily compact Riemannian manifold. Suppose N is equipped with a 1-dimensional smooth foliation. For a constant $\alpha > 0$ let $N^{\leq \alpha} \subset N$ denote the union of all compact leaves which are shorter than α .

Proposition 8.1. *There exists a constant $\mu_1 > 1$ such that:*

Given any large number $\alpha_0 > 0$ and any compact subset

$$N_2 \subset N - N^{\leq \mu_1 \alpha_0}$$

there exists a constant $\delta_0 > 0$ and compact subsets N_4 and N_3 with $N_2 \subset N_4 \subset N_3$ such that there exists a foliated absolute squeezing for (N, N_i) with a linear response function and a priori control (α_0, δ_0) .

The proof relies on two main ingredients: The squeezing theorem for subcomplexes of the unit cube 6.1 (and its Corollaries in particular 7.4) and the long and thin cell structure from Section 7 in [3].

Let N be an n -dimensional Riemannian manifold N equipped with a one-dimensional foliation. In Section 7 of [3] and the Appendix of [4] the authors construct a so called long and thin cell structure L with arbitrarily long cells (in the direction of the foliation) whose underlying set covers any given compact subset N_2 of N which does not meet the short closed leaves.

In Lemma 8.1 of [3] this long and thin cell structure is used to produce a cover \mathcal{U} of (a neighborhood) of N_2 that allows the application of the induction principle described before (cf. 5.4). The sets in \mathcal{U} are indexed by cells of L . In particular \mathcal{U} is a disjoint union

$$\mathcal{U} = \mathcal{U}^{(0)} \cup \mathcal{U}^{(1)} \cup \dots \cup \mathcal{U}^{(n)},$$

where the covering sets in $\mathcal{U}^{(q)}$ are indexed by the q -cells. For every set U of \mathcal{U} there is a diffeomorphism $g_U : X = X(U) \rightarrow U$ that respects the foliation, where $X(U)$ is one of the finitely many models M_q, M'_q . We set

$U_i := g_U(X_i)$. We proceed to state the crucial properties of \mathcal{U} in more detail. We will use the notation from the discussion preceding 5.4.

Proposition 8.2. *There exists a constant μ_1 such that, given any large number $\gamma > 0$ and any compact subset $N_2 \subset N - N^{\leq \mu_1 \gamma}$ there is a cover \mathcal{U} of a neighborhood of N_2 as above such that the following is satisfied:*

- (i) *The conditions stated before the induction principle 5.4 are satisfied. Moreover, we have $N_2 \subset |\mathcal{U}_2| \cap R^{(n)}$.*
- (ii) *There are $\eta, \beta > 1$, where β depends only on the dimension of N , such that we have the following estimates for all $U \in \mathcal{U}$.*
 - *For all tangent vectors v to X*

$$\eta^{-1}|v| \leq |dg_U(v)| \leq \eta|v|.$$

- *For all vectors v tangent to the foliation on X*

$$\beta^{-1}\gamma|v| \leq |dg_U(v)| \leq \beta\gamma|v|.$$

Proof. This is Lemma 8.1 in [3], see also 8.2 there. \square

Proof of 8.1. We use the cover described in the previous proposition. This together with 7.3 and 7.4 immediately implies:

- (A) There is a foliated squeezing for each (U, U_i) with a priori control $(\gamma\epsilon_0\beta^{-1}, \epsilon_0\eta^{-1})$ and response function $r(\alpha, \delta) = (\beta^2\lambda\alpha, \eta^2\lambda\delta)$. Here (ϵ_0, ϵ_0) is a common a priori control and $r(\alpha, \delta) = (\lambda\alpha, \lambda\delta)$ is a common response function for all the models M_j and M'_j as in Proposition 7.4.
- (B) Each (U, U_i) has $(\gamma\epsilon_s\beta^{-1}, \epsilon_s\eta^{-1})$ -security, where (ϵ_s, ϵ_s) is such that all models have ϵ_s -security.

Hence we can apply the foliated analogue of the Induction Principle 5.4. We obtain a foliated absolute squeezing with a priori control

$$((\beta^2\lambda)^{-n}\gamma\epsilon_0\beta^{-1}, (\eta^2\lambda)^{-n}\epsilon_0\eta^{-1})$$

and a linear response function. Note that ϵ_0, λ and β depend only on the dimension of N , while η depends (via the cell structure L) on the given arbitrary large γ . However η only appears on the δ -side of the (α, δ) -expression above. Hence 8.1 follows for arbitrary large α_0 . The Induction Principle also gives us explicitly the new response function. It is given by $r(\alpha, \delta) = ((\beta^2\lambda)^{n+1}\alpha, (\eta^2\lambda)^{n+1}\delta)$. Note that again the constant on the α -side only depends on the dimension of N . \square

9. FROM NON-EQUIVARIANT TO EQUIVARIANT

In [1] we need squeezing statements for automorphisms on the universal covering of a given space which are invariant under the action of the fundamental group. In this section we discuss the necessary easy modifications.

Let us first recall some definitions. Let X be a free Γ -space. A module M over X is Γ -invariant if for all $x \in X$ and $g \in \Gamma$ we have $M_x = M_{gx}$.

A morphism ϕ is Γ -invariant if for all $g \in \Gamma$ and all $x, y \in X$ we have $\phi_{y,x} = \phi_{gy,gx}$. Similarly there is the notion of an invariant deformation. The category of all locally finite Γ -invariant R -modules and Γ -invariant morphisms is denoted $\mathcal{C}^\Gamma(X)$.

An equivariant relative squeezing situation consist of a Γ -space U with a Γ -invariant metric equipped with Γ -invariant subsets U_i such that $U_2 \subset U_4 \subset U_3$.

There is an obvious equivariant version of the relative squeezing statement 4.4 where all modules, morphisms and deformations are Γ -invariant.

Let $\pi : \tilde{N} \rightarrow N$ be the universal covering of the Riemannian manifold N equipped with the lifted Riemannian metric and let Γ be the fundamental group. Let U be a subset of N and let (U, U_i) be a relative squeezing situation. One observes:

Note 9.1. Suppose $U \subset N$ is simply connected. Then $\pi^{-1}(U) \subset \tilde{N}$ is the disjoint union of Γ many copies of U . If there exists a relative squeezing for (U, U_i) , then there exists an equivariant relative squeezing for the preimage situation $(\pi^{-1}(U), \pi^{-1}(U_i))$ with the same a priori control and the same response function.

Note 9.2. If moreover N is equipped with a smooth 1-dimensional foliation and there exists a foliated relative squeezing for (U, U_i) then there exists an equivariant foliated relative squeezing for the preimage situation with the same a priori control and the same response function.

Note 9.3. If (U, U_i) has ϵ_s - or in the foliated context (α_s, δ_s) -security in N then the preimage situation $(\pi^{-1}(U), \pi^{-1}(U_i))$ has the same security in \tilde{N} . Compare Definition 4.2.

Hence every squeezing theorem which is proven via an induction over local relative squeezings or foliated local relative squeezings as described in Proposition 5.4 has an equivariant analogue if all the covering sets $U \in \mathcal{U}$ that occur are simply connected. In particular we obtain equivariant analogues of the squeezing theorem for compact Riemannian manifolds Theorem 6.6 and of the foliated squeezing theorem from the last subsection.

For convenience we formulate these two squeezing statements explicitly without referring to Statement 4.4. The following is Theorem 13.1 in [1].

Theorem 9.4. (Equivariant absolute squeezing for compact Riemannian manifolds.) *Let N be a compact connected Riemannian manifold. Let K and S be closed subsets of N with $S \cap K = \emptyset$. Then there is $\epsilon_0 = \epsilon_0(N, K, S)$ and a homeomorphism $r = r(N, K, S) : [0, \infty) \rightarrow [0, \infty)$ such that the following holds:*

Let $\epsilon_0 \geq \epsilon_g \geq \epsilon_{vg} \geq 0$ and $\alpha > r(\epsilon_g)$ and let $\phi : M \rightarrow M$ in $\mathcal{C}^\Gamma(\tilde{N})$ be an α -automorphism. Assume moreover that ϕ and ϕ^{-1} are ϵ_g -controlled over $\tilde{X} - \pi^{-1}(S)$.

Then there is a stabilizing module L and a deformation $\eta = (e_1, \dots, e_n)$ on $M \oplus L$ in $\mathcal{C}^\Gamma(\tilde{N})$ such that:

- (i) The deformed automorphism $\phi_{new} = (\varphi \oplus \text{id}_N)\eta$ is an α -automorphism. Moreover, ϕ_{new} and ϕ_{new}^{-1} are ϵ_{vg} -controlled over $\pi^{-1}(K)$.
- (ii) The deformation η is $r(\epsilon_g)$ -controlled and each e_i is the identity on $(M \oplus N)|_{\pi^{-1}(S)}$.

Proof. Construct a squeezing situation (N, N_i) with $N_2 = K$, $S \subset N - N_3$ and positive security. Apply the discussion above to Theorem 6.6. \square

Theorem 9.5 (Equivariant absolute foliated squeezing). *Let N be a not necessarily compact Riemannian manifold which is equipped with a smooth 1-dimensional foliation. Let $\pi : \tilde{N} \rightarrow N$ denote the universal cover and Γ the fundamental group. Equip \tilde{N} with the lifted Riemannian metric and the lifted foliation.*

There is a constant μ_1 which only depends on the dimension of N for which the following statement is true:

Let $\alpha_0 > 0$ be an arbitrarily large number. Let N_2 be an arbitrary compact subset of N which does not meet $N^{\leq \mu_1 \alpha_0}$.

Then there exist

numbers $\delta_0 > 0$ and $\mu_2 > 1$ depending on N_2 and

α_0

such that:

For any $\epsilon > 0$ and (α, δ) with

$$(\epsilon, \epsilon) \leq (\alpha, \delta) \leq (\alpha_0, \delta_0)$$

and every (α, δ) -automorphism $\phi : M \rightarrow M$ in $\mathcal{C}^\Gamma(\tilde{N})$ there exists a stabilizing module L over \tilde{N} and a deformation η on $M \oplus L$ in $\mathcal{C}^\Gamma(\tilde{N})$ such that:

- (i) *The deformed automorphism $\phi_{new} = \bar{\eta}(\phi \oplus \text{id}_L)$ and its inverse $\phi_{new}^{-1} = (\phi^{-1} \oplus \text{id}_L)\bar{\eta}^{-1}$ are both ϵ -controlled over $\pi^{-1}(N_2)$.*
- (ii) *The deformation η , ϕ_{new} and also its inverse ϕ_{new}^{-1} are all everywhere $(\mu_1 \alpha, \mu_2 \delta)$ -controlled.*

In particular we obtain the following statement: Given an arbitrary large α_0 , there exists a δ_0 such that given an arbitrary small ϵ we can achieve

the following control improvement:

control over	N_2	everywhere
before		$(\alpha, \delta) \leq (\alpha_0, \delta_0)$
after	ϵ	$(\mu_1\alpha, \mu_2\delta)$

The important point is that α_0 can be chosen arbitrarily large.

For Theorem 13.2 in [1] observe:

Addendum 9.6. *There is a version of 9.5 where all modules have Γ -compact support, i.e. we can work in the category $\mathcal{C}^\Gamma(\tilde{N}, \mathcal{F}_{\Gamma c})$ from [1].*

Proof. The working zone N_4 in 8.1 is compact and hence $\pi^{-1}(N_4)$ is Γ -compact. □

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