K- AND L-THEORY OF GROUP RINGS OVER $GL_n(\mathbf{Z})$

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ABSTRACT

We prove the K- and L-theoretic Farrell-Jones Conjecture (with coefficients in additive categories) for $GL_n(\mathbf{Z})$.

Introduction

The Farrell-Jones Conjecture predicts a formula for the K- and L-theory of group rings R[G]. This formula describes these groups in terms of group homology and K- and L-theory of group rings RV, where V varies over the family $\mathcal{VC}$yc of virtually cyclic subgroups of G.

Main Theorem. — Both the K*-theoretic and the* L*-theoretic Farrell-Jones Conjecture (see Definitions 0.1 and 0.2) hold for* $GL_n(\mathbf{Z})$.

We will generalize this theorem in the General Theorem below. In particular it also holds for arithmetic groups defined over number fields, compare Example 0.4, and extends to the more general version "with wreath products".

For cocompact lattices in almost connected Lie groups this result holds by Bartels-Farrell-Lück [1]. The lattice $GL_n(\mathbf{Z})$ has finite covolume but is not cocompact. It is a long standing question whether the Baum-Connes Conjecture holds for $GL_n(\mathbf{Z})$.

For torsion free discrete subgroups of $GL_n(\mathbf{R})$, or more generally, for fundamental groups of A-regular complete connected non-positive curved Riemannian manifolds, the Farrell-Jones Conjecture with coefficients in $\mathbf{Z}$ has been proven by Farrell-Jones [14].

The formulation of the Farrell-Jones Conjecture. —

Definition **0.1** *(*K*-theoretic FJC). — Let* G *be a group and let* $\mathcal{F}$ *be a family of subgroups. Then* G *satisfies the* K-theoretic Farrell-Jones Conjecture with respect to $\mathcal{F}$ *if for any additive* G*-category* $\mathcal{A}$ *the assembly map*

$$H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right)$$

induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$ *is bijective for all* $n \in \mathbf{Z}$. *If this map is bijective for all* $n \leq 0$ *and surjective for* $n = 1$, *then we say* G *satisfies the* K-theoretic Farrell-Jones Conjecture up to dimension 1 with respect to $\mathcal{F}$.

If the family $\mathcal{F}$ *is not mentioned, it is by default the family* $\mathcal{VC}$*yc of virtually cyclic subgroups.*

DOI 10.1007/s10240-013-0055-0

If one chooses $\mathcal{A}$ to be (a skeleton of) the category of finitely generated free R-modules with trivial G-action, then $K_n(\int_G \mathcal{A})$ is just the algebraic K-theory $K_n(RG)$ of the group ring RG.

If G is torsion free, R is a regular ring, and $\mathcal{F}$ is $\mathcal{VC}$yc, then the claim boils down to the more familiar statement that the classical assembly map $H_n(BG; \mathbf{K}_R) \to K_n(RG)$ from the homology theory associated to the (non-connective) algebraic K-theory spectrum of R applied to the classifying space BG of G to the algebraic K-theory of RG is a bijection. If we restrict further to the case $R = \mathbf{Z}$ and $n \leq 1$, then this implies the vanishing of the Whitehead group Wh(G) of G, of the reduced projective class group $\widetilde{K}_0(\mathbf{Z}G)$, and of all negative K-groups $K_n(\mathbf{Z}G)$ for $n \leq -1$.

Definition **0.2** *(L-theoretic FJC). — Let G be a group and let $\mathcal{F}$ be a family of subgroups. Then G satisfies the* L-theoretic Farrell-Jones Conjecture with respect to $\mathcal{F}$ *if for any additive G-category with involution $\mathcal{A}$ the assembly map*

$$H_n^G\big(E_{\mathcal{F}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}\big) \to H_n^G\big(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}\big) = L_n^{\langle -\infty \rangle}\left(\int_G \mathcal{A}\right)$$

induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$ is bijective for all $n \in \mathbf{Z}$.

If the family $\mathcal{F}$ is not mentioned, it is by default the family $\mathcal{VC}$yc of virtually cyclic subgroups.

Given a group G, a *family of subgroups* $\mathcal{F}$ is a collection of subgroups of G that is closed under conjugation and taking subgroups. For the notion of a *classifying space* $E_{\mathcal{F}}(G)$ *for a family* $\mathcal{F}$ we refer for instance to the survey article [20].

The natural choice for $\mathcal{F}$ in the Farrell-Jones Conjecture is the family $\mathcal{VC}$yc of virtually cyclic subgroups but for inductive arguments it is useful to consider other families as well.

Remark **0.3** *Relevance of the additive categories as coefficients.* — The versions of the Farrell-Jones Conjecture appearing in Definitions 0.1 and 0.2 are formulated and analyzed in [2], [7]. They encompass the versions for group rings RG over arbitrary rings R, where one can built in a twisting into the group ring or treat more generally crossed product rings $R * G$ and one can allow orientation homomorphisms $w\colon G \to \{\pm 1\}$ in the L-theory case. Moreover, inheritance properties, e.g., passing to subgroups, finite products, finite free products, and directed colimits, are built in and one does not have to pass to fibered versions anymore.

The original source for the (Fibered) Farrell-Jones Conjecture is the paper by Farrell-Jones [13, 1.6 on p. 257 and 1.7 on p. 262]. For more information about the Farrell-Jones Conjecture, its relevance and its various applications to prominent conjectures due to Bass, Borel, Kaplansky, Novikov and Serre, we refer to [6], [21].

We will often abbreviate Farrell-Jones Conjecture to FJC.

Extension to more general rings and groups. — We will see that it is not hard to generalize the Main Theorem as follows.

General Theorem. — *Let* R *be a ring whose underlying abelian group is finitely generated. Let* G *be a group which is commensurable to a subgroup of* $GL_n(\mathrm{R})$ *for some natural number* n.

Then G *satisfies both the* K*-theoretic and the* L*-theoretic Farrell-Jones Conjecture with wreath products Definition 6.1.*

Two groups G_1 and G_2 are called *commensurable* if they contain subgroups $G_1' \subseteq G_1$ and $G_2' \subseteq G_2$ of finite index such that G_1' and G_2' are isomorphic. In this case G_1 satisfies the FJC with wreath products if and only if G_2 does, see Remark 6.2.

Example **0.4** *(Ring of integers).* — Let K be an algebraic number field and $\mathcal{O}_K$ be its ring of integers. Then $\mathcal{O}_K$ considered as abelian group is finitely generated free (see [22, Chapter I, Proposition 2.10 on p. 12]). Hence by the General Theorem any group G which is commensurable to a subgroup of $GL_n(\mathcal{O}_K)$ for some natural number n satisfies both the K-theoretic and L-theoretic FJC with wreath products. This includes in particular arithmetic groups over number fields.

Discussion of the proof. — The proof of the FJC for $GL_n(\mathbf{Z})$ will use the transitivity principle [13, Theorem A.10], that we recall here.

Proposition **0.5** *Transitivity principle.* — *Let* $\mathcal{F} \subset \mathcal{H}$ *be families of subgroups of* G. *Assume that* G *satisfies the FJC with respect to* $\mathcal{H}$ *and that each* $H \in \mathcal{H}$ *satisfies the FJC with respect to* $\mathcal{F}$.

Then G *satisfies the FJC with respect to* $\mathcal{F}$.

This principle applies to all versions of the FJC discussed above. In this form it can be found for example in [1, Theorem 1.11].

The main step in proving the FJC for $GL_n(\mathbf{Z})$ is to prove that $GL_n(\mathbf{Z})$ satisfies the FJC with respect to a family $\mathcal{F}_n$. This family is defined at the beginning of Section 3. This family is larger than $\mathcal{VC}$yc and contains for example $GL_k(\mathbf{Z})$ for $k < n$. We can then use induction on n to prove that every group from $\mathcal{F}_n$ satisfies the FJC. At this point we also use the fact that virtually poly-cyclic groups satisfy the FJC.

To prove that $GL_n(\mathbf{Z})$ satisfies the FJC with respect to $\mathcal{F}_n$ we will apply two results from [4], [24]. Originally these results were used to prove that CAT(0)-groups satisfy the FJC. Checking that they are applicable to $GL_n(\mathbf{Z})$ is more difficult. While $GL_n(\mathbf{Z})$ is not a CAT(0)-group, it does act on a CAT(0)-space X. This action is proper and isometric, but not cocompact. Our main technical step is to show that the flow space associated to this CAT(0)-space admits *long* $\mathcal{F}_n$*-covers at infinity*, compare Definition 3.7.

In Section 1 we analyze the CAT(0)-space X. On it we introduce, following Grayson [16], certain volume functions and analyze them from a metric point of view. These functions will be used to cut off a suitable well-chosen neighborhood of infinity so that the $\mathrm{GL}_n(\mathbf{Z})$-action on the complement is cocompact. In Section 2 we study sublattices in $\mathbf{Z}^n$. This will be needed to find the neighborhood of infinity mentioned above. Here we prove a crucial estimate in Lemma 2.1.

As outlined this proof works best for the K-theoretic FJC up to dimension 1; this case is contained in Section 3. The modifications needed for the full K-theoretic FJC are discussed in Section 4 and use results of Wegner [24].

For L-theory the induction does not work quite as smoothly. The appearance of index 2 overgroups in the statement of [4, Theorem 1.1(ii)] force us to use a stronger induction hypothesis: we need to assume that finite overgroups of $GL_k(\mathbf{Z})$, $k < n$ satisfy the FJC. (It would be enough to consider overgroups of index 2, but this seems not to simplify the argument.) A good formalism to accommodate this is the FJC with wreath products (which implies the FJC). In Section 5 we provide the necessary extensions of the results from [4] for this version of the FJC. In Section 6 we then prove the L-theoretic FJC with wreath products for $GL_n(\mathbf{Z})$.

In Section 7 we give the proof of the General Theorem.

1. The space of inner products and the volume function

Throughout this section let V be an n-dimensional real vector space. Let $\widetilde{\mathrm{X}}(\mathrm{V})$ be the set of all inner products on V. We want to examine the smooth manifold $\widetilde{\mathrm{X}}(\mathrm{V})$ and equip it with an $\mathrm{aut}(\mathrm{V})$-invariant complete Riemannian metric with non-positive sectional curvature. With respect to this structure we will examine a certain volume function. We try to keep all definitions as intrinsic as possible and then afterward discuss what happens after choices of extra structures (such as bases).

1.1. *The space of inner products.* — We can equip V with the structure of a smooth manifold by requiring that any linear isomorphism $\mathrm{V} \to \mathbf{R}^n$ is a diffeomorphism with respect to the standard smooth structure on $\mathbf{R}^n$. In particular V carries a preferred structure of a (metrizable) topological space and we can talk about limits of sequences in V. We obtain a canonical trivialization of the tangent bundle TV

$$\phi_{\mathrm{V}} \colon \mathrm{V} \times \mathrm{V} \to \mathrm{TV} \tag{1.1}$$

which sends (x, v) to the tangent vector in $\mathrm{T}_x\mathrm{V}$ represented by the smooth path $\mathbf{R} \to \mathrm{V}, t \mapsto x + t \cdot v$. The inverse sends the tangent vector in TV represented by a path $w \colon (-\epsilon, \epsilon) \to \mathrm{V}$ to $(w(0), w'(0))$. If $f \colon \mathrm{V} \to \mathrm{W}$ is a linear map, the following diagram commutes

$$\begin{array}{ccc}
\mathrm{V} \times \mathrm{V} & \xrightarrow[\cong]{\phi_{\mathrm{V}}} & \mathrm{TV} \\
\Big\downarrow{\scriptstyle f \times f} & & \Big\downarrow{\scriptstyle \mathrm{T}f} \\
\mathrm{W} \times \mathrm{W} & \xrightarrow[\cong]{\phi_{\mathrm{W}}} & \mathrm{TW}
\end{array}$$

Let hom(V, V*) be the real vector space of linear maps $V \to V^*$ from V to the dual V^* of V. In the sequel we will always identify V and $(V^*)^*$ by the canonical isomorphism $V \to (V^*)^*$ which sends $v \in V$ to the linear map $V^* \to \mathbf{R}, \alpha \mapsto \alpha(v)$. Hence for $s \in \hom(V, V^*)$ its dual $s^* \colon (V^*)^* = V \to V^*$ belongs to $\hom(V, V^*)$ again. Let $\mathrm{Sym}(V) \subseteq \hom(V, V^*)$ be the subvector space of elements $s \in \hom(V, V^*)$ satisfying $s^* = s$. We can identify Sym(V) with the set of all bilinear symmetric pairings $V \times V \to \mathbf{R}$, namely, given $s \in \mathrm{Sym}(V)$ we obtain such a pairing by $(v, w) \mapsto s(v)(w)$. We will often write

$$s(v, w) := s(v)(w).$$

Under the identification above the set $\widetilde{X}(V)$ of inner products on V becomes the open subset of Sym(V) consisting of those elements $s \in \mathrm{Sym}(V)$ for which $s \colon V \to V^*$ is bijective and $s(v, v) \geq 0$ holds for all $v \in V$, or, equivalently, for which $s(v, v) \geq 0$ holds for all $v \in V$ and we have $s(v, v) = 0 \Leftrightarrow v = 0$. In particular $\widetilde{X}(V)$ inherits from the vector space Sym(V) the structure of a smooth manifold.

Given a linear map $f \colon V \to W$, we obtain a linear map $\mathrm{Sym}(f) \colon \mathrm{Sym}(W) \to \mathrm{Sym}(V)$ by sending $s \colon W \to W^*$ to $f^* \circ s \circ f$. A linear isomorphism $f \colon V \to W$ induces a bijection $\widetilde{X}(f) \colon \widetilde{X}(W) \to \widetilde{X}(V)$. Obviously this is a contravariant functor, i.e., $\widetilde{X}(g \circ f) = \widetilde{X}(f) \circ \widetilde{X}(g)$. If aut(V) is the group of linear automorphisms of V, we obtain a right aut(V)-action on $\widetilde{X}(V)$.

If $f \colon V \to W$ is a linear map and s_V and s_W are inner products on V and W, then the adjoint of f with respect to these inner products is $s_V^{-1} \circ f^* \circ s_W \colon W \to V$.

Consider a natural number $m \leq n := \dim(V)$. There is a canonical isomorphism

$$\beta_m(V) \colon \Lambda^m V^* \xrightarrow{\cong} \left(\Lambda^m V\right)^*$$

which maps $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$ to the map $\Lambda^m V \to \mathbf{R}$ sending $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ to $\sum_{\sigma \in S_m} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^m \alpha_i(v_{\sigma(i)})$. Let $s \colon V \to V^*$ be an inner product on V. We obtain an inner product s_{Λ^m} on $\Lambda^m V$ by the composite

$$s_{\Lambda^m} \colon \Lambda^m V \xrightarrow{\Lambda^m s} \Lambda^m V^* \xrightarrow{\beta_m(V)} \left(\Lambda^m V\right)^*.$$

One easily checks by a direct calculation for elements $v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m$ in V

$$s_{\Lambda^m}(v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_m) = \det\Big(\big(s(v_i, w_j)\big)_{i,j}\Big), \tag{1.2}$$

where $(s(v_i, w_j))_{i,j}$ is the obvious symmetric (m, m)-matrix.

Next we want to define a Riemannian metric g on $\widetilde{X}(V)$. Since $\widetilde{X}(V)$ is an open subset of Sym(V) and we have a canonical trivialization $\phi_{\mathrm{Sym}(V)}$ of T Sym(V) (see (**1.1**)), we have to define for every $s \in \widetilde{X}(V)$ an inner product g_s on Sym(V). It is given by

$$g_s(u, v) := \mathrm{tr}\big(s^{-1} \circ v \circ s^{-1} \circ u\big),$$

for $u, v \in \mathrm{Sym}(V)$. Here tr denotes the trace of endomorphisms of V. Obviously $g_s(-,-)$ is bilinear and symmetric since the trace is linear and satisfies $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. Since $s^{-1} \circ (s^{-1} \circ u)^* \circ s = s^{-1} \circ u$ holds, the endomorphism $s^{-1} \circ u \colon V \to V$ is selfadjoint with respect to the inner product s. Hence $\mathrm{tr}(s^{-1} \circ u \circ s^{-1} \circ u) \geq 0$ holds and we have $\mathrm{tr}(s^{-1} \circ u \circ s^{-1} \circ u) = 0$ if and only if $s^{-1} \circ u = 0$, or, equivalently, $u = 0$. Hence g_s is an inner product on $\mathrm{Sym}(V)$. We omit the proof that g_s depends smoothly on s. Thus we obtain a Riemannian metric g on $\widetilde{X}(V)$.

Lemma **1.1.** — *The Riemannian metric on* $\widetilde{X}$ *is* aut(V)*-invariant.*

Proof. — We have to show for an automorphism $f \colon V \to V$, an element $s \in \widetilde{X}(V)$ and two elements $u, v \in T_s\widetilde{X}(V) = T_s \mathrm{Sym}(V) = \mathrm{Sym}(V)$ that

$$g_{\widetilde{X}(f)(s)}\big(T_s\widetilde{X}(f)(u), T_s\widetilde{X}(f)(v)\big) = g_s(u, v)$$

holds. This follows from the following calculation:

$$\begin{aligned}
&g_{\widetilde{X}(f)(s)}\big(T_s\widetilde{X}(f)(u), T_s\widetilde{X}(f)(v)\big)\\
&= \mathrm{tr}\big(\widetilde{X}(f)(s)^{-1} \circ T_s\widetilde{X}(f)(v) \circ \widetilde{X}(f)(s)^{-1} \circ T_s\widetilde{X}(f)(u)\big)\\
&= \mathrm{tr}\Big(\big(f^* \circ s \circ f\big)^{-1} \circ \big(f^* \circ v \circ f\big) \circ \big(f^* \circ s \circ f\big)^{-1} \circ \big(f^* \circ u \circ f\big)\Big)\\
&= \mathrm{tr}\Big(f^{-1} \circ s^{-1} \circ \big(f^*\big)^{-1} \circ f^* \circ v \circ f \circ f^{-1} \circ s^{-1} \circ \big(f^*\big)^{-1} \circ f^* \circ u \circ f\Big)\\
&= \mathrm{tr}\big(f^{-1} \circ s^{-1} \circ v \circ s^{-1} \circ u \circ f\big)\\
&= \mathrm{tr}\big(s^{-1} \circ v \circ s^{-1} \circ u \circ f \circ f^{-1}\big)\\
&= \mathrm{tr}\big(s^{-1} \circ v \circ s^{-1} \circ u\big)\\
&= g_s(u, v).
\end{aligned}$$

□

Recall that so far we worked with the natural right action of aut(V) on $\widetilde{X}(V)$. In the sequel we prefer to work with the corresponding left action obtained by precomposing with $f \mapsto f^{-1}$ in order to match with standard notation, compare in particular diagram (**1.3**) below.

Fix a base point $s_0 \in \widetilde{X}(V)$. Choose a linear isomorphism $\mathbf{R}^n \xrightarrow{\cong} V$ which is isometric with respect to the standard inner product on $\mathbf{R}^n$ and s_0. It induces an isomorphism $GL_n(\mathbf{R}) \xrightarrow{\cong} \mathrm{aut}(V)$ and thus a smooth left action $\rho \colon GL_n(\mathbf{R}) \times \widetilde{X}(V) \to \widetilde{X}(V)$. Since for any two elements $s_1, s_2 \in \widetilde{X}(V)$ there exists an automorphism of V which is an isometry $(V, s_1) \to (V, s_2)$, this action is transitive. The stabilizer of $s_0 \in \widetilde{X}(V)$ is the compact subgroup $O(n) \subseteq GL_n(\mathbf{R})$. Thus we obtain a diffeomorphism $\tilde{\phi} \colon GL_n(\mathbf{R})/O(n) \xrightarrow{\cong} \widetilde{X}(V)$. Define smooth maps $p \colon \widetilde{X}(V) \to \mathbf{R}^{>0}$, $s \mapsto \det(s_0^{-1} \circ s)$ and $q \colon GL_n(\mathbf{R})/O(n) \to \mathbf{R}^{>0}$, $A \cdot O(n) \mapsto \det(A)^2$. Both maps are submersions. In particular the preimages of

$1 \in \mathbf{R}^{>0}$ under p and q are submanifolds of codimension 1. Denote by $\widetilde{X}(V)_1 := p^{-1}(1)$ and let $i\colon \widetilde{X}(V)_1 \to \widetilde{X}(V)$ be the inclusion. Set $SL_n^{\pm}(\mathbf{R}) = \{A \in GL_n(\mathbf{R}) \mid \det(A) = \pm 1\}$. The smooth left action $\rho\colon GL_n(\mathbf{R}) \times \widetilde{X}(V) \to \widetilde{X}(V)$ restricts to an action $\rho_1\colon SL_n^{\pm}(\mathbf{R}) \times \widetilde{X}(V)_1 \to \widetilde{X}(V)_1$. Since $f^* \circ s_0 \circ f \in \widetilde{X}(V)_1$ implies $1 = \det(s_0^{-1} \circ f^* \circ s_0 \circ f) = \det(f)^2$ this action is still transitive. The stabilizer of s_0 is still $O(n) \subseteq SL_n^{\pm}(\mathbf{R})$ and thus we obtain a diffeomorphism $\tilde{\phi}_1\colon SL_n^{\pm}(\mathbf{R})/O(n) \to \widetilde{X}(V)_1$. Note that the inclusion induces a diffeomorphism $SL_n(\mathbf{R})/SO(n) \cong SL_n^{\pm}(\mathbf{R})/O(n)$ but we prefer the right hand side because we are interested in the $GL_n(\mathbf{Z})$-action. The inclusion $SL_n^{\pm}(\mathbf{R}) \to GL_n(\mathbf{R})$ induces an embedding $j\colon SL_n^{\pm}(\mathbf{R})/O(n) \to GL_n(\mathbf{R})/O(n)$ with image $q^{-1}(1)$. One easily checks that the following diagram commutes

(**1.3**)
$$\begin{array}{ccccc}
\widetilde{X}(V)_1 & \xrightarrow{\;i\;} & \widetilde{X}(V) & \xrightarrow{\;p\;} & \mathbf{R}^{>0} \\
\uparrow{\scriptstyle \tilde{\phi}_1} & & \uparrow{\scriptstyle \tilde{\phi}} & & \uparrow{\scriptstyle \mathrm{id}} \\
SL_n^{\pm}(\mathbf{R})/O(n) & \xrightarrow{\;j\;} & GL_n(\mathbf{R})/O(n) & \xrightarrow{\;q\;} & \mathbf{R}^{>0}
\end{array}$$

Equip $\widetilde{X}(V)_1$ with the Riemannian metric g_1 obtained from the Riemannian metric g on $\widetilde{X}(V)$. We conclude from Lemma 1.1 that g_1 is $SL_n^{\pm}(\mathbf{R})$-invariant. Since $SL_n^{\pm}(\mathbf{R})$ is a semisimple Lie group with finite center and $O(n) \subseteq SL_n^{\pm}(\mathbf{R})$ is a maximal compact subgroup, $\widetilde{X}(V)_1$ is a symmetric space of non-compact type and its sectional curvature is non-positive (see [12, Section 2.2 on p. 70],[18, Theorem 3.1 (ii) in V.3 on p. 241]). Alternatively [8, Chapter II, Theorem 10.39 on p. 318 and Lemma 10.52 on p. 324] show that $\widetilde{X}(V)$ and $\widetilde{X}(V)_1$ are proper CAT(0) spaces.

We call two elements $s_1, s_2 \in \widetilde{X}(V)$ equivalent if there exists $r \in \mathbf{R}^{>0}$ with $r \cdot s_1 = s_2$. Denote by $X(V)$ the set of equivalence classes under this equivalence relation. Let $\mathrm{pr}\colon \widetilde{X}(V) \to X(V)$ be the projection and equip $X(V)$ with the quotient topology. The composite $\mathrm{pr} \circ i\colon \widetilde{X}(V)_1 \to X(V)$ is a homeomorphism. In the sequel we equip $X(V)$ with the structure of a Riemannian manifold for which $\mathrm{pr} \circ i$ is an isometric diffeomorphism. In particular $X(V)$ is a CAT(0)-space. The $GL_n(\mathbf{R})$-action on $\widetilde{X}(V)$ descends to an action on $X(V)$ and the diffeomorphism $\mathrm{pr} \circ i$ is $SL_n^{\pm}(\mathbf{R})$-equivariant.

1.2. *The volume function.* — Fix an integer m with $1 \leq m \leq n = \dim(V)$. In this section we investigate the following volume function.

Definition **1.2** *(Volume function).* — *Consider* $\xi \in \Lambda^m V$ *with* $\xi \neq 0$. *Define the* volume function associated to ξ *by*

$$\mathrm{vol}_\xi\colon \widetilde{X}(V) \to \mathbf{R}, \qquad s \mapsto \sqrt{(s_{\Lambda^m})(\xi, \xi)},$$

i.e., the function vol_ξ *sends an inner product* s *on* V *to the length of* ξ *with respect to* s_{Λ^m}.

Fix $\xi \neq 0$ in $\Lambda^m V$ of the form $\xi = v_1 \wedge v_2 \cdots \wedge v_m$ for the rest of this section. This means that the line $\langle \xi \rangle$ lies in the image of the Plücker embedding, compare [17, Chapter 1.5 p. 209–211]. Then there is precisely one m-dimensional subvector space $V_\xi \subseteq V$ such that the image of the map $\Lambda^m V_\xi \to \Lambda^m V$ induced by the inclusion is the 1-dimensional subvector space spanned by ξ. The subspace V_ξ is the span of the vectors $v_1, v_2, \ldots, v_m$. It can be expressed as $V_\xi = \{v \in V | v \wedge \xi = 0\}$. This shows that V_ξ depends on the line $\langle \xi \rangle$ but is independent of the choice of $v_1, \ldots, v_m$.

Given an inner product s on V, we obtain an orthogonal decomposition $V = V_\xi \oplus V_\xi^\perp$ of V with respect to s and we define $s_\xi \in \mathrm{Sym}(V)$ to be the element which satisfies $s_\xi(v + v^\perp, w + w^\perp) = s(v, w)$ for all $v, w \in V_\xi$ and $v^\perp, w^\perp \in V_\xi^\perp$.

Theorem **1.3** *(Gradient of the volume function). — The gradient of the square* vol_ξ^2 *of the volume function* vol_ξ *is given for* $s \in \widetilde{X}(V)$ *by*

$$\nabla_s\left(\mathrm{vol}_\xi^2\right) = \mathrm{vol}_\xi^2(s) \cdot s_\xi \in \mathrm{Sym}(V) = T_s\widetilde{X}(V).$$

Proof. — Let A be any (m, m)-matrix. Then

$$\lim_{t \to 0} \frac{\det(I_m + t \cdot A) - \det(I_m)}{t} = \mathrm{tr}(A). \tag{1.4}$$

This follows because

$$\det(I_m + t \cdot A) = 1 + t\,\mathrm{tr}(A) \bmod t^2$$

by the Leibniz formula for the determinant.

Consider $s \in \widetilde{X}(V)$ and $u \in T_s(\widetilde{X}(V)) = \mathrm{Sym}(V)$. Notice that there exists $\epsilon > 0$ such that $s + t \cdot u$ lies in $\widetilde{X}(V)$ for all $t \in (-\epsilon, \epsilon)$. Fix $v_1, v_2, \ldots, v_m \in V$ with $\xi = v_1 \wedge \cdots \wedge v_m$. We compute using (**1.2**) and (**1.4**)

$$\begin{aligned}
&\lim_{t \to 0} \frac{\mathrm{vol}_\xi^2(s + t \cdot u) - \mathrm{vol}_\xi^2(s)}{t} \\
&\quad = \lim_{t \to 0} \frac{\det(((s + t \cdot u)(v_i, v_j))_{i,j}) - \det((s(v_i, v_j))_{i,j})}{t} \\
&\quad = \lim_{t \to 0} \frac{\det((s(v_i, v_j))_{i,j} + t \cdot (u(v_i, v_j))_{i,j}) - \det((s(v_i, v_j))_{i,j})}{t} \\
&\quad = \det\left(\left(s(v_i, v_j)\right)_{i,j}\right) \\
&\qquad \cdot \lim_{t \to 0} \frac{\det(I_m + t \cdot (s(v_i, v_j))_{i,j}^{-1} \cdot (u(v_i, v_j))_{i,j}) - \det(I_m)}{t} \\
&\quad = \det\left(\left(s(v_i, v_j)\right)_{i,j}\right) \cdot \mathrm{tr}\left(\left(s(v_i, v_j)\right)_{i,j}^{-1} \cdot \left(u(v_i, v_j)\right)_{i,j}\right).
\end{aligned}$$

The gradient $\nabla_s(\mathrm{vol}_\xi^2)$ is uniquely determined by the property that for all $u \in \mathrm{Sym}(V)$ we have

$$g_s\big(\nabla_s\big(\mathrm{vol}_\xi^2\big), u\big) = \lim_{t\to 0} \frac{\mathrm{vol}_\xi^2(s + t\cdot u) - \mathrm{vol}_\xi^2(s)}{t}.$$

Hence it remains to show for every $u \in \mathrm{Sym}(V)$

$$g_s\big(\mathrm{vol}_\xi^2(s)\cdot s_\xi, u\big) = \det\Big(\big(s(v_i, v_j)\big)_{i,j}\Big)\cdot \mathrm{tr}\Big(\big(s(v_i, v_j)\big)_{i,j}^{-1}\cdot \big(u(v_i, v_j)\big)_{i,j}\Big).$$

Since $\mathrm{vol}_\xi^2(s) = \det((s(v_i, v_j))_{i,j})$ by (**1.2**) and $g_s(s_\xi, u) = \mathrm{tr}(s^{-1}\circ s_\xi \circ s^{-1}\circ u)$ by definition, it remains to show

$$\mathrm{tr}\big(s^{-1}\circ s_\xi\circ s^{-1}\circ u\big) = \mathrm{tr}\Big(\big(s(v_i, v_j)\big)_{i,j}^{-1}\cdot\big(u(v_i, v_j)\big)_{i,j}\Big).$$

We obtain a decomposition $s = \left(\begin{smallmatrix} s_\xi & 0 \\ 0 & s_\xi^\perp\end{smallmatrix}\right)$ for an inner product $s_\xi^\perp\colon V_\xi^\perp \to (V_\xi^\perp)^*$, where we will here and in the sequel identify $(V_\xi)^*\oplus(V_\xi^\perp)^*$ and $(V_\xi\oplus V_\xi^\perp)^*$ by the canonical isomorphism. We decompose

$$u = \begin{pmatrix} u_\xi & u' \\ u'' & u''' \end{pmatrix} : V_\xi \oplus V_\xi^\perp \to (V_\xi)^* \oplus \big(V_\xi^\perp\big)^*$$

for a linear map $u_\xi\colon V_\xi \to V_\xi^*$. One easily checks

$$\mathrm{tr}\big(s^{-1}\circ s_\xi\circ s^{-1}\circ u\big) = \mathrm{tr}\big(s_\xi^{-1}\circ u_\xi\big).$$

The set $\{v_1, v_2, \ldots, v_m\}$ is a basis for V_ξ. Let $\{v_1^*, v_2^*, \ldots, v_m^*\}$ be the dual basis of V_ξ^*. Then the matrix of u_ξ with respect to these basis is $(u(v_i, v_j))_{i,j}$ and the matrix of s_ξ with respect to these basis is $(s(v_i, v_j))_{i,j}$. Hence the matrix of $s_\xi^{-1}\circ u_\xi\colon V_\xi \to V_\xi$ with respect to the basis $\{v_1, v_2, \ldots v_m\}$ is $(s(v_i, v_j))_{i,j}^{-1}\cdot(u(v_i, v_j))_{i,j}$. This implies

$$\mathrm{tr}\big(s_\xi^{-1}\circ u_\xi\big) = \mathrm{tr}\Big(\big(s(v_i, v_j)\big)_{i,j}^{-1}\cdot\big(u(v_i, v_j)\big)_{i,j}\Big).$$

This finishes the proof of Theorem 1.3. □

Corollary **1.4.** — *The gradient of the function*

$$\ln\circ\mathrm{vol}_\xi^2\colon \widetilde{X}(V) \to \mathbf{R}$$

at $s \in \widetilde{X}(V)$ is given by the tangent vector $s_\xi \in T_s\widetilde{X}(V) = \mathrm{Sym}(V)$. In particular its norm with respect to the Riemannian metric g on $\widetilde{X}(V)$ is independent of s, namely, $\sqrt{m}$.

Proof. — Since the derivative of $\ln(x)$ is x^{-1}, the chain rule implies together with Theorem 1.3 for $s \in \widetilde{X}(V)$

$$\nabla_s\bigl(\ln \circ \mathrm{vol}_\xi^2\bigr) = \nabla_s\bigl(\mathrm{vol}(\xi)^2\bigr) \cdot \frac{1}{\mathrm{vol}_\xi^2(s)} = \mathrm{vol}_\xi^2(s) \cdot s_\xi \cdot \frac{1}{\mathrm{vol}_\xi^2(s)} = s_\xi.$$

We use the orthogonal decomposition $V = V_\xi \oplus V_\xi^\perp$ with respect to s to compute

$$\begin{aligned}\bigl(\bigl\|\nabla_s\bigl(\ln \circ \mathrm{vol}_\xi^2\bigr)\bigr\|_s\bigr)^2 &= (\|s_\xi\|_s)^2 = g_s(s_\xi, s_\xi) = \mathrm{tr}\bigl(s^{-1} \circ s_\xi \circ s^{-1} \circ s_\xi\bigr)\\ &= \mathrm{tr}\bigl(\mathrm{id}_{V_\xi} \oplus 0\colon V_\xi \oplus V_\xi^\perp \to V_\xi \oplus V_\xi^\perp\bigr)\\ &= \dim(V_\xi) = m.\end{aligned}$$

□

2. Sublattices of $\mathbf{Z}^n$

A *sublattice* W of $\mathbf{Z}^n$ is a $\mathbf{Z}$-submodule $W \subseteq \mathbf{Z}^n$ such that $\mathbf{Z}^n/W$ is a projective $\mathbf{Z}$-module. Equivalently, W is a $\mathbf{Z}$-submodule $W \subseteq \mathbf{Z}^n$ such that for $x \in \mathbf{Z}^n$ for which $k \cdot x$ belongs to W for some $k \in \mathbf{Z}$, $k \neq 0$ we have $x \in W$. Let $\mathcal{L}$ be the set of sublattices L of $\mathbf{Z}^n$.

Consider $W \in \mathcal{L}$. Let m be its rank as an abelian group. Let $\Lambda^m_{\mathbf{Z}} W \to \Lambda^m_{\mathbf{Z}}\mathbf{Z}^n \to \Lambda^m\mathbf{R}^n$ be the obvious map. Let $\xi(W) \in \Lambda^m\mathbf{R}^n$ be the image of a generator of the infinite cyclic group $\Lambda^m_{\mathbf{Z}}W$. We have defined the map $\mathrm{vol}_{\xi(W)}\colon \widetilde{X}(\mathbf{R}^n) \to \mathbf{R}$ above. Obviously it does not change if we replace $\xi(W)$ by $-\xi(W)$. Hence it depends only on W and not on the choice of generator $\xi(W)$. Notice that for $W = 0$ we have by definition $\Lambda^0_{\mathbf{Z}}W = \mathbf{Z}$ and $\Lambda^0\mathbf{R}^n = \mathbf{R}$ and $\xi(W)$ is $\pm 1 \in \mathbf{R}$. In that case vol_ξ is the constant function with value 1.

We will abbreviate

$$\widetilde{X} = \widetilde{X}\bigl(\mathbf{R}^n\bigr), \quad \widetilde{X}_1 = \widetilde{X}\bigl(\mathbf{R}^n\bigr)_1, \quad X = X\bigl(\mathbf{R}^n\bigr) \quad \text{and} \quad \mathrm{vol}_W = \mathrm{vol}_{\xi(W)}$$

for $W \in \mathcal{L}$.

Given a chain $W_0 \subsetneq W_1$ of elements $W_0, W_1 \in \mathcal{L}$, we define a function

$$\tilde{c}_{W_0 \subsetneq W_1}\colon \widetilde{X} \to \mathbf{R}$$

by

$$\tilde{c}_{W_0 \subsetneq W_1}(s) := \frac{\ln(\mathrm{vol}_{W_1}(s)) - \ln(\mathrm{vol}_{W_0}(s))}{\mathrm{rk}_{\mathbf{Z}}(W_1) - \mathrm{rk}_{\mathbf{Z}}(W_0)}.$$

Obviously this can be rewritten as

$$\tilde{c}_{W_0 \subsetneq W_1}(s) = \frac{1}{2} \cdot \left(\frac{\ln(\mathrm{vol}_{W_1}(s)^2) - \ln(\mathrm{vol}_{W_0}(s)^2)}{\mathrm{rk}_{\mathbf{Z}}(W_1) - \mathrm{rk}_{\mathbf{Z}}(W_0)}\right).$$

Hence we get for the norm of the gradient of this function at $s \in \widetilde{X}$

$$\begin{aligned}
&\left\| \nabla_s(\widetilde{c}_{W_0 \subsetneq W_1}) \right\| \\
&= \left\| \frac{1}{2} \cdot \left(\frac{\nabla_s(\ln \circ \mathrm{vol}^2_{W_1}) - \nabla_s(\ln \circ \mathrm{vol}^2_{W_0})}{\mathrm{rk}_{\mathbf{Z}}(W_1) - \mathrm{rk}_{\mathbf{Z}}(W_0)} \right) \right\| \\
&\leq \frac{1}{2} \cdot \left(\frac{\|\nabla_s(\ln \circ \mathrm{vol}^2_{W_1})\| + \|\nabla_s(\ln \circ \mathrm{vol}^2_{W_0})\|}{\mathrm{rk}_{\mathbf{Z}}(W_1) - \mathrm{rk}_{\mathbf{Z}}(W_0)} \right) \\
&\leq \frac{1}{2} \cdot \left(\left\| \nabla_s\big(\ln \circ \mathrm{vol}^2_{W_1}\big) \right\| + \left\| \nabla_s\big(\ln \circ \mathrm{vol}^2_{W_0}\big) \right\| \right).
\end{aligned}$$

We conclude from Corollary 1.4 for all $s \in \widetilde{X}$.

$$\left\| \nabla_s(\widetilde{c}_{W_0 \subsetneq W_1}) \right\| \leq \frac{\sqrt{\mathrm{rk}_{\mathbf{Z}}(W_1)} + \sqrt{\mathrm{rk}_{\mathbf{Z}}(W_0)}}{2} \leq \sqrt{n}.$$

If $f \colon M \to \mathbf{R}$ is a differentiable function on a Riemannian manifold M and $C = \sup\{\|\nabla_x f\| \mid x \in M\}$ we always have

$$\big|f(x_1) - f(x_2)\big| \leq C d_M(x_1, x_2),$$

where d_M denotes the metric associated to the Riemannian metric. In particular we get for any two elements $s_0, s_1 \in \widetilde{X}$

$$\left|\widetilde{c}_{W_0 \subsetneq W_1}(s_1) - \widetilde{c}_{W_0 \subsetneq W_1}(s_0)\right| \leq \sqrt{n} \cdot d_{\widetilde{X}}(s_0, s_1),$$

where $d_{\widetilde{X}}$ is the metric on $\widetilde{X}$ coming from the Riemannian metric g on $\widetilde{X}$. Recall that the Riemannian metric g_1 on $\widetilde{X}_1$ is obtained by restricting the metric g. Let $d_{\widetilde{X}_1}$ be the metric on $\widetilde{X}_1$ coming from the Riemannian metric g_1 on $\widetilde{X}_1$. Recall that $i \colon \widetilde{X}_1 \to \widetilde{X}$ is the inclusion. Then we get for $s_0, s_1 \in \widetilde{X}_1$

$$d_{\widetilde{X}}\big(i(s_0), i(s_1)\big) \leq d_{\widetilde{X}_1}(s_0, s_1).$$

Indeed $\widetilde{X}_1$ is a geodesic submanifold of $\widetilde{X}$ ([8, Chapter II, Lemma 10.52 on p. 324]). So both sides are even equal. We obtain for $s_0, s_1 \in \widetilde{X}_1$

$$\left|\widetilde{c}_{W_0 \subsetneq W_1} \circ i(s_1) - \widetilde{c}_{W_0 \subsetneq W_1} \circ i(s_0)\right| \leq \sqrt{n} \cdot d_{\widetilde{X}_1}(s_0, s_1). \tag{2.1}$$

Put

$$\mathcal{L}' = \left\{ W \in \mathcal{L} \mid W \neq 0, W \neq \mathbf{Z}^n \right\}.$$

Define for $W \in \mathcal{L}'$ functions

$$\widetilde{c}^{\,i}_W, \widetilde{c}^{\,s}_W \colon \widetilde{X} \to \mathbf{R}$$

by

$$\tilde{c}^i_W(x) := \inf\{\tilde{c}_{W\subsetneq W_2}(x) \mid W_2 \in \mathcal{L}, W \subsetneq W_2\};$$
$$\tilde{c}^s_W(x) := \sup\{\tilde{c}_{W_0\subsetneq W}(x) \mid W_0 \in \mathcal{L}, W_0 \subsetneq W\}.$$

In order to see that infimum and supremum exist we use the fact that for fixed $x \in \widetilde{X}$ there are at most finitely many $W \in \mathcal{L}$ with $\mathrm{vol}_W(x) \leq 1$, compare [16, Lemma 1.15]. Put

$$(\mathbf{2.2}) \qquad \tilde{d}_W : \widetilde{X} \to \mathbf{R}, \quad x \mapsto \exp\big(\tilde{c}^i_W(x) - \tilde{c}^s_W(x)\big).$$

Since $\mathrm{vol}_W(r\cdot s) = r^{\mathrm{rk}_{\mathbf{Z}}(W)} \cdot \mathrm{vol}_W(s)$ holds for $r \in \mathbf{R}^{>0}$, $W \in \mathcal{L}$ and $s \in \widetilde{X}$, we have $\tilde{c}_{W_0\subsetneq W_1}(r\cdot s) = \tilde{c}_{W_0\subsetneq W_1}(s) + \ln(r)$ and the function $\tilde{d}_W$ factorizes over the projection $\mathrm{pr}\colon \widetilde{X} \to X$ to a function

$$d_W : X \to \mathbf{R}.$$

Lemma **2.1.** — *Consider $x \in X$, $W \in \mathcal{L}'$, and $\alpha \in \mathbf{R}^{>0}$. Then we get for all $y \in X$ with $d_X(x, y) \leq \alpha$*

$$d_W(y) \in \big[d_W(x)/e^{2\sqrt{n}\alpha}, d_W(x)\cdot e^{2\sqrt{n}\alpha}\big].$$

Proof. — By the definition of the structure of a smooth Riemannian manifold on X, it suffices to show for all $s_0, s_1 \in \widetilde{X}_1$ with $d_{\widetilde{X}_1}(s_0, s_1) \leq \alpha$

$$\tilde{d}_W(s_1) \in \big[\tilde{d}_W(s_0)/e^{2\sqrt{n}\alpha}, \tilde{d}_W(s_0)\cdot e^{2\sqrt{n}\alpha}\big].$$

One concludes from (**2.1**) for $s_0, s_1 \in \widetilde{X}_1$ with $d_{\widetilde{X}_1}(s_0, s_1) \leq \alpha$ and $W \in \mathcal{L}'$

$$\tilde{c}^s_W(s_1) \in \big[\tilde{c}^s_W(s_0) - \sqrt{n}\cdot\alpha, \tilde{c}^s_W(s_0) + \sqrt{n}\cdot\alpha\big],$$
$$\tilde{c}^i_W(s_1) \in \big[\tilde{c}^i_W(s_0) - \sqrt{n}\cdot\alpha, \tilde{c}^i_W(s_0) + \sqrt{n}\cdot\alpha\big].$$

Now the claim follows. □

In particular the function $d_W : X \to \mathbf{R}$ is continuous.

Define the following open subset of X for $W \in \mathcal{L}'$ and $t \geq 1$

$$X(W, t) := \{x \in X \mid d_W(x) > t\}.$$

There is an obvious $GL_n(\mathbf{Z})$-action on $\mathcal{L}$ and $\mathcal{L}'$ and in the discussion following Lemma 1.1 we have already explained how $GL_n(\mathbf{Z}) \subset SL_n^{\pm}(\mathbf{R})$ acts on $\widetilde{X}_1$ and hence on X (choosing the standard inner product on $\mathbf{R}^n$ as the basepoint s_0).

Lemma **2.2.** — *For any $t \geq 1$ we get:*

(i) $X(gW, t) = gX(W, t)$ *for* $g \in GL_n(\mathbf{Z})$, $W \in \mathcal{L}'$;

(ii) *The complement of the* $GL_n(\mathbf{Z})$*-invariant open subset*

$$|\mathcal{W}(t)| := \bigcup_{W \in \mathcal{L}'} X(W, t)$$

in X *is a cocompact* $GL_n(\mathbf{Z})$*-set;*

(iii) *If* $X(W_1, t) \cap X(W_2, t) \cap \cdots \cap X(W_k, t) \neq \emptyset$ *for* $W_i \in \mathcal{L}'$, *then we can find a permutation* $\sigma \in \Sigma_k$ *such that* $W_{\sigma(1)} \subseteq W_{\sigma(2)} \subseteq \cdots \subseteq W_{\sigma(k)}$ *holds.*

Proof. — This follows directly from Grayson [16, Lemma 2.1, Cor. 5.2] as soon as we have explained how our setup corresponds to the one of Grayson.

We are only dealing with the case $\mathcal{O} = \mathbf{Z}$ and $F = \mathbf{Q}$ of [16]. In particular there is only one archimedian place, namely, the absolute value on $\mathbf{Q}$ and for it $\mathbf{Q}_\infty = \mathbf{R}$. So an element $s \in \widetilde{X}$ corresponds to the structure of a lattice which we will denote by $(\mathbf{Z}^n, s)$ with underlying $\mathbf{Z}$-module $\mathbf{Z}^n$ in the sense of [16]. Given $s \in \widetilde{X}$, an element $W \in \mathcal{L}$ defines a sublattice of the lattice $(\mathbf{Z}^n, s)$ in the sense of [16] which we will denote by $(\mathbf{Z}^n, s) \cap W$. The volume of a sublattice $(\mathbf{Z}^n, s) \cap W$ of $(\mathbf{Z}^n, s)$ in the sense of [16] is $\mathrm{vol}_W(s)$.

Given $W \in \mathcal{L}'$, we obtain a $\mathbf{Q}$-subspace in the sense of [16, Definition 2.1] which we denote again by W, and vice versa. It remains to explain why our function d_W of (**2.2**) agrees with the function d_W of [16, Definition 2.1] which is given by

$$d_W(s) = \exp\big(\min\big((\mathbf{Z}^n, s)/(\mathbf{Z}^n, s) \cap W\big) - \max\big((\mathbf{Z}^n, s) \cap W\big)\big),$$

see [16, Definition 1.23, 1.9]. This holds by the following observation.

Consider $s \in \widetilde{X}$ and $W \in \mathcal{L}'$. Consider the canonical plot and the canonical polygon of the lattice $(\mathbf{Z}^n, s) \cap W$ in the sense of [16, Definition 1.10 and Discussion 1.16]. The slopes of the canonical polygon are strictly increasing when going from the left to the right because of [16, Corollary 1.30]. Hence $\max((\mathbf{Z}^n, s) \cap W)$ in the sense of [16, Definition 1.23] is the slope of the segment of the canonical polygon ending at $(\mathbf{Z}^n, s) \cap W$. Consider any $W_0 \in \mathcal{L}$ with $W_0 \subsetneq W$. Obviously the slope of the line joining the plot point of $(\mathbf{Z}^n, s) \cap W_0$ and $(\mathbf{Z}^n, s) \cap W$ is less than or equal to the slope of the segment of the canonical polygon ending at $(\mathbf{Z}^n, s) \cap W$. If $(\mathbf{Z}^n, s) \cap W_0$ happens to be the starting point of this segment, then this slope agrees with the slope of the segment of the canonical plot ending at $(\mathbf{Z}^n, s) \cap W$. Hence

$$\begin{aligned} &\max\big((\mathbf{Z}^n, s) \cap W\big) \\ &= \max\left\{ \frac{\ln(\mathrm{vol}_W(s)) - \ln(\mathrm{vol}_{W_0}(s))}{\mathrm{rk}_{\mathbf{Z}}(W) - \mathrm{rk}_{\mathbf{Z}}(W_0)} \;\middle|\; W_0 \in \mathcal{L}, W_0 \subsetneq W \right\} = \widetilde{c}^{\,s}_W(s). \end{aligned} \tag{2.3}$$

We have the formula

$$\mathrm{vol}\big((\mathbf{Z}^n, s) \cap W_2\big) = \mathrm{vol}\big((\mathbf{Z}^n, s) \cap W\big) \cdot \mathrm{vol}\big((\mathbf{Z}^n, s) \cap W_2/(\mathbf{Z}^n, s) \cap W\big)$$

for any $W_2 \in \mathcal{L}$ with $W \subsetneq W_2$ (see [16, Lemma 1.8]). Hence

$$\frac{\ln(\mathrm{vol}_{W_2}(s)) - \ln(\mathrm{vol}_W(s))}{\mathrm{rk}_{\mathbf{Z}}(W_2) - \mathrm{rk}_{\mathbf{Z}}(W)} = \frac{\ln(\mathrm{vol}_{W_2/W}(s))}{\mathrm{rk}_{\mathbf{Z}}(W_2/W)}.$$

There is an obvious bijection of the set of direct summands in $\mathbf{Z}^n/W$ and the set of direct summand in $\mathbf{Z}^n$ containing W. Now analogously to the proof of (**2.3**) one shows

$$\min\big((\mathbf{Z}^n, s)/((\mathbf{Z}^n, s) \cap W)\big) \tag{2.4}$$

$$= \min\left\{ \frac{\ln(\mathrm{vol}_{W_2}(s)) - \ln(\mathrm{vol}_W(s))}{\mathrm{rk}_{\mathbf{Z}}(W_2) - \mathrm{rk}_{\mathbf{Z}}(W)} \,\middle|\, W_2 \in \mathcal{L}, W \subsetneq W_2 \right\} = \widetilde{c}^{\,i}_W(s). \tag{2.5}$$

Now the equality of the two versions for d_W follows from (**2.3**) and (**2.5**). This finishes the poof of Lemma 2.2. □

3. Transfer reducibility of $GL_n(\mathbf{Z})$

Let $\mathcal{F}_n$ be the family of those subgroups H of $GL_n(\mathbf{Z})$, which are virtually cyclic or for which there exists a finitely generated free abelian group P, natural numbers r and n_i with $n_i < n$ and an extension of groups

$$1 \to P \to K \to \prod_{i=1}^{r} GL_{n_i}(\mathbf{Z}) \to 1$$

such that H is isomorphic to a subgroup of K.

In this section we prove the following theorem, which by [4, Theorem 1.1] implies the K-theoretic FJC up to dimension 1 for $GL_n(\mathbf{Z})$ with respect to the family $\mathcal{F}_n$. The notion of *transfer reducibility* has been introduced in [4, Definition 1.8]. Transfer reducibility asserts the existence of a compact space Z and certain equivariant covers of $G \times Z$. A slight modification of transfer reducibility is discussed in Section 5, see Definition 5.3.

Here our main work is to verify the conditions formulated in Definition 3.7 for our situation, see Lemma 3.8. Once this has been done the verification of transfer reducibility proceeds as in [3], as is explained after Lemma 3.8.

Theorem **3.1.** — *The group $GL_n(\mathbf{Z})$ is transfer reducible over $\mathcal{F}_n$.*

To prove this we will use the space $X = X(\mathbf{R}^n)$ and its subsets $X(W, t)$ considered in Sections 1 and 2.

For a G-space X and a family of subgroups $\mathcal{F}$, a subset $U \subseteq X$ is called an $\mathcal{F}$-subset if $G_U := \{g \in G \mid g(U) = U\}$ belongs to $\mathcal{F}$ and $gU \cap U = \emptyset$ for all $g \in G \setminus G_U$. An open G-invariant cover consisting of $\mathcal{F}$-subsets is called an $\mathcal{F}$-cover.

Lemma **3.2.** — *Consider for* $t \geq 1$ *the collection of subsets of* X

$$\mathcal{W}(t) = \left\{X(W, t) \mid W \in \mathcal{L}'\right\}.$$

It is a $GL_n(\mathbf{Z})$*-invariant set of open* $\mathcal{F}_n$*-subsets of* X *whose covering dimension is at most* $(n-2)$.

Proof. — The set $\mathcal{W}(t)$ is $GL_n(\mathbf{Z})$-invariant because of Lemma 2.2 (i) and its covering dimension is bounded by $(n-2)$ because of Lemma 2.2 (iii) since for any chain of sublattices $\{0\} \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r \subsetneq \mathbf{Z}^n$ of $\mathbf{Z}^n$ we have $r \leq n-2$.

It remains to show that $\mathcal{W}(t)$ consists of $\mathcal{F}_n$-subsets. Consider $W \in \mathcal{L}'$ and $g \in GL_n(\mathbf{Z})$. Lemma 2.2 implies:

$$X(W, t) \cap gX(W, t) \neq \emptyset \iff X(W, t) \cap X(gW, t) \neq \emptyset \iff gW = W.$$

Put $GL_n(\mathbf{Z})_W = \{g \in GL_n(\mathbf{Z}) \mid gW = W\}$. Choose $V \subset \mathbf{Z}^n$ with $W \oplus V = \mathbf{Z}^n$. Under this identification every element $\phi \in GL_n(\mathbf{Z})_W$ is of the shape

$$\begin{pmatrix} \phi_W & \phi_{V,W} \\ 0 & \phi_V \end{pmatrix}$$

for $\mathbf{Z}$-automorphism ϕ_V of V, ϕ_W of W and a $\mathbf{Z}$-homomorphism $\phi_{V,W}\colon V \to W$. Define

$$p\colon GL_n(\mathbf{Z})_W \to \operatorname{aut}_{\mathbf{Z}}(W) \times \operatorname{aut}_{\mathbf{Z}}(V), \quad \begin{pmatrix} \phi_W & \phi_{V,W} \\ 0 & \phi_V \end{pmatrix} \mapsto (\phi_W, \phi_V)$$

and

$$i\colon \hom_{\mathbf{Z}}(V, W) \to GL_n(\mathbf{Z})_W, \quad \psi \mapsto \begin{pmatrix} \mathrm{id}_W & \psi \\ 0 & \mathrm{id}_V \end{pmatrix}.$$

Then we obtain an exact sequence of groups

$$1 \to \hom_{\mathbf{Z}}(V, W) \xrightarrow{i} GL_n(\mathbf{Z})_W \xrightarrow{\mathrm{pr}} \operatorname{aut}_{\mathbf{Z}}(W) \times \operatorname{aut}_{\mathbf{Z}}(V) \to 1,$$

where $\hom_{\mathbf{Z}}(V, W)$ is the abelian group given by the obvious addition. Since both V and W are different from $\mathbf{Z}^n$, we get $\operatorname{aut}_{\mathbf{Z}}(V) \cong GL_{n(V)}(\mathbf{Z})$ and $\operatorname{aut}_{\mathbf{Z}}(W) \cong GL_{n(W)}(\mathbf{Z})$ for $n(V), n(W) < n$. Hence each element in $\mathcal{W}(t)$ is an $\mathcal{F}_n$-subset with respect to the $GL_n(\mathbf{Z})$-action. □

For a subset A of a metric space X and $\alpha \in \mathbf{R}^{>0}$ we denote by

$$B_\alpha(A) := \left\{x \in X \mid d_X(x, a) < \alpha \text{ for some } a \in A\right\}$$

the α-neighborhood of A.

Lemma **3.3.** — *For every* $\alpha > 0$ *and* $t \geq 1$ *there exists* $\beta > 0$ *such that for every* $W \in \mathcal{L}'$ *we have* $B_\alpha(X(W, t+\beta)) \subseteq X(W, t)$.

Proof. — This follows from Lemma 2.1 if we choose $\beta > (e^{2\sqrt{n}\alpha} - 1) \cdot t$. □

Let $FS(X)$ be the flow space associated to the CAT(0)-space $X = X(\mathbf{R}^n)$ in [3, Section 2]. It consists of generalized geodesics, i.e., continuous maps $c \colon \mathbf{R} \to X$ for which there is a closed subinterval I of $\mathbf{R}$ such that $c|_I$ is a geodesic and $c|_{\mathbf{R}\setminus I}$ is locally constant. The flow Φ on $FS(X)$ is defined by the formula $(\Phi_\tau(c))(t) = c(\tau + t)$.

Lemma **3.4.** — *Consider* $\delta, \tau > 0$ *and* $c \in FS(X)$. *Then we get for* $d \in B_\delta(\Phi_{[-\tau,\tau]}(c))$

$$d_X\big(d(0), c(0)\big) < 4 + \delta + \tau.$$

Proof. — Choose $s \in [-\tau, \tau]$ with $d_{FS(X)}(d, \Phi_s(c)) < \delta$. We estimate using [3, Lemma 1.4 (i)] and a special case of [3, Lemma 1.3]

$$\begin{aligned} d_X\big(d(0), c(0)\big) &\le d_X\big(d(0), \Phi_s(c)(0)\big) + d_X\big(\Phi_s(c)(0), c(0)\big) \\ &\le d_{FS(X)}\big(d, \Phi_s(c)\big) + 2 + d_{FS(X)}\big(\Phi_s(c), c\big) + 2 \\ &< \delta + 2 + |s| + 2 \le 4 + \delta + \tau. \end{aligned}$$ □

Let $ev_0 \colon FS(X) \to X$, $c \mapsto c(0)$ be the evaluation map at 0. It is $GL_n(\mathbf{Z})$-equivariant, uniform continuous and proper [3, Lemma 1.10]. Define subsets of $FS(X)$ by

$$Y(W, t) := ev_0^{-1}\big(X(W, t)\big) \quad \text{for } t \ge 1, \quad W \in \mathcal{L}',$$

and set $\mathcal{V}(t) := \{Y(W, t) \mid W \in \mathcal{L}'\}$, $|\mathcal{V}(t)| := \bigcup_{W\in\mathcal{L}'} Y(W, t) \subseteq FS(X)$.

Lemma **3.5.** — *Let* $\tau, \delta > 0$ *and* $t \ge 1$ *and set* $\alpha := 4 + \delta + \tau$. *If* $\beta > 0$ *is such that for every* $W \in \mathcal{L}'$ *we have* $B_\alpha(X(W, t + \beta)) \subseteq X(W, t)$ *then the following holds*

(i) *The set* $\mathcal{V}(t)$ *is* $GL_n(\mathbf{Z})$*-invariant;*
(ii) *Each element in* $\mathcal{V}(t)$ *is an open* $\mathcal{F}_n$*-subset with respect to the* $GL_n(\mathbf{Z})$*-action;*
(iii) *The dimension of* $\mathcal{V}(t)$ *is less or equal to* $(n - 2)$*;*
(iv) *The complement of* $|\mathcal{V}(t + \beta)|$ *in* $FS(X)$ *is a cocompact* $GL_n(\mathbf{Z})$*-subspace.*
(v) *For every* $c \in |\mathcal{V}(t + \beta)|$ *there exists* $W \in \mathcal{L}'$ *with*

$$B_\delta\big(\Phi_{[-\tau,\tau]}(c)\big) \subset Y(W, t).$$

The existence of a suitable β is the assertion of Lemma 3.3.

Proof. — (i), (ii) and (iii) follow from Lemma 3.2 since ev_0 is $GL_n(\mathbf{Z})$-equivariant. (iv) follows from Lemma 2.2 (ii) since the map $ev_0 \colon FS(X) \to X$ is proper and $ev_0^{-1}(X - |\mathcal{W}(t + \beta)|) = FS(X) - |\mathcal{V}(t + \beta)|$.

For (v) consider $c \in |\mathcal{V}(t+\beta)|$. Choose $W \in \mathcal{L}'$ with $c \in Y(W, t+\beta)$. Then $c(0) \in X(W, t+\beta)$. Consider $d \in B_\delta(\Phi_{[-\tau,\tau]}(c))$. Lemma 3.4 implies $d_X(d(0), c(0)) < \alpha$. Hence $d(0) \in B_\alpha(c(0))$. We conclude $d(0) \in B_\alpha(X(W, t+\beta))$. This implies $d(0) \in X(W, t)$ and hence $d \in Y(W, t)$. This shows $B_\delta(\Phi_{[-\tau,\tau]}(c)) \subseteq Y(W, t)$. □

Let $FS_{\leq\gamma}(X)$ be the subspace of $FS(X)$ of those generalized geodesics c for which there exists for every $\epsilon > 0$ a number $\tau \in (0, \gamma+\epsilon]$ and $g \in G$ such that $g \cdot c = \Phi_\tau(c)$ holds. As an instance of [3, Theorem 4.2] we obtain the following in our situation.

Theorem **3.6.** — *There is a natural number* M *such that for every* $\gamma > 0$ *and every compact subset* $L \subseteq X$ *there exists a* $GL_n(\mathbf{Z})$*-invariant collection* $\mathcal{U}$ *of subsets of* $FS(X)$ *satisfying:*

(i) *Each element* $U \in \mathcal{U}$ *is an open* $\mathcal{VC}yc$*-subset of the* $GL_n(\mathbf{Z})$*-space* $FS(X)$*;*
(ii) *We have* $\dim \mathcal{U} \leq M$*;*
(iii) *There is* $\epsilon > 0$ *with the following property: for* $c \in FS_{\leq\gamma}(X)$ *such that* $c(t) \in GL_n(\mathbf{Z}) \cdot L$ *for some* $t \in \mathbf{R}$ *there is* $U \in \mathcal{U}$ *such that* $B_\epsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq U$.

Definition **3.7** *([3, Definition 5.5]).* — *Let* G *be a group,* $\mathcal{F}$ *be a family of subgroups of* G*,* (FS, d_{FS}) *be a locally compact metric space with a proper isometric* G*-action and* $\Phi \colon FS \times \mathbf{R} \to FS$ *be a* G*-equivariant flow.*

We say that FS admits long $\mathcal{F}$-covers at infinity and at periodic flow lines *if the following holds:*

There is $N > 0$ *such that for every* $\gamma > 0$ *there is a* G*-invariant collection of open* $\mathcal{F}$*-subsets* $\mathcal{V}$ *of FS and* $\varepsilon > 0$ *satisfying:*

(i) $\dim \mathcal{V} \leq N$*;*
(ii) *there is a compact subset* $K \subseteq FS$ *such that*
   (a) $FS_{\leq\gamma} \cap G \cdot K = \emptyset$*;*
   (b) *for* $z \in FS - G \cdot K$ *there is* $V \in \mathcal{V}$ *such that* $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(z)) \subset V$.

We remark that it is natural to think of this definition as requiring two conditions, the first dealing with everything outside some cocompact subset ("at infinity") and the second dealing with (short) periodic orbits of the flow that meet a given cocompact subset ("at periodic flow lines"). In proving that this condition is satisfied in our situation in the next lemma we deal with these conditions separately. For the first condition we use the sets $Y(W, t)$ introduced earlier; for the second the theorem cited above.

Lemma **3.8.** — *The flow space* $FS(X)$ *admits long* $\mathcal{F}_n$*-covers at infinity and at periodic flow lines.*

Proof. — Fix $\gamma > 0$. Choose $t \geq 1$. Put $\delta := 1$ and $\tau := \gamma$. Let $\beta > 0$ be the number appearing in Lemma 3.5 and let $M \in \mathbf{N}$ be the number appearing in Theorem 3.6. Since by Lemma 2.2 (ii) the complement of $|\mathcal{W}(t+\beta)|$ in X is cocompact, we can find a compact subset L of this complement such that

$$GL_n(\mathbf{Z}) \cdot L = X \setminus \big|\mathcal{W}(t+\beta)\big|.$$

For this compact subset L we obtain a real number $\epsilon > 0$ and a set $\mathcal{U}$ of subsets of $FS(X)$ from Theorem 3.6. We can arrange that $\epsilon \leq 1$.

Consider $\mathcal{V} := \mathcal{U} \cup \mathcal{V}(t)$, where $\mathcal{V}(t)$ is the collection of open subsets defined before Lemma 3.5. We want to show that $\mathcal{V}$ satisfies the conditions appearing in Definition 3.7 with respect to the number $N := M + n - 1$.

Since the covering dimension of $\mathcal{U}$ is less or equal to M by Theorem 3.6 (ii) and the covering dimension of $\mathcal{V}(t)$ is less or equal to $n-2$ by Lemma 3.5 (iii), the covering dimension of $\mathcal{U} \cup \mathcal{V}(t)$ is less or equal to N.

Since $\mathcal{U}$ and $\mathcal{V}(t)$ are $GL_n(\mathbf{Z})$-invariant by Theorem 3.6 and Lemma 3.5 (i), $\mathcal{U} \cup \mathcal{V}(t)$ is $GL_n(\mathbf{Z})$-invariant.

Since each element of $\mathcal{U}$ is an open $\mathcal{VC}$yc-set by Theorem 3.6 (i) and each element of $\mathcal{V}(t)$ is an open $\mathcal{F}_n$-subset by Lemma 3.5 (ii), each element of $\mathcal{U} \cup \mathcal{V}(t)$ is an open $\mathcal{F}_n$-subset, as $\mathcal{VC}\text{yc} \subset \mathcal{F}_n$. Define

$$S := \big\{c \in FS(X) \mid \exists Z \in \mathcal{U} \cup \mathcal{V}(t) \text{ with } \overline{B}_\epsilon\big(\Phi_{[-\gamma,\gamma]}(c)\big) \subseteq Z\big\}.$$

This set S contains $FS(X)_{\leq\gamma} \cup |\mathcal{V}(t+\beta)|$ by the following argument. If $c \in |\mathcal{V}(t+\beta)|$, then $c \in S$ by Lemma 3.5(v). If $c \in FS(X)_{\leq\gamma}$ and $c \notin |\mathcal{V}(t+\beta)|$, then $c \in FS(X)_{\leq\gamma}$ and $c(0) \in GL_n(\mathbf{Z}) \cdot L$ and hence $c \in S$ by Theorem 3.6 (iii).

The subset $S \subseteq FS(X)$ is $GL_n(\mathbf{Z})$-invariant because $\mathcal{U} \cup \mathcal{V}(t)$ is $GL_n(\mathbf{Z})$-invariant.

Next we prove that S is open. Assume that this is not the case. Then there exists $c \in S$ and a sequence $(c_k)_{k\geq 1}$ of elements in $FS(X) - S$ such that $d_{FS(X)}(c, c_k) < 1/k$ holds for $k \geq 1$. Choose $Z \in \mathcal{U} \cup \mathcal{V}(t)$ with $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq Z$. Since $FS(X)$ is proper as metric space by [3, Proposition 1.9] and $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c))$ has bounded diameter, $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c))$ is compact. Hence we can find $\mu > 0$ with $B_{\epsilon+\mu}(\Phi_{[-\gamma,\gamma]}(c)) \subseteq Z$. We conclude from [3, Lemma 1.3] for all $s \in [-\gamma, \gamma]$

$$d_{FS(X)}\big(\Phi_s(c), \Phi_s(c_k)\big) \leq e^{|s|} \cdot d_{FS(X)}(c, c_k) < e^\tau \cdot 1/k.$$

Hence we get for $k \geq 1$

$$B_\epsilon\big(\Phi_{[-\gamma,\gamma]}(c_k)\big) \subseteq B_{\epsilon+e^\tau\cdot 1/k}\big(\Phi_{[-\gamma,\gamma]}(c)\big)$$

Since c_k does not belong to S, we conclude that $B_{\epsilon+e^\tau\cdot 1/k}(\Phi_{[-\gamma,\gamma]}(c))$ is not contained in Z. This implies $e^\tau \cdot 1/k \geq \mu$ for all $k \geq 1$, a contradiction.

The $GL_n(\mathbf{Z})$-set $FS(X) - |\mathcal{V}(t+\beta)|$ is cocompact by Lemma 3.5 (iv). Since S is an open $GL_n(\mathbf{Z})$-subset of $FS(X)$ and contains $|\mathcal{V}(t+\beta)|$, the $GL_n(\mathbf{Z})$-set $FS(X) - S$ is cocompact. Hence we can find a compact subset $K \subseteq FS(X) - S$ satisfying

$$GL_n(\mathbf{Z}) \cdot K = FS(X) - S.$$

Obviously $FS(X)_{\leq\gamma} \cap GL_n(\mathbf{Z}) \cdot K = \emptyset$. □

Proof of Theorem 3.1. — The group $GL_n(\mathbf{Z})$ and the associated flow space $FS(X)$ satisfy [3, Convention 5.1] by the argument of [3, Section 6.2]. Notice that in [3, Convention 5.1] it is *not* required that the action is cocompact. The argument of [3, Section 6.2] showing that there is a constant k_G such that the order of any finite subgroup of G is bounded by k_G uses that the action is cocompact. But such a number exists for $GL_n(\mathbf{Z})$ as well, since $GL_n(\mathbf{Z})$ is virtually torsion free (see [9, Exercise II.3 on p. 41]).

Because of [3, Proposition 5.11] it suffices to show that $FS(X)$ admits long $\mathcal{F}_n$-covers at infinity and at periodic flow lines in the sense of Definition 3.7 and admits contracting transfers in the sense of [3, Definition 5.9]. For the first condition this has been done in Lemma 3.8, while the second condition follows from the argument given in [3, Section 6.4]. □

Proposition **3.9.** — *The* K*-theoretic FJC up to dimension* 1 *holds for* $GL_n(\mathbf{Z})$.

Proof. — We proceed by induction over n. As $GL_1(\mathbf{Z})$ is finite, the initial step of the induction is trivial.

Since $GL_n(\mathbf{Z})$ is transfer reducible over $\mathcal{F}_n$ by Theorem 3.1 it follows from [4, Theorem 1.1] that $GL_n(\mathbf{Z})$ satisfies the K-theoretic FJC up to dimension 1 with respect to $\mathcal{F}_n$. It remains to replace $\mathcal{F}_n$ by the family $\mathcal{VC}$yc. Because of the Transitivity Principle 0.5 it suffices to show that each $H \in \mathcal{F}_n$ satisfies the FJC up to dimension 1 (with respect to $\mathcal{VC}$yc). Combining the induction assumption for $GL_k(\mathbf{Z})$, $k < n$ with well known inheritance properties for direct products, exact sequences of groups and subgroups (see for example [1, Theorem 1.10, Corollary 1.13, Theorem 1.9]) it is easy to reduce the K-theoretic FJC up to dimension 1 for members of $\mathcal{F}_n$ to the class of virtually poly-cyclic groups. Finally, for virtually poly-cyclic groups the FJC holds by [1]. □

4. Strong transfer reducibility of $GL_n(\mathbf{Z})$

In this section we will discuss the modifications needed to extend Proposition 3.9 to higher K-theory. The necessary tools for this extension have been developed by Wegner [24].

Theorem **4.1.** — *The group* $GL_n(\mathbf{Z})$ *is strongly transfer reducible over* $\mathcal{F}_n$ *in the sense of [24, Definition 3.1].*

Wegner proves in [24, Theorem 3.4] that CAT(0)-groups are strongly transfer reducible over $\mathcal{VC}$yc. As $GL_n(\mathbf{Z})$ does not act cocompactly on X, we cannot use Wegner's result directly. However, in combination with Lemma 3.8 his method yields a proof of Theorem 4.1.

Proof of Theorem 4.1. — The only place where Wegner uses cocompactness of the action is when he verifies the assumptions of [3, Theorem 5.7], see [24, Proof of Theorem 3.4].

We know by Lemma 3.8 that $FS(X)$ admits long $\mathcal{F}_n$-covers at infinity and periodic flow lines. Wegner cites [3, Section 6.3] for this assumption. In the cocompact setting the family can even be chosen to be $\mathcal{VC}$yc.

That the assumptions of [3, Convention 5.1], which are used implicitly in [3, Theorem 5.7], are satisfied has already been explained in the proof of Theorem 3.1. □

Theorem **4.2.** — *The* K*-theoretic FJC holds for* $GL_n(\mathbf{Z})$.

Proof. — Theorem 4.1 together with [24, Theorem 1.1] imply that $GL_n(\mathbf{Z})$ satisfies the K-theoretic FJC with respect to the family $\mathcal{F}_n$.

Using the induction from the proof of Proposition 3.9 the family $\mathcal{F}_n$ can be replaced by $\mathcal{VC}$yc. □

5. Wreath products and transfer reducibility

Our main result in this section is the following variation of [4, Theorem 1.1].

Theorem **5.1.** — *Let* $\mathcal{F}$ *be a family of subgroups of the group* G *and let* F *be a finite group. Denote by* $\mathcal{F}^{\wr}$ *the family of subgroups* H *of* $G \wr F$ *that contain a subgroup of finite index that is isomorphic to a subgroup of* $H_1 \times \cdots \times H_n$ *for some* n *and* $H_1, \ldots, H_n \in \mathcal{F}$.

(i) *If* G *is transfer reducible over* $\mathcal{F}$, *then the wreath product* $G \wr F$ *satisfies the* K*-theoretic FJC up to dimension* 1 *with respect to* $\mathcal{F}^{\wr}$ *and the* L*-theoretic FJC with respect to* $\mathcal{F}^{\wr}$;

(ii) *If* G *is strongly transfer reducible over* $\mathcal{F}$, *then* $G \wr F$ *satisfies the* K*-theoretic and* L*-theoretic FJC in all dimensions with respect to* $\mathcal{F}^{\wr}$.

The idea of the proof of this result is very easy. We only need to show that $G \wr F$ is transfer reducible over $\mathcal{F}^{\wr}$ and apply [4, Theorem 1.1]. However, it will be easier to verify a slightly weaker condition for $G \wr F$.

Definition **5.2** *(Homotopy* S*-action; [4, Definition 1.4]).* — *Let* S *be a finite subset of a group* G *(containing the identity element* $e \in G$*). Let* X *be a space.*

(i) *A* homotopy S-action (φ, H) on X *consists of continuous maps* $\varphi_g : X \to X$ *for* $g \in S$ *and homotopies* $H_{g,h} : X \times [0, 1] \to X$ *for* $g, h \in S$ *with* $gh \in S$ *such that* $H_{g,h}(-, 0) = \varphi_g \circ \varphi_h$ *and* $H_{g,h}(-, 1) = \varphi_{gh}$ *holds for* $g, h \in S$ *with* $gh \in S$. *Moreover, it is required that* $H_{e,e}(-, t) = \varphi_e = \mathrm{id}_X$ *for all* $t \in [0, 1]$;

(ii) *For* $g \in S$ *let* $F_g(\varphi, H)$ *be the set of all maps* $X \to X$ *of the form* $x \mapsto H_{r,s}(x, t)$ *where* $t \in [0, 1]$ *and* $r, s \in S$ *with* $rs = g$;

(iii) *Given a subset* $A \subset G \times X$ *let* $S^1_{\varphi,H}(A) \subset G \times X$ *denote the set*

$$\left\{\left(ga^{-1}b, y\right) \mid \exists x \in X,\ a, b \in S,\ f \in F_a(\varphi, H),\ \tilde{f} \in F_b(\varphi, H) \text{ satisfying } (g, x) \in A,\ f(x) = \tilde{f}(y)\right\}.$$

Then define inductively $S^n_{\varphi,H}(A) := S^1_{\varphi,H}(S^{n-1}_{\varphi,H}(A))$*;*

(iv) *Let* (φ, H) *be a homotopy* S*-action on* X *and* $\mathcal{U}$ *be an open cover of* $G \times X$*. We say that* $\mathcal{U}$ *is* S-long with respect to (φ, H) *if for every* $(g, x) \in G \times X$ *there is* $U \in \mathcal{U}$ *containing* $S^{|S|}_{\varphi,H}(g, x)$ *where* $|S|$ *is the cardinality of* S.

We will use the following variant of [4, Definition 1.8].

Definition **5.3** *(Almost transfer reducible). — Let* G *be a group and* $\mathcal{F}$ *be a family of subgroups. We will say that* G *is* almost transfer reducible *over* $\mathcal{F}$ *if there is a number* N *such that for any finite subset* S *of* G *we can find*

(i) *a contractible, compact, controlled* N*-dominated, metric space* X *([3, Definition 0.2]), equipped with a homotopy* S*-action* (φ, H) *and*
(ii) *a* G*-invariant cover* $\mathcal{U}$ *of* $G \times X$ *of dimension at most* N *that is* S*-long. Moreover we require that for all* $U \in \mathcal{U}$ *the subgroup* $G_U := \{g \in G \mid gU = U\}$ *belongs to* $\mathcal{F}$*. (Here we use the* G*-action on* $G\times X$ *given by* $g \cdot (h, x) = (gh, x)$*.)*

The original definition for transfer reducibility requires in addition that gU and U are disjoint if $U \in \mathcal{U}$ and $g \notin G_U$, in other words each U is required to be an $\mathcal{F}$-subset. One can also drop this condition from the notion of "strongly transfer reducible" introduced in [24, Definition 3.1]. A group satisfying this weaker version will be called *almost strongly transfer reducible.*

The result corresponding to [4, Theorem 1.1] (respectively [24, Theorem 1.1]) is as follows.

Proposition **5.4.** *— Let* $\mathcal{F}$ *be a family of subgroups of a group* G *and let* $\mathcal{F}'$ *be the family of subgroups of* G *that contain a member of* $\mathcal{F}$ *as a finite index subgroup.*

(i) *If* G *is almost transfer reducible over* $\mathcal{F}$*, it satisfies the* K*-theoretic FJC up to dimension* 1 *with respect to* $\mathcal{F}'$ *and the* L*-theoretic FJC with respect to* $\mathcal{F}'$*.*
(ii) *If* G *is almost strongly transfer reducible over* $\mathcal{F}$*, it satisfies the* K*-theoretic FJC with respect to* $\mathcal{F}'$ *and the* L*-theoretic FJC with respect to* $\mathcal{F}'$*.*

Proof. — (i) The proof can be copied almost word for word from the proof of [4, Theorem 1.1]. The only difference is that we no longer know that the isotropy groups of the action of G on the geometric realization Σ of the nerve of $\mathcal{U}$ belong to $\mathcal{F}$, and this action may no longer be cell preserving. But it is still simplicial and therefore we can just replace Σ by its barycentric subdivision by the following Lemma 5.5 provided we replace $\mathcal{F}$ by $\mathcal{F}'$. (The precise place where this makes a difference is in the proof of [4, Proposition 3.9].)

(ii) The L-theory part follows from part (i) since almost strongly transfer reducible implies almost transfer reducible.

For the proof of the K-theory part we have to adapt Wegner's proof of [24, Theorem 1.1]. The necessary changes concern [24, Proposition 3.6] and are similar to the changes discussed above, but as Wegner's argument is somewhat differently organized we have to be a little more careful. First we observe that for any fixed $M > 0$ the second assertion in [24, Proposition 3.6] can be strengthened to

$$\text{(5.1)} \qquad Mk \cdot d^1_\Sigma\big(f(g,x), f(h,y)\big) \leq d_{\Psi,S,k,\Lambda}\big((g,x),(h,y)\big) \quad \text{for all}(g,x),(h,y) \in G\times X.$$

To do so we only need to set $n := M \cdot 4Nk$ instead of $4Nk$ in the second line of Wegner's proof; then k can be replaced by $M \cdot k$ in the denominator of the final expression on p.786. (Of course X, Ψ, Λ, Σ and f depend now also on M.) Then we can replace Σ by its barycentric subdivision Σ'. Using Lemma 5.5 we conclude from (**5.1**) with sufficiently large $M > 0$ that

- $d^1_{\Sigma'}(f(g,x), f(h,y)) \leq \frac{1}{k}$ for all $(g,x),(h,y) \in G\times X$ satisfying the inequality $d_{\Psi,S,k,\Lambda}((g,x),(h,y)) \leq k$.

This assertion still guarantees that the maps (f_n) induce a functor as needed on the right hand side of the diagram on p. 789 of [24]. With this change Wegner's argument proves the K-theory part of (ii).

(Alternatively one can strengthen Lemma 5.5 below and check that the l^1-metric under barycentric subdivision changes only up to Lipschitz equivalence. Thus [24, Proposition 3.6] remains in fact true for Σ'.) ☐

Lemma **5.5.** — *Let G be a group, and let $\mathcal{F}$ be the family of subgroups of G. Let $\mathcal{F}'$ be the family of subgroups of G that contain a member of $\mathcal{F}$ as a finite index subgroup. Let Σ be a simplicial complex with a simplicial G-action such that the isotropy group of each vertex is contained in $\mathcal{F}$. Let Σ' be the barycentric subdivision. Denote by d^1_Σ the l^1-metric on Σ and by $d^1_{\Sigma'}$ the l^1-metric on Σ'.*

(i) *The group G acts cell preserving on Σ'. All isotropy groups of Σ' lie in $\mathcal{F}'$. In particular Σ' is a G-CW-complex whose isotropy groups belong to $\mathcal{F}'$;*
(ii) *Given a number $\epsilon' > 0$ and a natural number N, there exists a number $\epsilon > 0$ depending only on ϵ' and N such that the following holds: If $\dim(\Sigma) \leq N$ and $x, y \in \Sigma$ satisfy $d^1_\Sigma(x,y) < \epsilon$, then $d^1_{\Sigma'}(x,y) < \epsilon'$.*

Proof. — The elementary proof of assertion (i) is left to the reader. The proof of assertion (ii) is an obvious variation of the proof of [4, Lemma 9.4 (ii)]. Namely, any simplicial complex can be equipped with the l^1-metric. With those metrics the inclusion of a subcomplex is an isometric embedding. Let $\widehat{\Sigma}$ be the simplicial complex with the same vertices as Σ, such that any finite subset of the vertices spans a simplex. The inclusions $i\colon \Sigma \hookrightarrow \widehat{\Sigma}$ and $i'\colon \Sigma' \hookrightarrow \widehat{\Sigma}'$ are isometric embeddings. The images $i(x)$, $i(y)$ of any two points $x, y \in \Sigma$ are contained in a closed simplex of dimension at most $2N+1$. Thus it suffices to consider the case where Σ is replaced by the standard $(2N+1)$-simplex. A compactness argument gives the result in that case. □

Remark **5.6.** — Often the family $\mathcal{F}$ is closed under finite overgroups. In this case $\mathcal{F} = \mathcal{F}'$ and it is really easier to work with the weaker notion of almost transfer reducible instead of transfer reducible. However, there are situations in which a family that is not closed under finite overgroups is important. For example, in [10] the family of virtually cyclic groups of type I is considered.

Let G be a group and F be finite group. We will think of elements of the F-fold product G^F as functions $g\colon F \to G$. For $a \in F$, $g \in G^F$ we write $l_a(g) \in G^F$ for the function $b \mapsto g(ba)$; this defines a left action of F on G^F and the corresponding semi-direct product is the wreath product $G \wr F$ with multiplication $gag'a' = gl_a(g')aa'$ for $g, g' \in G^F$ and $a, a' \in F$. If G acts on a set X, then we obtain an action of $G \wr F$ on X^F. In formulas this action is given by

$$(g \cdot x)(b) := g(b) \cdot x(b);$$
$$(a \cdot x)(b) := x(ba),$$

and hence

$$(ga \cdot x)(b) = g(b) \cdot x(ba)$$

for $g \in G^F$, $x \in X^F$ and $a, b \in F$. We will sometimes also write $l_a(x)$ for $a \cdot x$. Let now $S \subset G$ and (φ, H) be a homotopy S-action on a space X. Set $S \wr F := \{sa \mid s \in S^F, a \in F\} \subset G \wr F$. Then we obtain a homotopy $S \wr F$-action $(\hat{\varphi}, \widehat{H})$ on X^F. In formulas this is given by

$$\big[\hat{\varphi}_{sa}(x)\big](b) := \varphi_{s(b)}\big(x(ba)\big);$$
$$\big[\widehat{H}_{sa,s'a'}(x,t)\big](b) := H_{s(b),s'(ba)}\big(x\big(baa'\big), t\big),$$

for $t \in [0,1]$, $s, s' \in S^F$, $a, a', b \in F$ with $sas'a' = sl_a(s')aa' \in S \wr F$. Hence $(sl_a(s')(b) = s(b)s'(ba) \in S$ and the right hand side is defined. It is easy to check that if X is a contractible, compact, controlled N-dominated, metric space, then the same is true for X^F provided we replace N by $N \cdot |F|$.

Proof of Theorem 5.1. — Let us postpone the "strong"-case until the end of the proof. Because of Proposition 5.4 it suffices to show that $G \wr F$ is almost transfer reducible over $\mathcal{F}^{\wr}$. Let $\widehat{S}$ be a finite subset of $G \wr F$. By enlarging it we can assume that it has the form $S \wr F$

for some finite subset $S \subset G$. Pick a finite subset $S' \subset G$ such that $S \subset S'$ and $|S'| \geq |S \wr F|$. (If G is finite, then $G \wr F \in \mathcal{F}^{\wr}$ and there is nothing to prove.) As G is transfer reducible and hence in particular almost transfer reducible, there is a number N, (depending only on G, not on $\widehat{S}$ or S) a compact, contractible, controlled N-dominated, metric space X, a homotopy S'-action (φ, H) on X, and a G-invariant S'-long open cover $\mathcal{U}$ of $G \times X$ of dimension at most N such that for all $U \in \mathcal{U}$ we have $G_U \in \mathcal{F}$. As pointed out before, X^F is a compact, contractible, controlled $N \cdot |F|$-dominated, metric space. For $u\colon F \to \mathcal{U}$, let $V^u := \{(g, x) \in G^F \times X^F \mid (g(b), x(b)) \in u(b) \quad \text{for all } b \in F\}$. We obtain an open cover $\mathcal{V} := \{V^u \mid u\colon F \to \mathcal{U}\}$ of $G^F \times X^F$ of dimension at most $(N+1)^{|F|} - 1$.

This cover is invariant for the $G \wr F$-action defined by

$$\begin{aligned}\big[g \cdot (h, x)\big](b) &:= \big(g(b)h(b), x(b)\big);\\ \big[a \cdot (h, x)\big](b) &:= \big(h(ba), x(ba)\big),\end{aligned}$$

for $g, h \in G^F$, $x \in X^F$ and $a, b \in F$. As we have $(G^F)_{V^u} = \prod_{b \in F} G_{u(b)}$, it follows that $(G \wr F)_V \in \mathcal{F}^{\wr}$ for all $V \in \mathcal{V}$. Now we pull back $\mathcal{V}$ to a cover $\widehat{\mathcal{U}} := \{p^{-1}(V) \mid V \in \mathcal{V}\}$ of $G \wr F \times X^F$ along the $G \wr F$-equivariant map $p\colon G \wr F \times X^F \to G^F \times X^F$, $(ga, x) \mapsto (g, a \cdot x)$. Here $G \wr F$ operates on $G \wr F \times X^F$ via left multiplication on the first factor and on $G^F \times X^F$ via the operation defined above.

The definition of the homotopy $S \wr F$-action on X^F gives

$$\widehat{H}_{sa, s'a'}(-, t) = \widehat{H}_{s, l_a(s')}(-, t) \circ l_{aa'} = l_{aa'} \circ \widehat{H}_{l_{a'^{-1}a^{-1}}(s), l_{a'^{-1}}(s')}(-, t)$$

for $s, s' \in S^F$, $a, a' \in F$. For $\bar{s} := s l_a(s')$, $\bar{a} := aa'$ we have $\bar{s}\bar{a} = sas'a'$ and consequently

$$F_{\bar{s}\bar{a}}(\hat{\varphi}, \widehat{H}) = F_{\bar{s}}(\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F}) \circ l_{\bar{a}} = l_{\bar{a}} \circ F_{l_{\bar{a}^{-1}}(\bar{s})}(\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F}). \tag{5.2}$$

Let us insert this into the definition of $(S \wr F)^1_{\hat{\varphi}, \widehat{H}}(hb, x)$ with $h \in G^F$, $b \in F$, $x \in X^F$.

Pick any $(h'b', x') \in (S \wr F)^1_{\hat{\varphi}, \widehat{H}}(hb, x)$ with $h' \in G^F$, $b' \in F$. Then there are elements $s, s' \in S^F$, $a, a' \in F$ and $f \in F_{sa}(\hat{\varphi}, \widehat{H})$, $\tilde{f} \in F_{s'a'}(\hat{\varphi}, \widehat{H})$ such that

$$f(x) = \tilde{f}\big(x'\big), \quad \text{and} \quad h'b' = hb(sa)^{-1}s'a'.$$

Using the first equality in (**5.2**) we find $\bar{f} \in F_s(\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F})$ such that $f = \bar{f} \circ l_a$. Using the second equality in (**5.2**) we find $f' \in F_{l_{ba^{-1}}(s)}(\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F})$ such that $\bar{f} \circ l_{ab^{-1}} = l_{ab^{-1}} \circ f'$. Then $f \circ l_{b^{-1}} = l_{ab^{-1}} \circ f'$; equivalently $l_{ba^{-1}} \circ f = f' \circ l_b$. Similarly, using both equations in (**5.2**) again, we find $\tilde{f}' \in F_{l_{ba^{-1}}(s')}(\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F})$ such that $l_{ba^{-1}} \circ \tilde{f} = \tilde{f}' \circ l_{ba^{-1}a'}$.

We claim that $p(h'b', x')$ belongs to $(S^F)^1_{\hat{\varphi}|_{G^F}, \widehat{H}|_{G^F}}(p(hb, x))$. We have $p(h'b', x') = (h', b' \cdot x')$ and $p(hb, x) = (h, b \cdot x)$. From $h'b' = hb(sa)^{-1}s'a'$ we conclude $h' = h l_{ba^{-1}}(s^{-1}s')$ and $b' = ba^{-1}a'$. Now the equations

$$f'(b \cdot x) = l_{ba^{-1}}\big(f(x)\big) = l_{ba^{-1}}\big(\tilde{f}\big(x'\big)\big) = \tilde{f}'\big(ba^{-1}a' \cdot x'\big) = \tilde{f}'\big(b'x'\big)$$

$$h' = hl_{ba^{-1}}\big(s^{-1}s'\big) = h\big(l_{ba^{-1}}(s)\big)^{-1} l_{ba^{-1}}\big(s'\big)$$

prove our claim.

Summarizing we have shown that for any $A \subset G \wr F \times X^F$ we have

$$p\big((S \wr F)^1_{\hat{\varphi},\widehat{H}}(A)\big) \subset \big(S^F\big)^1_{\hat{\varphi}|_{G^F},\widehat{H}|_{G^F}}\big(p(A)\big).$$

By induction then for all n

$$p\big((S \wr F)^n_{\hat{\varphi},\widehat{H}}(A)\big) \subset \big(S^F\big)^n_{\hat{\varphi}|_{G^F},\widehat{H}|_{G^F}}\big(p(A)\big).$$

Since the S^F-homotopy action on X^F is defined componentwise we have

$$\big(S^F\big)^n_{\hat{\varphi}|_{G^K},\widehat{H}|_{G^K}}(g, x) \subset \prod_{a\in F} S^n_{\varphi,H}\big(g(a), x(a)\big).$$

Recall the definition of S' from the beginning of the proof. Since the cover $\mathcal{U}$ of $G \times X$ is S'-long, there is for each $a \in F$ a $u(a) \in \mathcal{U}$ with $(S')^{|S'|}_{\varphi,H}(h(a), x(ab)) \subset u(a)$. Thus we obtain

$$p\big((S \wr F)^{|S \wr F|}_{\hat{\varphi},\widehat{H}}(hb, x)\big) \subset \big(S^F\big)^{|S\wr F|}_{\hat{\varphi}|_{G^F},\widehat{H}|_{G^F}}(h, b\cdot x) \subset \prod_{a\in F} S^{|S\wr F|}_{\varphi,H}\big(h(a), x(ab)\big)$$
$$\subset \prod_{a\in F} S'^{|S'|}_{\varphi,H}\big(h(a), x(ab)\big) \subset \prod_{a\in F} u(a) = V^u.$$

So $(S \wr F)^{|S\wr F|}_{\hat{\varphi},\widehat{H}}(hb, x) \subset p^{-1}(V^u)$. Hence the cover $\widehat{\mathcal{U}}$ is $S \wr F$-long.

If X is equipped with a strong homotopy action Ψ (see [24, Section 2]), we obtain a strong homotopy action $\widehat{\Psi}$ on X^F. In formulas it is given by

$$\begin{aligned}&\widehat{\Psi}(g_n a_n, t_n, \dots, t_1, g_0 a_0, x)(b)\\ &\quad := \Psi\big(g_n(b), t_n, g_{n-1}(ba_n), t_{n-1}, g_{n-2}(ba_n a_{n-1}), \dots,\\ &\qquad\qquad g_0(ba_n \dots a_1), x(ba_n \dots a_0)\big)\end{aligned}$$

for $g_0, \dots, g_n \in G^F$, $a_0, \dots, a_n, b \in F$, $t_1, \dots, t_n \in [0, 1]$, $x \in X^F$. We also have here

$$\begin{aligned}&\widehat{\Psi}(g_n a_n, t_n, \dots, t_1, g_0 a_0, -)\\ &= \widehat{\Psi}\big(g_n, t_n, l_{a_n}(g_{n-1}), t_{n-1}, l_{a_n a_{n-1}}(g_{n-2}), t_{n-2}, \dots, l_{a_n\dots a_1}(g_0), -\big) \circ l_{a_n\dots a_0}\\ &= l_{a_n\dots a_0} \circ \widehat{\Psi}\big(l_{(a_n\dots a_0)^{-1}} g_n, t_n, l_{(a_{n-1}\dots a_0)^{-1}}(g_{n-1}), t_{n-1}, \dots, l_{a_0^{-1}}(g_0), -\big).\end{aligned}$$

For $\bar{g} := g_n l_{a_n}(g_{n-1}) l_{a_n a_{n-1}}(g_{n-2}) \dots l_{a_n\dots a_1}(g_0) \in G^F$, $\bar{a} := a_n \dots a_0 \in F$, $n \in \mathbf{N}$ we have $\bar{g}\bar{a} = g_n a_n g_{n-1} a_{n-1} \dots g_0 a_0$ and consequently we have analogously to (**5.2**)

$$F_{\bar{g}\bar{a}}(\widehat{\Psi}, S \wr F, n) = F_{\bar{g}}\big(\widehat{\Psi}|_{G^F}, S^F, n\big) \circ l_{\bar{a}} = l_{\bar{a}} \circ F_{l_{\bar{a}^{-1}}(\bar{g})}\big(\widehat{\Psi}|_{G^F}, S^F, n\big).$$

With this observation the proof can be carried out in exactly the same way as in the case of a homotopy action. □

Remark **5.7.** — The proof of Theorem 5.1 given above only uses that G is almost (strongly) transfer reducible over $\mathcal{F}$, not that G is (strongly) transfer reducible. Consequently, Theorem 5.1 remains true if we replace the assumption "(strongly) transfer reducible" by the weaker assumption "almost (strongly) transfer reducible".

6. The Farrell-Jones conjecture with wreath products

Definition **6.1.** — *A group* G *is said to satisfy the* L*-theoretic Farrell-Jones Conjecture* with wreath products *with respect to the family* $\mathcal{F}$ *if for any finite group* F *the wreath product* $G \wr F$ *satisfies the* L*-theoretic Farrell-Jones Conjecture with respect to the family* $\mathcal{F}$ *(in the sense of Definition 0.2).*

If the family $\mathcal{F}$ *is not mentioned, it is by default the family* $\mathcal{VC}yc$ *of virtually cyclic subgroups.*

There are similar versions with wreath products of the K*-theoretic Farrell-Jones Conjecture and the* K*-theoretic Farrell-Jones Conjecture up to dimension* 1.

The FJC with wreath products has first been used in [15] to deal with finite extensions, see also [23, Definition 2.1].

Remark **6.2.** — The inheritance properties of the FJC for direct products, subgroups, exact sequences and directed colimits hold also for the FJC with wreath products and can be deduced from the corresponding properties of the FJC itself. See for example [19, Lemma 3.2, 3.15, 3.16, Satz 3.5].

For a group G and two finite groups F_1 and F_2 we have $(H \wr F_1) \wr F_2 \subset H \wr (F_1 \wr F_2)$ and $F_1 \wr F_2$ is finite. In particular, if G satisfies the FJC with wreath products, then the same is true for any wreath product $G \wr F$ with F finite.

The main advantage of the FJC with wreath products is that in addition it passes to overgroups of finite index. Let G' be an overgroup of G of finite index, i.e., $G \subset G'$, $[G' : G] < \infty$. Let S denote a system of representatives of the cosets G'/G. Then $N := \bigcap_{s \in S} sGs^{-1}$ is a finite index, normal subgroup of G'. Now G' can be embedded in $N \wr G'/N$ (see [11, Section 2.6], [15, Section 2]). This implies that G' satisfies the FJC with wreath products, because $N \wr G'/N$ does by the inheritance properties discussed before.

Theorem **6.3.** — *The* L*-theoretic FJC with wreath products holds for* $GL_n(\mathbf{Z})$.

Proof. — We proceed by induction over n. As $GL_1(\mathbf{Z})$ is finite, the induction beginning is trivial.

Let F be a finite group. Since $GL_n(\mathbf{Z})$ is transfer reducible over $\mathcal{F}_n$ by Theorem 3.1 it follows from Theorem 5.1 (i) that $GL_n(\mathbf{Z}) \wr F$ satisfies the L-theoretic FJC with respect to $(\mathcal{F}_n)^{\wr}$. It remains to replace $(\mathcal{F}_n)^{\wr}$ by the family $\mathcal{VC}yc$. By the Transitivity Principle 0.5 it suffices to prove the L-theoretic FJC (with respect to $\mathcal{VC}yc$) for all groups $H \in (\mathcal{F}_n)^{\wr}$.

Because the FJC with wreath products passes to products and finite index overgroups, see Remark 6.2 it suffices to consider $H \in \mathcal{F}_n$.

Combining the induction assumption for $GL_k(\mathbf{Z})$, $k < n$ with the inheritance properties for direct products, exact sequences of groups and subgroups (see Remark 6.2) it is easy to reduce the FJC with wreath products for members of $\mathcal{F}_n$ to the class of virtually poly-cyclic groups. Wreath products of virtually poly-cyclic groups with finite groups are again virtually poly-cyclic. Thus the result follows in this case from [1]. □

Because for finite F the wreath product $GL_n(\mathbf{Z}) \wr F$ can be embedded into $GL_m(\mathbf{Z})$ for some $m > n$, there is really no difference between the FJC and the FJC with wreath products for the collection of groups $GL_n(\mathbf{Z})$, $n \in \mathbf{N}$. Nevertheless, as discussed in the introduction, for L-theory the induction only works for the FJC with wreath products.

Remark **6.4.** — We also conclude that a hyperbolic group G satisfies the K- and L-theoretic FJC with wreath products. For K-theory (without finite wreath products) this has already been proved in [5]. A hyperbolic group is strongly transfer reducible over $\mathcal{VC}$yc by [24, Example 3.2] and in particular transfer reducible over $\mathcal{VC}$yc. Hence it satisfies the K-theoretic FJC in all dimensions and the L-theoretic FJC with respect to the family $\mathcal{VC}\mathrm{yc}^{\wr}$ by Theorem 5.1 (ii). By the transitivity principle 0.5 it suffices to show the FJC for all groups from $\mathcal{VC}\mathrm{yc}^{\wr}$. Since those groups are virtually polycyclic the FJC holds for them by [1, Theorem 0.1].

Notice that a group is hyperbolic if a subgroup of finite index is hyperbolic. Nevertheless, it is desirable to have the wreath product version for hyperbolic groups also since it inherits to colimits of hyperbolic groups and many constructions of groups with exotic properties occur as colimits of hyperbolic groups.

7. Proof of the general theorem

Lemma **7.1.** — *Let* R *be a ring whose underlying abelian group is finitely generated. Then both the* K*-theoretic and the* L*-theoretic FJC hold for* $GL_n(R)$ *and* $SL_n(R)$.

Proof. — Since the FJC passes to subgroups (by [7]) we only need to treat $GL_n(R)$. Choose an isomorphism of abelian groups $h' \colon R^n \xrightarrow{\cong} \mathbf{Z}^k \times T$ for some natural number k and a finite abelian group T. We obtain an injection of groups

$$GL_n(R) \cong \mathrm{aut}_R\big(R^n\big) \xrightarrow{f} \mathrm{aut}_{\mathbf{Z}}\big(R^n\big) \xrightarrow{h} \mathrm{aut}_{\mathbf{Z}}\big(\mathbf{Z}^k \times T\big),$$

where f is the forgetful map and h comes by conjugation with h'. Since the FJC passes to subgroups, it suffices to prove the FJC for $\mathrm{aut}_{\mathbf{Z}}(\mathbf{Z}^k \times T)$. There is an obvious exact sequence of groups

$$1 \to \hom_{\mathbf{Z}}\big(\mathbf{Z}^k, T\big) \to \mathrm{aut}_{\mathbf{Z}}\big(\mathbf{Z}^k \times T\big) \to GL_k(\mathbf{Z}) \times \mathrm{aut}_{\mathbf{Z}}(T) \to 1.$$

Since $\hom_{\mathbf{Z}}(\mathbf{Z}^k, T)$ and $\mathrm{aut}_{\mathbf{Z}}(T)$ are finite and $GL_k(\mathbf{Z})$ satisfies the FJC, Lemma 7.1 follows from [1, Corollary 1.12]. □

Proof of General Theorem. — Let G be a group which is commensurable to a subgroup $H \subseteq GL_n(R)$ for some natural number n. We have to show that G satisfies the FJC with wreath products. We have explained in Remark 6.2 that the FJC with wreath products passes to overgroups of finite index and all subgroups. Therefore it suffices to show that the FJC with wreath products holds for $GL_n(R)$.

Consider a finite group H. Let $R \wr H$ be the twisted group ring of H with coefficients in $\prod_H R$ where the H-action on $\prod_H R$ is given by permuting the factors. Since R is finitely generated as abelian group by assumption, the same is true for $R \wr H$. Hence $GL_n(R \wr H)$ satisfies the FJC by Lemma 7.1.

There is an obvious injective group homomorphism $GL_n(R) \wr H = GL_n(R)^H \rtimes H \to GL_n(R \wr H)$. It extends the obvious group monomorphism $GL_n(R)^H = GL_n(R^H) \to GL_n(R \wr H)$ via $H \to GL_n(R \wr H)$, $h \mapsto h \cdot I_n$. Since the FJC passes to subgroups it holds for $GL_n(R) \wr H$. □

Acknowledgements

We are grateful to Dan Grayson for fruitful discussions about his paper [16]. We also thank Enrico Leuzinger for answering related questions.

The work was financially supported by SFB 878 *Groups, Geometry and Actions* in Münster, the HCM (Hausdorff Center for Mathematics) in Bonn, and the Leibniz-Preis of the second author. Parts of the paper were developed during the Trimester Program *Rigidity* at the HIM (Hausdorff Research Institute for Mathematics) in Bonn in the fall of 2009.

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Manuscrit reçu le 11 avril 2012
Manuscrit accepté le 6 mai 2013
publié en ligne le 25 mai 2013.