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K_0 AND THE PASSAGE FROM INTEGRAL TO RATIONAL COEFFICIENTS

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We are outlining the following conjecture, already stated in [2].

CONJECTURE 54.1. *For every group G the map $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Q}G)$ induced from the natural inclusion $\mathbf{Z}G \to \mathbf{Q}G$ is the zero-map.*

Here for a ring R we denote by $\widetilde{K}_0(R)$ the *reduced projective class group*, i.e. the cokernel of the map $K_0(\mathbf{Z}) \to K_0(R)$ induced from the natural map $\mathbf{Z} \to R$. Hence the conjecture says that for every finitely generated projective $\mathbf{Z}G$-module P the module $P \otimes_{\mathbf{Z}} \mathbf{Q}$ is a stably free $\mathbf{Q}G$-module.

Swan proved the conjecture in the case where G is a finite group (see [3], Theorem 8.1). I think that somehow the conjecture above should be implied by the K-theoretic Farrell–Jones Conjecture, which is discussed in the contribution by Wolfgang Lück. Indeed if G is torsion-free the Farrell–Jones Conjecture predicts that $\widetilde{K}_0(\mathbf{Z}G) = 0$ and hence in that case the conjecture above is clearly implied by the Farrell–Jones Conjecture. But in the general case I do not know how to derive it from the Farrell–Jones Conjecture. One can however show that the Farrell–Jones Conjecture implies that the map

$$\widetilde{K}_0(\mathbf{Z}G) \otimes \mathbf{Q} \to \widetilde{K}_0(\mathbf{Q}G) \otimes \mathbf{Q}$$

is the zero-map, see [2], Proposition 3.11 (iii), for a more precise statement and a proof.

Further evidence for the conjecture above is given by the following fact, which is true for all groups. If we compose the map in the conjecture above with the map induced from the inclusion of $\mathbf{Q}G$ into the group von Neumann algebra $\mathcal{N}G$ then the resulting composition

$$\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Q}G) \to \widetilde{K}_0(\mathcal{N}G),$$

is the zero-map. This is proven in [1], Theorem 9.62.

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