Optimal Investment Decision Under Switching regimes of Subsidy Support

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Abstract

We address the problem of making a managerial decision when the investment project is subsidized, which results in the resolution of an infinite-horizon optimal stopping problem of a switching diffusion driven by either an homogeneous or an inhomogeneous continuous-time Markov chain.

We provide a characterization of the value function (and optimal strategy) of the optimal stopping problem. On the one hand, broadly, we can prove that the value function is the unique viscosity solution to a system of HJB equations. On the other hand, when the Markov chain is homogeneous and the switching diffusion is one-dimensional, we obtain stronger results: the value function is the difference between two convex functions.

Keywords. Optimal stopping, Switching diffusions, Investment Decisions.

1 Introduction

The optimal time to make managerial decisions has been broadly studied in the context of Real Options since the pioneering works of Dixit and Pindyck [1] and Trigeorgis [2]. Over time, while trying to fit the market's necessities, this type of models has become more and more complex from both the economic and the mathematical point of view. From the economic side, the number of sequential decisions studied in these models has increased and, from the mathematical angle, the associated stochastic control problems have become progressively more difficult to solve.

In the past few years, several authors have introduced in real options models the existence of temporary subsidy support schemes in order to study their influence in the optimal investment time. This is particularly important in subsidized fields such as renewable energies, where there is an intense research activity (see, for instance, Boomsma, Meade and Fleten [3]), Boomsma and Linnerud [4], Adkins and Paxson [5], Fleten, Linnerud, Molnár and Nygaard [6], Kitzing, Juul, Drud, Boomsma [7] and Guerra, Kort, Nunes and Oliveira [8]).

Following the previously cited authors, we formulate an investment model in a more general sense, where we assume that: (1) there are various different levels of subsidy, (2) the coefficients of

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the dynamic relative to the economic indicator change with the level of subsidy and (3) the follow-up of the firm's situation is influenced by the time since the previous evaluation. In consequence, we formulate our model as an infinite-horizon optimal stopping problem where the uncertainty is generally modeled by a switching diffusion driven by an inhomogeneous continuous-time Markov chain.

There are a few articles on optimal stopping problems for switching diffusions, covering different topics of financial mathematics. On the one hand, Eloe, Liu, Yatsuki, Yin and Zhang [9], Guo [10], and Guo and Zhang [11] give explicit solutions for a few particular problems; on the other hand, Pemy [12], Pemy and Zhang [13], and Liu [14] show that, in certain conditions, the value function for the correspondent optimal stopping problem is a viscosity solution to a system of Hamilton-Jacobi-Bellman (HJB) equations. Very recently, Egami and Kevkhishvili [15] show that these type of problems can be reduced to a set of optimal stopping problems without a switching regime.

In this work, we show that, in general, the value function is time-dependent and the unique viscosity solution to a system of HJB equations. Additionally, when the continuous-time Markov chain is homogeneous and when the diffusion is one-dimensional, the value function is the difference of two convex functions and the time-dependence is lost.

We organize the text as follows: in Section 2, we describe the stochastic process that we consider; in Section 3, we define the optimal stopping problem and some of the required assumptions; in Section 4 we prove that the value function is the unique viscosity solution to a system of HJB equations and, finally, in Section 5, we discuss the optimal stopping problem in the homogeneous and one-dimensional case.

2 The stochastic process

We consider an investment project enrolled in an assistant program where there are k different levels of subsidy. The process $\theta = \{\theta_s : s \geq 0\}$, which provides the information concerning the level of subsidy at the current moment, is such that

$$\begin{cases} \theta_s \in \Theta \equiv \{1, \ldots, k\}, & \text{for each } s \geq 0, \\ \theta \text{ is a càdlàg process.} \end{cases} \tag{1}$$

To completely characterize the Markov chain θ, we introduce the process $\{\nu_n : n \in \mathbb{N}_0\}$, where ν_n, with $n \in \mathbb{N}$, is the time until the $n^{\text{th}}-$transition of Markov chain θ, defined by

$$\nu_1 = \inf\{s > 0 : \theta_{s^-} \neq \theta_s\} \quad \text{and} \quad \nu_n = \inf\{s > \nu_{n-1} : \theta_{s^-} \neq \theta_s\}.$$

We assume that for every $j, m \in \Theta$

$$\begin{aligned} &P(\nu_n - \nu_{n-1} \leq s \,|\, \theta_{\nu_{n-1}} = j) = 1 - e^{-\int_0^s \lambda_j(u)du}, \quad \text{for all } s \geq 0, \\ &P(\theta_{\nu_n} = m \,|\, \theta_{\nu_{n-1}} = j) = p_{j,m}(\nu_n - \nu_{n-1}), \end{aligned}$$

where $\lambda_j : [0, \infty) \to [0, \infty)$ is continuous and $p_{j,k} : [0, \infty) \to [0, 1]$ is a continuous function such that $\sum_{m=1}^{k} p_{j,m}(s) = 1$ and $p_{j,j}(s) = 0$, for all $s > 0$. Additionally, we consider that, for every $n_1 \neq n_2 \in \mathbb{N}$, the random variables $(\nu_{n_1} - \nu_{n_1-1})$ and $(\nu_{n_2} - \nu_{n_2-1})$ are independent.

The investment project operates in a random environment characterized by an economic indicator, which is modeled by a $n-$dimensional stochastic process $X = \{X_s : s \geq 0\}$. This process solves the switching stochastic differential equation (SDE)

$$dX_s = \alpha(X_s, \theta_s)ds + \sigma(X_s, \theta_s)dW_s, \tag{2}$$

taking values in the open set $D \subseteq \mathbb{R}^n$, where $W = \{W_s, s \geq 0\}$ is an $m-$dimensional Brownian motion independent of θ and where $\alpha : D \to \mathbb{R}^n$ and $\sigma : D \to \mathbb{R}^{n\times m}$ are Borel measurable functions. Therefore, we build this model on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P)$ satisfying the usual conditions and supporting the independent processes θ and W.

The next assumption characterizes the solution of the switching SDE (2). Some results concerning the existence and uniqueness of solutions to switching diffusions may be found in Mao and Yuan [16], and Yin and Zhu [17]. Additionally, in Kallenberg [18], Karatzas and Shreve [19] and Krylov [20], one can find results concerning the existence and uniqueness of SDEs without switching regimes.

Assumption 2.1. *The Borel measurable functions $\alpha : D \times \Theta \to \mathbb{R}^n$ and $\sigma : D \times \Theta \to \mathbb{R}^{n\times m}$ are such that the SDE* (2)*, for each initial condition, has a unique strong solution (W, X) on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P)$ that remains in D for all times. Additionally, we assume that*

$$|\alpha(x, i) - \alpha(y, i)| + \|\sigma(x, i) - \sigma(y, i)\| \leq L|x - y|.$$

For any set $I \subseteq D$ we define the $\mathcal{F}_s-$stopping time

$$T^I \equiv \inf\{s \geq 0 : X_s \in \partial I\} \text{ with } X_0 = x \in I,$$

where ∂I is obtained by considering the topology on D, which is the trace of the usual topology on $\mathbb{R}^n$. If $I = D$, then $\partial I = \emptyset$, since D is open in the usual topology on $\mathbb{R}^n$. In addition, we assume that D is such that $P(T^A < \infty) > 0$, for all open $A \subsetneq D$ and $x \in A$.

The process (X, θ) is not, in general, a Markov process, because it is never known how much time was spent since the last transition in the Markov chain. Therefore, we introduce the process $\zeta = \{\zeta_s,\ s \geq t\}$, which represents the time spent from the last change in the level of subsidy until the moment s, defined by

$$\zeta_s = s - \nu^s,\ s > 0,$$

where $\nu^s \equiv \sup\{\nu_n : \nu_n \leq s, n \in \mathbb{N}\}$ is an $\mathcal{F}_s-$stopping time. Unless otherwise stated, we will work with the process (X, θ, ζ), which is the Markovian representation of the process (X, θ).

3 Optimal stopping problem

In this section, we formulate the stochastic optimization problem that we are interested in. Thus, we consider that the cash-flow associated with the investment project is different in the k different levels of subsidy. Therefore, the running payoff is represented by the function $\Pi : D \times \Theta \to \mathbb{R}$, and the cost of abandonment is represented by $h : D \times \Theta \to \mathbb{R}$. Additionally, we represent the instantaneous interest rate with $r : D \times \Theta \to \mathbb{R}$.

Assumption 3.1. *The functions* $\Pi, h, r : D \times \Theta \to \mathbb{R}$ *are such that*

$$\begin{aligned} &\Pi(\cdot, i), h(\cdot, i), r(\cdot, i) \in C(D), \quad \textit{for all } i \in \Theta \\ &\exists \epsilon_i > 0 \textit{ such that } r(\cdot, i) > \epsilon_i, \quad \textit{for all } i \in \Theta. \end{aligned}$$

If the investment project is permanently abandoned at the moment τ, where τ is a $\mathcal{F}_s-$stopping time, its revenue is given by

$$\begin{aligned} &\int_0^\tau e^{-\rho_s}\Pi(X_s, \theta_s)ds - e^{-\rho_\tau} h(X_\tau, \theta_\tau)1_{\{\tau<\infty\}}, \\ &\rho_s = \int_0^s r(X_u, \theta_u)du, \quad s \geq 0. \end{aligned}$$

Therefore, the expected outcome associated with the project, when the initial observation is $(X_0, \zeta_0, \theta_0) = (x, t, i)$, is given by the functional

$$J(x, t, i, \tau) = E_{x,t,i}\left[\int_0^\tau e^{-\rho_s}\Pi(X_s, \theta_s)ds - e^{-\rho_\tau} h(X_\tau, \theta_\tau)1_{\{\tau<\infty\}}\right]. \tag{3}$$

Here, $E_{x,t,i}[\cdot]$ is the expected value conditional on $X_0 = x$, $\zeta_0 = t$ and $\theta_0 = i$[1]. Our main goal is to seek the $\mathcal{F}_s-$stopping time, τ^*, maximizing the expected outcome (3) in a certain open and connected set I, which should satisfy the following property: the set ∂I is regular for the process X in the sense that,

$$T^I = 0,\ P - \text{almost surely, for all } x \in \partial I.$$

Notice that, in this formulation, the project is necessarily abandoned for $s > T^I$. Therefore, if $\mathcal{T}$ is the set of all $\mathcal{F}_s-$stopping times and $\mathcal{S} = \{\tau \wedge T^I : \tau \in \mathcal{T}\}$, we intend to find the value function V^*, verifying

$$V^*(x, t, i) = \sup_{\tau\in\mathcal{S}} J(x, t, i, \tau), \quad (x, t, i) \in \overline{I} \times [0, \infty) \times \Theta.^2 \tag{4}$$

Since the strategy $\tau \equiv 0$ (to stop immediately, regardless of the current state (X_0, ζ_0, θ_0)) verifies $J(x, t, i, 0) = -h(x, i)$, it is obvious that $V^* \geq -h$. Thus, an optimal stopping time is given by the rule

$$\tau^* = \inf\{s \geq 0 : V^*(X_s, \zeta_s, \theta_s) \leq -h(X_s, \theta_s)\}.$$

In what follows, for every real function f, we set $f^+ = \max(0, f)$, $f^- = \max(0, -f)$. Thus $f = f^+ - f^-$ and $|f| = f^+ + f^-$. The problem's well-posedness is guaranteed by introducing the following integrability conditions:

Assumption 3.2. *The functions* $\Pi, h, r : D \times \Theta \to \mathbb{R}$ *are such that*

$$\begin{aligned} &E_{x,t,i}\left[\int_0^{T^I} e^{-\rho_s}\Pi^+(X_t, \theta_t)ds\right] < \infty \quad \textit{and} \\ &\{h(X_\tau, \theta_\tau)\}_{\tau\in\mathcal{S}} \quad \textit{is a uniformly integrable family of random variables.} \end{aligned}$$

[1] Throughout the paper, we also use the notation $E_{x,i}[\cdot]$ (resp., $E_{t,i}[\cdot]$) representing the expected value conditional on $X_0 = x$ and $\theta_0 = i$ (resp., $\zeta_0 = t$ and $\theta_0 = i$).

[2] From now on, we use the following notation: $\overline{I} = I \cup \partial I$.

For future reference, we notice that according to Assumption 3.2, for any initial condition $(X_0, \zeta_0, \theta_0) = (x, t, i)$,

$$\left\{ \int_0^\tau e^{-\rho s} \Pi^+(X_t, \theta_t) ds - e^{-\rho\tau} h(X_\tau, \theta_\tau) \right\}_{\tau \in \mathcal{S}}$$

is a uniformly integrable family of random variables, meaning that there is a uniform integrability test function $f : [0, \infty) \to [0, \infty)$ (see Definition C.2 and Theorem C.3 in Øksendal [21]) such that

$$\sup_{\tau \in \mathcal{S}} E_{x,t,i} \left[f \left(\left| \int_0^\tau e^{-\rho s} \Pi^+(X_t, \theta_t) ds - e^{-\rho\tau} h(X_\tau, \theta_\tau) \right| \right) \right] < \infty. \tag{5}$$

To finalize this section, in the next proposition we establish that under the assumptions considered in this section, $\{V^*(X_\tau, \zeta_\tau, \theta_\tau)\}_{\tau \in \mathcal{S}}$ is a uniformly integrable family of random variables.

Proposition 3.1. *Let V^* be the value function defined as in (4). Then, $\{V^*(X_\tau, \zeta_\tau, \theta_\tau)\}_{\tau \in \mathcal{S}}$ is a uniformly integrable family of random variables, for every initial condition $(X_0, \zeta_0, \theta_0) = (x, t, i)$.*

Proof. We start by noting that, by definition,

$$\begin{aligned} V^*(x, t, i) &= E_{x,t,i} \left[\int_0^{\tau^*} e^{-\rho s} \Pi(X_s, \theta_s) ds - e^{-\rho\tau^*} h(X_{\tau^*}, \theta_{\tau^*}) \right] \\ &\leq E_{x,t,i} \left[\int_0^{\tau^*} e^{-\rho s} \Pi^+(X_s, \theta_s) ds - e^{-\rho\tau^*} h(X_{\tau^*}, \theta_{\tau^*}) \right]. \end{aligned}$$

Consequently, choosing a function $f : [0, \infty) \to [0, \infty)$ as in (5), which is convex, we get, for any $\tau \in \mathcal{S}$,

$$\begin{aligned} &E_{x,t,i} \left[f \left(|V^*(X_\tau, \zeta_\tau, \theta_\tau)| \right) \right] \leq \\ &\qquad \leq E_{x,t,i} \left[f \left(\left| E_{X_\tau, \zeta_\tau, \theta_\tau} \left[\int_0^{\tau^*} e^{-\rho s} \Pi^+(X_s, \theta_s) ds - e^{-\rho\tau^*} h(X_{\tau^*}, \theta_{\tau^*}) \right] \right| \right) \right] \\ &\qquad \leq E_{x,t,i} \left[f \left(\left| \int_0^{\tau^*} e^{-\rho s} \Pi^+(X_s, \theta_s) ds - e^{-\rho\tau^*} h(X_{\tau^*}, \theta_{\tau^*}) \right| \right) \right] < \infty. \end{aligned} \tag{6}$$

The first inequality in (6) follows from the strong Markov property and the Jensen's inequality, while the second inequality follows from Equation (5). ∎

4 HJB equations

In this section, our main goal is to provide the system of HJB equations associated with the optimal stopping problem (4). Furthermore, we will prove that, under certain conditions, the value function V^* is the unique viscosity solution to this system of HJB equations.

In Section 4.1, we present a weak version of the dynamic programming principle (DPP) that we will use in the following sections. A general formulation of this DPP can be found in Bouchard and Touzi [22].

4.1 Dynamic programming principle

Consider the Markov process (X, θ, ζ), and its infinitesimal generator $\mathcal{L}$, defined by

$$(\mathcal{L}\varphi)(x,t,i) = \lim_{u \downarrow 0} \frac{1}{u} E_{x,t,i}\left[\varphi(X_u, \zeta_u, \theta_u) - \varphi(x,t,i)\right], \tag{7}$$

for all φ in the domain of $\mathcal{L}$. In the next proposition, we present an expression for $\mathcal{L}$.

Proposition 4.1. *Let (X, θ, ζ) be the $(n+1+1)$-dimensional process defined as in Section 2. Then, the infinitesimal generator $\mathcal{L}$ defined in* (7) *is such that*

$$\begin{aligned}
(\mathcal{L}\varphi)(x,t,i) &= \frac{\partial \varphi}{\partial t}(x,i,t) + \alpha(x,i) D\varphi(x,t,i) + \frac{1}{2} Tr\left[\sigma\sigma^T(x,i) D^2\varphi(x,t,i)\right] \\
&\quad + (Q\varphi)(x,t,i) \\
(Q\varphi)(x,t,i) &= \sum_{j \neq i} \lambda_{i,j}(t)\left(\varphi(x,0,j) - \varphi(x,t,i)\right), \quad \text{for a fixed } i \in \Theta,
\end{aligned}$$

where $\lambda_{i,j}(t) = p_{i,j}(t)\lambda_i(t)$, for every $t \geq 0$ and $\varphi : \mathbb{R}^n \times \Theta \times [0,\infty) \to \mathbb{R}$ is such that $\varphi(\cdot,\cdot,i) \in C_0^{2,1}(D \times [0,\infty))^3$, for $i \in \Theta$.

Before we prove Proposition 4.1, we note that the process (X, ζ, θ) is a semimartingale (indeed X is the sum of a martingale and a finite variation process, and ζ and θ are finite variation processes) and, consequently, admits a generalized Itô decomposition (see Theorem II.33 from Protter [23]). Indeed, for any function $\varphi : D \times [0,\infty) \times \Theta \to \mathbb{R}$ such that $\varphi(\cdot,\cdot,i) \in C^{2,1}(D \times [0,\infty))$ and any $u > 0$,

$$\begin{aligned}
\varphi(X_u, \zeta_u, \theta_u) - \varphi(x,t,i) &= \int_0^u \alpha(X_s, \theta_s) \cdot D\varphi(X_s, \zeta_s, \theta_s) + \frac{1}{2} Tr\left[\sigma\sigma^T(X_s, \theta_s) D^2\varphi(X_s, \zeta_s, \theta_s)\right] ds \\
&\quad + \int_0^u \frac{\partial \varphi}{\partial t}(X_s, \zeta_s, \theta_s) ds + \int_0^u D\varphi(X_s, \zeta_s, \theta_s)\sigma(X_s, \theta_s) dW_s \\
&\quad + \sum_{n \in \mathbb{N}} \left(\varphi\left(X_{\nu_n}, 0, \theta_{\nu_n}\right) - \varphi\left(X_{\nu_n^-}, \zeta_{\nu_n^-}, \theta_{\nu_n^-}\right)\right) 1_{\{u \geq \nu_n\}}.
\end{aligned} \tag{8}$$

Proof of Proposition 4.1. Taking into account the Itô formula presented in Equation (8), we get that

$$\begin{aligned}
&E_{x,t,i}\left[\varphi(X_u, \zeta_u, \theta_u) - \varphi(x,t,i)\right] = \\
&\quad = E_{x,t,i}\Bigg[\int_0^u \alpha(X_s, \theta_s) \cdot D\varphi(X_s, \zeta_s, \theta_s) + \frac{1}{2} Tr\left[\sigma\sigma^T(X_s, \theta_s) D^2\varphi(X_s, \zeta_s, \theta_s)\right] ds \\
&\quad + \sum_{n \in \mathbb{N}} \left(\varphi\left(X_{\nu_n}, 0, \theta_{\nu_n}\right) - \varphi\left(X_{\nu_n^-}, \zeta_{\nu_n^-}, \theta_{\nu_n^-}\right)\right) 1_{\{u \geq \nu_n\}}\Bigg].
\end{aligned}$$

[3] A function $\varphi \in C_0^{2,1}(I \times [0,\infty))$ (resp., $\varphi \in C_0^2(I \times [0,\infty))$) if $\varphi \in C^{2,1}(I \times [0,\infty))$ (resp., $\varphi \in C^2(I \times [0,\infty))$) and has compact support.

Furthermore, we note that

$$E_{x,t,i}\left[\sum_{n\in\mathbb{N}}\left(\varphi\left(X_{\nu_n},0,\theta_{\nu_n}\right)-\varphi\left(X_{\nu_n^-},\zeta_{\nu_n^-},\theta_{\nu_n^-}\right)\right)1_{\{u\geq\nu_n\}}\right]= \tag{9}$$

$$=E_{x,t,i}\left[\left(\varphi\left(X_{\nu_1},0,\theta_{\nu_1}\right)-\varphi\left(X_{\nu_1^-},\zeta_{\nu_1^-},\theta_{\nu_1^-}\right)\right)1_{\{u\geq\nu_1\}}\right] \tag{10}$$

$$+E_{x,t,i}\left[\sum_{n>1}E_{x,t,i}\left[\left(\varphi\left(X_{\nu_n},0,\theta_{\nu_n}\right)-\varphi\left(X_{\nu_n^-},\zeta_{\nu_n^-},\theta_{\nu_n^-}\right)\right)1_{\{u\geq\nu_n>\nu_{n-1}\}}\left|(\nu_{n-1},\theta_{\nu_{n-1}})\right.\right]\right]$$

where, for $n>1$,

$$E_{x,t,i}\left[\left(\varphi\left(X_{\nu_n},0,\theta_{\nu_n}\right)-\varphi\left(X_{\nu_n^-},\zeta_{\nu_n^-},\theta_{\nu_n^-}\right)\right)1_{\{u\geq\nu_n>\nu_{n-1}\}}|(\nu_{n-1},\theta_{\nu_{n-1}})\right]= \tag{11}$$
$$=\sum_{m=1}^{k}E_{x,t,i}\left[\left(\varphi\left(X_{\nu_n},0,m\right)-\varphi\left(X_{\nu_n^-},\zeta_{\nu_n^-},\theta_{\nu_{n-1}}\right)\right)p_{\theta_{\nu_{n-1}},m}(\nu_n-\nu_{n-1})1_{\{u\geq\nu_n\}}|(\nu_{n-1},\theta_{\nu_{n-1}})\right].$$

Since, for $n>1$,

$$P\left(\nu_n<s\middle|\left(\nu_{n-1},\theta_{\nu_{n-1}}\right)\right)=\begin{cases}1-e^{-\int_{\nu_{n-1}}^{s}\lambda_{\theta_{\nu_{n-1}}}(\omega-\nu_{n-1})d\omega}, & \text{if } s>\nu_{n-1}\\ 0, & \text{if } s\leq\nu_{n-1}\end{cases}.$$

Equation (11) can be given by

$$E_{x,t,i}\left[\int_{\nu_{n-1}\wedge u}^{u}f_{\theta_{\nu_{n-1}}}(s)\sum_{m=1}^{k}\left(\varphi\left(X_s,0,m\right)-\varphi\left(X_s,s,\theta_{\nu_{n-1}}\right)\right)p_{\theta_{\nu_{n-1}},m}(\nu_n-\nu_{n-1})ds|(\nu_{n-1},\theta_{\nu_{n-1}})\right],$$

where $f_{\theta_{\nu_{n-1}}}(s)=\lambda_{\theta_{\nu_{n-1}}}(s-\nu_{n-1})e^{\int_{\nu_{n-1}}^{s}\lambda_{\theta_{\nu_{n-1}}}(\omega-\nu_{n-1})d\omega}$, for $s>\nu_{n-1}$. Furthermore, as $E[1_{\{\nu_n\geq s\}}|(\nu_{n-1},\theta_{\nu_{n-1}})]=e^{\int_{\nu_{n-1}}^{s}\lambda_{\theta_{\nu_{n-1}}}(\omega-\nu_{n-1})d\omega}$, for $s>\nu_{n-1}$, Equation (11) can also be given by

$$E_{x,t,i}\left[\int_{\nu_{n-1}\wedge u}^{\nu_n\wedge u}\lambda_{\theta_{\nu_{n-1}}}(s-\nu_{n-1})p_{\theta_{\nu_{n-1}},m}(\nu_n-\nu_{n-1})\times\right.$$
$$\left.\times\sum_{m=1}^{k}\left(\varphi\left(X_s,0,m\right)-\varphi\left(X_s,s,\theta_{\nu_{n-1}}\right)\right)ds|(\nu_{n-1},\theta_{\nu_{n-1}})\right]. \tag{12}$$

A similar representation can be found for the expected value in (10), since we have

$$P\left(\nu_1<s\middle|\left(\zeta_0,\theta_0\right)=(t,i)\right)=\begin{cases}1-e^{-\int_0^{s+t}\lambda_i(\omega)d\omega}, & \text{if } s>0\\ 0, & \text{if } s\leq 0\end{cases}. \tag{13}$$

Then, by using the definition of the infinitesimal generator in (7), the result is straightforward. ∎

It can be useful to consider the operator $\tilde{\mathcal{L}}$ given by

$$\begin{aligned}(\tilde{\mathcal{L}}\varphi)(x,t,i) &= \lim_{h\downarrow 0}\frac{1}{h}E_{x,t,i}\left[e^{-\rho_h}\varphi(X_h,\zeta_h,\theta_h)-\varphi(x,t,i)\right]\\ &= -r(x,i)\varphi(x,t,i)+(\mathcal{L}\varphi)(x,t,i).\end{aligned}$$

For future reference, we note that along the same lines of Proposition 4.1, we can prove that the Dynkin's formula, for the $(n+1+1)-$dimensional process (X,ζ,θ), holds true and verifies

$$E_{x,t,i}\left[e^{-\rho_h}\varphi(X_u,\zeta_u,\theta_u)\right]=\varphi(x,t,i)+E_{x,t,i}\left[\int_t^u e^{-\rho_s}(\tilde{\mathcal{L}}\varphi)(X_s,\zeta_s,\theta_s)ds\right].$$

In Proposition 4.2, a weak version of the DPP for the optimal stopping problem (4) is presented. The proof relies on the Markov structure of the process (X,ζ,θ) and we follow the exposition of Guerra [24] (pages 143-167) and Touzi [25]. Before we introduce the DPP, we state an auxiliary result concerning the continuity of the function $(x,t)\to J(x,t,i,\tau)$. If necessary, to highlight the dependence of X_s on the initial condition $X_0=x$ and on the element $\omega\in\Omega$, we will write X_s^x and $X_s^x(\omega)$, respectively.

Lemma 4.1. *The function $(x,t)\to J(x,t,i,\tau)$ is continuous, for every $\tau\in\mathcal{S}$ and $i\in\Theta$.*

Proof. Firstly, by definition of a solution to a switching SDE, the function $s\to X_s^x$ is continuous. Additionally, we prove that the function $x\to X_s^x$ is $P-$almost surely continuous. To do this, we note that

$$|X_s^x-X_s^{x'}|^2\le 3|x-x'|^2+3Ls\int_0^s|X_u^x-X_u^{x'}|^2ds+3\left|\int_0^s\sigma(X_u^x,\theta_u)-\sigma(X_u^{x'},\theta_u)dW_u\right|^2.$$

From the Doob's maximal inequality, we get that, for all $s'>0$

$$E_{t,i}\left[\sup_{0\le s\le s'}\left|\int_0^s\sigma(X_u^x,\theta_u)-\sigma(X_u^{x'},\theta_u)dW_u\right|^2\right]\le 4E_{t,i}\left[\left|\int_0^{s'}\sigma(X_u^x,\theta_u)-\sigma(X_u^{x'},\theta_u)dW_u\right|^2\right].$$

Therefore, by using the Itô isometry,

$$\begin{aligned}E_{t,i}\left[\sup_{0\le s\le s'}|X_s^x-X_s^{x'}|^2\right] &\le 3|x-x'|^2+3Ls'E_{t,i}\left[\int_0^{s'}|X_u^x-X_u^{x'}|^2du\right]\\ &\quad+12LE_{t,i}\left[\int_0^{s'}\left|X_u^x-X_u^{x'}\right|^2du\right]\\ &\le 3|x-x'|^2+(3Ls'+12L)\int_0^{s'}E_{t,i}\left[|X_u^x-X_u^{x'}|^2\right]du\\ &\le 3|x-x'|^2+(3Ls'+12L)\int_0^{s'}E_{t,i}\left[\sup_{0\le s\le u}|X_s^x-X_s^{x'}|^2\right]du.\end{aligned}$$

By the Grönwall's inequality, we get

$$E_{t,i}\left[\sup_{0\le s\le s'}|X_s^x - X_s^{x'}|^2\right] \le 3|x-x'|^2 e^{3Ls'+12L},$$

which proves the first statement after an application of Kolmogorov's continuity criterion.

To show that $(x,t)\to J(x,t,i,\tau)$ is continuous, we note that

$$\begin{aligned}
J(x,t,i,\tau) &= E_{x,t,i}\left[\int_0^\tau e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\infty\}}\right]\\
&= E_{x,t,i}\Bigg[\int_0^{\tau\wedge\nu_1} e^{-\rho_s}\Pi(X_s,i)ds + \int_{\tau\wedge\nu_1}^{\tau} e^{-\rho_s}\Pi(X_s,\theta_s)ds\\
&\qquad - e^{-\rho_\tau}\left(h(X_\tau,i)1_{\{\tau<\nu_1\}} + h(X_\tau,\theta_\tau)1_{\{\nu_1\le\tau<\infty\}}\right)\Bigg]\\
&= \int_0^\infty \lambda_i(u)e^{-\int_0^{u+t}\lambda_i(s)ds}E_{x,i}\Bigg[\Bigg(\int_0^{\tau\wedge u} e^{-\rho_s}\Pi(X_s,i)ds + \int_{\tau\wedge u}^{\tau} e^{-\rho_s}\Pi(X_s,\theta_s)ds\\
&\qquad - e^{-\rho_\tau}\left(h(X_\tau,i)1_{\{\tau<u\}} + h(X_\tau,\theta_\tau)1_{\{u\le\tau<\infty\}}\right)\Bigg)\Bigg|\nu_1=u\Bigg]du,
\end{aligned}$$

where the last equality follows in light of Equation (13) and Fubini's Theorem.

Let $U_N\subset I\times[0,\infty)$ be a compact set, such that $U_N\nearrow \overline{I}\times[0,\infty)$, and fix $\omega\in\Omega$. Due to the continuity of $(s,x)\to X_s^x(\omega)$, the functions $(s,x)\to r(X_s^x(\omega),i)$, $(s,x)\to\Pi(X_s^x(\omega),i)$ and $(s,x)\to h(X_s^x(\omega),i)$ have maximum and minimum on the set U_N, namely

$$\begin{aligned}
r(X_s^x(\omega),i) &\in [\epsilon_i, \tilde{r}_N(\omega,i)],\\
\Pi(X_s^x(\omega),i) &\in [\utilde{\Pi}_N(\omega,i), \tilde{\Pi}_N(\omega,i)],\\
h(X_s^x(\omega),i) &\in [\utilde{h}_N(\omega,i), \tilde{h}_N(\omega,i)].
\end{aligned}$$

Let $(x',t',i)\in I\times[0,\infty)\times\Theta$, then, for a fixed τ, it follows from the Dominated Convergence Theorem that

$$\begin{aligned}
\lim_{(x,t)\to(x',t')} J(x,t,i,\tau) &= \int_0^\infty \lambda_i(u)e^{-\int_0^{u+t'}\lambda_i(s)ds}E_{x',i}\Bigg[\Bigg(\int_0^{\tau\wedge u} e^{-\rho_s}\Pi(X_s(\omega),i)ds\\
&\quad + \int_{\tau\wedge u}^{\tau} e^{-\rho_s}\Pi(X_s(\omega),\theta_s)ds - e^{-\rho_\tau}\Big(h(X_\tau(\omega),i)1_{\{\tau<u\}}\\
&\quad + h(X_\tau(\omega),\theta_\tau)1_{\{u\le\tau<\infty\}}\Big)\Bigg)\Bigg|\nu_1=u\Bigg]du = J(x',t',i,\tau_N^{x',t',i}).
\end{aligned}$$

Let $\tau^{x',t',i}$ and τ_{U_N} be $\mathcal{F}_s-$stopping times, assuming that $(X_0,\zeta_0,\theta_0)=(x',t',i)$. Then, if $\tau_{U_N}=\inf\{s\ge 0:(s,X_s)\notin U_N\}$, and $\tau_N^{x',t',i}=\tau^{x',t',i}\wedge\tau_{U_N}$, The result holds true if

$$\lim_{U_N\nearrow\overline{I}\times[0,\infty)} J\left(x',t',i,\tau_N^{x',t',i}\right) = J\left(x',t',i,\tau^{x',t',i}\right). \tag{14}$$

To prove Equation (14), we fix $\tau \in \mathcal{S}$ and we notice that, as $U_N \nearrow I \times [0,\infty)$,

$$0 \leq \int_t^{\tau\wedge\tau_{U_N}} e^{-\rho_s}\Pi^{\pm}(X_s,\theta_s)ds \nearrow \int_t^{\tau\wedge T^I} e^{-\rho_s}\Pi^{\pm}(X_s,\theta_s)ds.$$

Then, by utilizing the Monotone Convergence Theorem, it follows that

$$\lim_{U_N\nearrow \overline{I}\times[0,\infty)} E_{x',t',i}\left[\int_0^{\tau\wedge\tau_{U_N}} e^{-\rho_s}\Pi(X_s,\theta_s)ds\right] = E_{x',t',i}\left[\int_0^{\tau\wedge T^I} e^{-\rho_s}\Pi(X_s,\theta_s)ds\right]. \tag{15}$$

Furthermore, since $\{h(X_\tau,\theta_\tau)\}_{\tau\in\mathcal{S}}$ is a uniformly integrable family of random variables and $e^{-\rho_{\tau\wedge\tau_{U_N}}}h(X_{\tau\wedge\tau_{U_N}},\theta_{\tau\wedge\tau_{U_N}}) \to e^{-\rho_{\tau\wedge T^I}}h(X_{\tau\wedge T^I},\theta_{\tau\wedge T^I})$, $P-$almost surely, then,

$$\lim_{U_N\nearrow \overline{I}\times[0,\infty)} E_{x',t',i}\left[e^{-\rho_{\tau\wedge\tau_{U_N}}}h(X_{\tau\wedge\tau_{U_N}},\theta_{\tau\wedge\tau_{U_N}})\right] = E_{x',t',i}\left[e^{-\rho_{\tau\wedge T^I}}h(X_{\tau\wedge T^I},\theta_{\tau\wedge T^I})\right]. \tag{16}$$

Since $\tau \in \mathcal{S}$ is arbitrary, Equation (14) holds true and, thus, we finish the proof. ∎

To prove the DPP, we will introduce the following concept:

Definition 4.1. *Given the initial condition $(X_0,\zeta_0,\theta_0) = (x,t,i)$, the $\mathcal{F}_s-$stopping time $\tau_\epsilon^{x,t,i}$ is an $\epsilon-$optimal strategy if*

$$J(x,t,i,\tau_\epsilon^{x,t,i}) \geq V^*(x,t,i) - \epsilon, \text{ for some } \epsilon \geq 0.$$

We note that, for each $(x,t,i) \in I \times [0,\infty) \times \Theta$, an $\epsilon-$optimal strategy always exists in light of the definition of the value function and Assumption 3.2.

Henceforward, we denote the lower and upper semicontinuous envelopes of a locally bounded function $\varphi : I \times [0,\infty) \times \Theta \to \mathbb{R}$, with respect to the variables x and t, by:

$$\underline{\varphi}(x,t,i) \equiv \liminf_{(y,s)\to(x,t)} \varphi(y,s,i),$$
$$\overline{\varphi}(x,t,i) \equiv \limsup_{(y,s)\to(x,t)} \varphi(y,s,i).$$

Proposition 4.2. *Let $(x,t,i) \in I \times [0,\infty) \times \Theta$ and $\delta \in \mathcal{S}$ be such that $\delta < \infty$. Then*

$$V^*(x,t,i) \leq \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau\wedge\delta}}\left(h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + \overline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\tau\geq\delta\}}\right)\right],$$

and

$$V^*(x,t,i) \geq \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau\wedge\delta}}\left(h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + \underline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\tau\geq\delta\}}\right)\right].$$

Proof. The first inequality can be easily obtained, since for any $\delta \in \mathcal{S}$, such that $\delta < \infty$

$$\int_0^{\tau} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\infty\}} = \int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}}$$
$$+ e^{-\rho_\delta}\left(\int_\delta^{\tau} e^{-(\rho_s-\rho_\delta)}\Pi(X_s,\theta_s)ds - e^{-(\rho_\tau-\rho_\delta)}h(X_\tau,\theta_\tau)1_{\{\tau<\infty\}}\right)1_{\{\delta\leq\tau\}}.$$

Due to the strong Markov property, it follows that

$$J(x,t,i,\tau) \leq E_{x,t,i}\left[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau\wedge\delta}}\left(h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + J(X_\delta,\zeta_\delta,\theta_\delta,\tau)1_{\{\tau\geq\delta\}}\right)\right].$$

Since $J(X_\delta,\zeta_\delta,\theta_\delta,\tau) \leq V^*(X_\delta,\zeta_\delta,\theta_\delta) \leq \overline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)$, the result follows by applying the supremum over $\tau \in \mathcal{S}$ to the previous inequality.

To prove the second inequality, we fix $i \in \Theta$. Additionally, we note that $V^*(x,t,i) \geq -h(x,i)$, for every $(x,t) \in I \times [0,\infty)$, which is also true for $\underline{V}^*$ ($\underline{V}^*(x,t,i) \geq -h(x,i)$), due to the continuity of the function $h(\cdot,i)$. Consequently, from Assumption 3.2, there is a bounded continuous function $\varphi(\cdot,\cdot,i) : I \times [0,\infty) \to \mathbb{R}$ such that $\underline{V}^*(x,t,i) \geq \varphi(x,t,i)$.

Fix $(x,t) \in I \times [0,\infty)$ and let $\tau_\epsilon^{x,t,i}$ be an $\epsilon-$optimal strategy, for some $\epsilon > 0$, as defined in Definition 4.1. Taking into account Lemma 4.1 and the continuity of $\varphi(\cdot,\cdot,i)$, there is a sequence $\{\gamma_{(x,t,i)}\}_{(x,t)\in I\times[0,\infty)} \subset]0,\infty[$, such that, for every $(x',t') \in B_{\gamma_{(x,t,i)}}(x,t)$,

$$J(x',t',i,\tau_\epsilon^{x,t,i}) - J(x,t,i,\tau_\epsilon^{x,t,i}) > -\epsilon \quad \text{and} \quad \varphi(x',t',i) - \varphi(x,t,i) < \epsilon.$$

Naturally, $\{B_{\gamma_{(x,t,i)}}(x,t) : (x,t) \in I \times [0,\infty)\}$ is an open cover of $I \times [0,\infty)$, and, therefore, in light of the Lindelöf's Covering Theorem, there is a sequence $\{(x_j,t_j)\}_{j\in\mathbb{N}} \subset I \times [0,\infty)$, such that $\{B_{\gamma_{(x_j,t_j,i)}}(x_j,t_j)\}_{j\in\mathbb{N}}$ forms an open subcover of $I \times [0,\infty)$. Therefore, for $(x,t) \in B_{\gamma_{(x_j,t_j,i)}}(x_j,t_j)$, with $j \in \mathbb{N}$, we have

$$J(x,t,i,\tau_\epsilon^{x,t,i}) > J(x_j,t_j,i,\tau_\epsilon^{x_j,t_j,i}) - \epsilon \geq V^*(x_j,t_j,i) - 2\epsilon \geq \varphi(x_j,t_j,i) - 2\epsilon > \varphi(x,t,i) - 3\epsilon.$$

From the previous arguments, it is clear that $\{A_j\}_{j\in\mathbb{N}}$, with

$$A_j = B_{\gamma_{(x_j,t_j,i)}}(x_j,t_j) \setminus \bigcup_{n=1}^{j-1} B_{\gamma_{(x_n,t_n,i)}}(x_n,t_n),$$

is also a finite subcover of $I \times [0,\infty)$, verifying $A_j \cap A_n = \emptyset$, with $j \neq n$. Consider $\delta, \tau \in \mathcal{S}$ with $\delta < \infty$, then one can build the strategy

$$\overline{\tau} = \tau 1_{\{\tau<\delta\}} + \tau_\epsilon 1_{\{\tau\geq\delta\}}, \text{ with } \tau_\epsilon = \sum_{i=1}^k \left(\sum_{j=1}^n \tau_\epsilon^{x_j,t_j,i} 1_{\{(X_\delta,\zeta_\delta)\in A_j\}}\right) 1_{\{\theta_\delta=i\}} + \delta 1_{\{(X_\delta,\zeta_\delta)\notin\cup_{j=1}^n A_j\}},$$

which trivially belongs to $\mathcal{S}$. Taking into account the decomposition

$$\int_0^{\overline{\tau}} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_{\tau_\epsilon}}h(X_{\overline{\tau}},\theta_{\overline{\tau}})1_{\{\overline{\tau}<\infty\}} = \int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}}$$
$$+ e^{-\rho_\delta}\left(\int_\delta^{\tau_\epsilon} e^{-(\rho_s-\rho_\delta)}\Pi(X_s,\theta_s)ds - e^{-(\rho_{\tau_\epsilon}-\rho_\delta)}h(X_{\tau_\epsilon},\theta_{\tau_\epsilon})1_{\{\tau_\epsilon<\infty\}}\right)1_{\{\delta\leq\tau\}},$$

we have

$$V^*(x,t,i) \geq J(x,t,i,\overline{\tau}) \geq E_{x,t,i}\left[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}}\right] +$$
$$+ E_{x,t,i}\left[\left(e^{-\rho_\delta}\left(\varphi(X_\delta,\zeta_\delta,\theta_\delta) - 3\epsilon\right)1_{(X_\delta,\zeta_\delta)\in\cup_{j=1}^n A_j} - e^{-\rho_\delta}h(X_\delta,\theta_\delta)1_{(X_\delta,\zeta_\delta)\notin\cup_{j=1}^n A_j}\right)1_{\{\delta\leq\tau\}}\right].$$

Since $\varphi(\cdot,\cdot,i)$ is a bounded continuous function, for every $i\in\Theta$, and $\{h(X_\delta,\theta_\delta)\}_{\{\delta\in\mathcal{S}\}}$ is a uniformly integrable family of random variables, from the Dominated Convergence Theorem we get

$$\begin{aligned}V^*(x,t,i) &\geq E_{x,t,i}\Bigg[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}}+\\ &\qquad\qquad + e^{-\rho_\delta}\left(\varphi(X_\delta,\zeta_\delta,\theta_\delta)-3\epsilon\right)1_{\{\delta\leq\tau\}}\Bigg]\\ &\geq E_{x,t,i}\left[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + e^{-\rho_\delta}\varphi(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\delta\leq\tau\}}\right]-3\epsilon.\end{aligned}$$

Now, we pick a monotonically increasing sequence of bounded continuous functions $\{\varphi_n\}_{n\in\mathbb{N}}$, such that $\varphi_n(x,t,i)\to\underline{V}^*(x,t,i)$ as $n\to\infty$ (this sequence exists in light of Urysohn's Lemma) and, consequently, from the Monotones Convergence Theorem we get

$$\lim_{n\to\infty} E_{x,t,i}\left[e^{-\rho_\delta}\varphi_n(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\delta\leq\tau\}}\right] = E_{x,t,i}\left[e^{-\rho_\delta}\underline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\delta\leq\tau\}}\right].$$

As ϵ is arbitrary, we obtain the result. ∎

4.2 Viscosity solutions

Assuming that V^* is sufficiently regular, it is not difficult to show that V^*, in the classical sense, satisfies the system of HJB equations

$$\left\{\begin{array}{l} F_i(x,t,\{v(x,t,j):j\in\Theta\},\partial_t v(x,t,i),Dv(x,t,i),D^2v(x,t,i))=0\\ (x,t,i)\in I\times\Theta\times(0,\infty)\end{array}\right., \tag{17}$$

where

$$\begin{aligned}&F_i(x,t,\{v(x,t,j):j\in\Theta\},\partial_t v(x,t,i),Dv(x,t,i),D^2v(x,t,i))\equiv\\ &\qquad\qquad\equiv\min\Big\{-(\tilde{\mathcal{L}}v)(x,t,i)-\Pi(x,i),\ v(x,t,i)+h(x,i)\Big\},\end{aligned}$$

and $\partial_t v$, Dv and D^2v are, respectively, the first derivative of v in t, the vector of first derivatives of v in x and the matrix of second derivatives of v in x. Furthermore, the following boundary condition must also be satisfied

$$v(x,t,i)=-h(x,i),\quad \text{for all } x\in\partial I. \tag{18}$$

One can note that this boundary condition is trivially satisfied when $I=D$ since, in this case, $\partial I=\emptyset$. Throughout this section, we will prove that the value function, V^*, is a viscosity solution to the system of coupled HJB equations (17) and the boundary condition (18).

Before we state the main result of this section, we introduce the definition of viscosity solutions for systems of variational inequalities, following the work of Ishii and Koike [26] (see also Crandall, Ishii and Lions [27]).

Definition 4.2. *Consider a locally bounded function $v : I \times (0,\infty) \times \Theta \to \mathbb{R}$. Then, v is a*

(a) viscosity subsolution to (17) *if whenever $\psi \in C^2(I \times [0,\infty))$, $i \in \Theta$ and $\overline{v}(\cdot,\cdot,i) - \psi(\cdot,\cdot)$ has a local maximum at $(x,t) \in I \times [0,\infty)$, such that $\overline{v}(x,t,i) = \psi(x,t)$, then*

$$F_i(x,t,\overline{v}(x,t,i), D\psi(x,t), D^2\psi(x,t); \{\overline{v}(x,t,j) : j \in \Theta,\, j \neq i\}) \leq 0.$$

(b) viscosity supersolution to (17) *if whenever $\psi \in C^2(I \times [0,\infty))$, $i \in \Theta$ and $\underline{v}(\cdot,\cdot,i) - \psi(\cdot,\cdot)$ has a local minimum at $(x,t) \in I \times [0,\infty)$, such that $\underline{v}(x,t,i) = \psi(x,t)$, then*

$$F_i(x,t,\underline{v}(x,t,i), D\psi(x,t), D^2\psi(x,t); \{\underline{v}(x,t,j) : j \in \Theta,\, j \neq i\}) \geq 0.$$

(c) viscosity solution to (17) *if it is simultaneously a viscosity subsolution and a viscosity supersolution to* (17).

We note that in Definition 4.2, without loss of generality, we can consider functions $\psi \in C_0^2(I \times [0,\infty))$, instead of $\psi \in C^2(I \times [0,\infty))$. Additionally, in the previous definition the term "local" can be replaced by either "strict local" or "global".

Proposition 4.3. *Let V^* be the value function defined as in* (4). *Then V^* is a viscosity solution to the system of equations* (17) *and the boundary condition* (18) *is satisfied.*

Proof. In light of to Assumption 3.2, the function V^* is locally bounded. Therefore, we will proceed through the viscosity supersolution and subsolution properties and the boundary condition.

Supersolution property: To see that V^* is a viscosity supersolution to the system of equations (17), we fix $i \in \Theta$, and let $(\overline{x},\overline{t}) \in I \times [0,\infty)$ and $\psi \in C_0^2(I \times [0,\infty))$ be such that $(\overline{x},\overline{t})$ is a local minimizer of $\underline{V}^*(\cdot,\cdot,i) - \psi(\cdot,\cdot)$ and $\underline{V}^*(\overline{x},\overline{t},i) - \psi(\overline{x},\overline{t}) = 0$.

Fix a sufficiently small $\epsilon > 0$ and let $B_\epsilon(\overline{x},\overline{t})$ be a ball centered in $(\overline{x},\overline{t})$ with radius ϵ. Furthermore, let $\{(x_n,t_n)\}_{n\in\mathbb{N}} \subset B_\epsilon(\overline{x},\overline{t})$ be such that

$$(x_n, t_n, \underline{V}^*(x_n,t_n,i)) \to \left(\overline{x}, \overline{t}, V^*(\overline{x},\overline{t},i)\right),$$

as $n \to \infty$. Naturally, such a sequence exists in light of the definition of $\underline{V}^*$. Throughout the proof, we are interested in the process $(X_s^n, \zeta_s^n, \theta_s)$ which represents (X_s,ζ_s,θ_s) when $(X_0,\zeta_0,\theta_0) = (x_n,t_n,i)$. Whenever there is no risk of misunderstanding, we will simple write (X_s,ζ_s,θ_s).

If $\{\eta_n\}_{n\in\mathbb{N}}$ is such that $\eta_n \to 0$ and

$$\tau_{n,\epsilon} \equiv \inf\{s > 0 : |X_s^n - x_n| \geq \epsilon,\, \zeta_s^n \geq t_n + \sqrt{\eta_n}\} \wedge \inf\{s > 0 : \theta_s - \theta_{s^-} \neq 0\},$$

for some $\epsilon > 0$, then, it follows from the DPP that

$$0 \geq E_{x_n,t_n,i}\left[\int_0^{\tau_{n,\epsilon}} e^{-\rho s}\Pi(X_s,\theta_s)ds + e^{-\rho\tau_{n,\epsilon}}\underline{V}^*(X_{\tau_{n,\epsilon}}, \zeta_{\tau_{n,\epsilon}}, \theta_{\tau_{n,\epsilon}})\right] - V^*(x_n.t_n,i).$$

Now, consider the auxiliary function Ψ given by

$$\Psi(x,t,j) = \begin{cases} \psi(x,t), & \text{if } j = i \\ \underline{V}^*(\overline{x},\overline{t},j), & \text{if } j \neq i \end{cases}.$$

From the Dynkin's formula we get that

$$
\begin{aligned}
E_{x_n,t_n,i}[e^{-\rho\tau_{n,\epsilon}}\Psi(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}},\theta_{\tau_{n,\epsilon}})] &= E_{x_n,t_n,i}[e^{-\rho\tau_{n,\epsilon}}\Psi(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}},i)] \\
&= \psi(x_n,t_n) + E_{x_n,t_n,i}\left[\int_0^{\tau_{n,\epsilon}} e^{-\rho_s}(\tilde{\mathcal{L}}\Psi)(X_s,\zeta_s,i)ds\right],
\end{aligned}
$$

where

$$
\begin{aligned}
(\tilde{\mathcal{L}}\Psi)(\overline{x},\overline{t},i) = & \frac{\partial\varphi}{\partial t}(\overline{x},\overline{t}) - r(\overline{x},i)\underline{V}^*(\overline{x},\overline{t},i) + \alpha(\overline{x},i)\cdot D\varphi(\overline{x},\overline{t}) + \frac{1}{2}Tr\left[\sigma\sigma^T(\overline{x},i)D^2\varphi(\overline{x},\overline{t})\right] \\
& + \sum_{j\neq i}\lambda_{i,j}(\overline{t})\left(\underline{V}^*(\overline{x},\overline{t},j) - \underline{V}^*(\overline{x},\overline{t},i)\right).
\end{aligned}
$$

Since there are $\epsilon > 0$ such that $\underline{V}^*(x,t,i) \geq \psi(x,t) = \Psi(x,t,i)$, in $B_\epsilon(\overline{x},\overline{t})$, we can choose n,ϵ such that

$$
V^*(x_n,t_n,i) \geq E_{x_n,t_n,i}\left[\int_0^{\tau_{n,\epsilon}} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho\tilde{\tau}_{n,\epsilon}}\Psi(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}},\theta_{\tau_{n,\epsilon}})\right].
$$

Therefore, fixing $\eta_n \equiv V^*(x_n,t_n,i) - \Psi(x_n,t_n,i) \to 0$ as $n \to \infty$, we obtain

$$
\begin{aligned}
\eta_n &\geq E_{x_n,t_n,i}\left[\int_0^{\tau_{n,\epsilon,h}} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho\tilde{\tau}_{n,\epsilon}}\Psi(X_{\tau_{n,\epsilon,h}},\zeta_{\tau_{n,\epsilon,h}},\theta_{\tau_{n,\epsilon,h}})\right] - \Psi(x_n,t_n,i) \\
&= E_{x_n,t_n,i}\left[\int_0^{\tau_{n,\epsilon,h}} e^{-\rho_s}\left(\Pi(X_s,i) + (\tilde{\mathcal{L}}\Psi)(X_s,\zeta_s,i)\right)ds\right].
\end{aligned}
$$

Assuming, without loss of generality, that $\eta_n \to 0$ but $\eta_n \neq 0$, we have

$$
\sqrt{\eta_n} \geq E_{x_n,t_n,i}\left[\frac{1}{\sqrt{\eta_n}}\int_0^{\tau_{n,\epsilon}} e^{-\rho_s}\left(\Pi(X_s,i) + (\tilde{\mathcal{L}}\Psi)(X_s,\zeta_s,i)\right)ds\right].
$$

By letting $n \to \infty$, we obtain

$$
0 \geq -(\tilde{\mathcal{L}}v)(\overline{x},\overline{t},i) - \Pi(\overline{x},i), \quad \text{for } i \in \Theta.
$$

To finish this part of the proof, we note that $V^*(\overline{x},\overline{t},i) \geq J(\overline{x},\overline{t},i,0) = -h(\overline{x},i) \Rightarrow \underline{V}^*(\overline{x},\overline{t},i) \geq -h(\overline{x},i)$, because h is continuous.

Subsolution property: To prove that V^* is a viscosity subsolution to the system of HJB equations (17), we argue by contradiction.

Fix $i \in \Theta$, and let $(\overline{x},\overline{t}) \in I \times [0,\infty)$ and $\psi \in C_0^2(I \times [0,\infty))$ be such that $(\overline{x},\overline{t})$ is a strict maximizer of $\overline{V}^*(\cdot,\cdot,i) - \psi(\cdot,\cdot)$ and $\overline{V}^*(\overline{x},\overline{t},i) - \psi(\overline{x},\overline{t}) = 0$. To get a contradiction, we also assume that

$$
F_i(\overline{x},\overline{t},\{\overline{V}^*(\overline{x},\overline{t},j) : j \in \Theta\}, \partial_t\psi(\overline{x},\overline{t},i), D\psi(\overline{x},\overline{t},i), D^2\psi(\overline{x},\overline{t},i)) > 0. \tag{19}
$$

Taking into account that $h(\cdot,i)$ is continuous, for all $i \in \Theta$, there is $\epsilon > 0$ such that

$$
\psi(\overline{x},\overline{t}) + h(\overline{x},i) \geq \epsilon \quad \text{and} \quad -(\tilde{\mathcal{L}}\psi)(\overline{x},\overline{t},i) - \Pi(\overline{x},i) \geq 0, \quad \text{for all } (\overline{x},\overline{t}) \in \mathcal{V}_\epsilon(\overline{x},\overline{t}), \tag{20}
$$

where, $\mathcal{V}_\epsilon(\overline{x},\overline{t})$ is a neighborhood of $(\overline{x},\overline{t})$ of the form $\mathcal{V}_\epsilon(\overline{x},\overline{t}) = B_\epsilon(\overline{x}) \times [\overline{t},\overline{t}+\epsilon[$, and B_ϵ is a ball centered in $\overline{x}$ with radius ϵ. Additionally, since $(\overline{x},\overline{t})$ is a strict maximizer, there is $\delta < 0$ such that

$$\max_{(x,t)\in\partial\mathcal{V}_\epsilon(\overline{x},\overline{t})} \left(\overline{V}^*(x,t,i) - \psi(x,t)\right) = \delta. \tag{21}$$

In light of the definition of $\overline{V}^*$, there is $\{(x_n,t_n)\}_{n\in\mathbb{N}} \subset I \times [0,\infty)$, such that

$$\left(x_n, t_n, \overline{V}^*(x_n,t_n,i)\right) \to \left(\overline{x},\overline{t}, V^*(\overline{x},\overline{t},i)\right).$$

Now, we define the stopping time $\tau_{n,\epsilon} \equiv \inf\{s \geq 0 : (X_s^n, \zeta_s^n) \notin \mathcal{V}_\epsilon(\overline{x},\overline{t})\}$ and the function

$$\Psi(x,t,j) = \begin{cases} \psi(x,t), & j = i \\ \overline{V}^*(\overline{x},\overline{t},j), & j \neq i \end{cases}.$$

If $\eta_n = V^*(\overline{x},\overline{t},i) - \psi(\overline{x},\overline{t})$, from the Dynkin's formula, it follows that

$$\begin{aligned} V^*(\overline{x},\overline{t},i) &= \eta_n + \psi(\overline{x},\overline{t}) = \eta_n + \Psi(\overline{x},\overline{t},i) \\ &= \eta_n + E\left[e^{-\rho_{\tau\wedge\tau_{n,\epsilon}}}\Psi(X_{\tau\wedge\tau_{n,\epsilon}}, \zeta_{\tau\wedge\tau_{n,\epsilon}}, \theta_{\tau\wedge\tau_{n,\epsilon}}) - \int_0^{\tau\wedge\tau_{n,\epsilon}} e^{-\rho_s}(\tilde{\mathcal{L}}\Psi)(X_s,\zeta_s,\theta_s)ds\right] \\ &\geq \eta_n + E\left[e^{-\rho_{\tau\wedge\tau_{n,\epsilon}}}\Psi(X_{\tau\wedge\tau_{n,\epsilon}}, \zeta_{\tau\wedge\tau_{n,\epsilon}}, \theta_{\tau\wedge\tau_{n,\epsilon}}) + \int_0^{\tau\wedge\tau_{n,\epsilon}} e^{-\rho_s}\Pi(X_s,\theta_s)ds\right]. \end{aligned} \tag{22}$$

The inequality follows in light of the right-hand side of (20).

Choosing $\epsilon > 0$ such that $\tau_{n,\epsilon} < \inf\{s \geq 0 : \theta_s \neq i\}$, from the left-hand side of (20), we get that

$$\begin{aligned} &E\left[e^{-\rho_{\tau\wedge\tau_{n,\epsilon}}}\Psi(X_{\tau\wedge\tau_{n,\epsilon}}, \zeta_{\tau\wedge\tau_{n,\epsilon}}, \theta_{\tau\wedge\tau_{n,\epsilon}})\right] = \\ &= E\left[e^{-\rho_\tau}\psi(X_\tau,\zeta_\tau)1_{\{\tau<\tau_{n,\epsilon}\}}\right] + E\left[e^{-\rho_{\tau_{n,\epsilon}}}\psi(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}})1_{\{\tau\geq\tau_{n,\epsilon}\}}\right] \\ &= E\left[e^{-\rho_\tau}(-h(X_\tau,\theta_\tau)+\epsilon)1_{\{\tau<\tau_{n,\epsilon}\}} + e^{-\rho_{\tau_{n,\epsilon}}}\left(\overline{V}^*(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}},i) - \delta\right)1_{\{\tau\geq\tau_{n,\epsilon}\}}\right] \\ &\geq E\left[-e^{-\rho_\tau}h(X_\tau,\theta_\tau)1_{\{\tau<\tau_{n,\epsilon}\}} + e^{-\rho_{\tau_{n,\epsilon}}}\overline{V}^*(X_{\tau_{n,\epsilon}},\zeta_{\tau_{n,\epsilon}},i)1_{\{\tau\geq\tau_{n,\epsilon}\}}\right] + \min(\epsilon,-\delta)E\left[e^{-\rho_{\tau\wedge\tau_{n,\epsilon}}}\right]. \end{aligned} \tag{23}$$

Since $E\left[e^{-\rho_{\tau\wedge\tau_{n,\epsilon}}}\right] > 0$, by combining the calculations made in (22) and (23), and taking into account that $\eta_n \to 0$ as $n \to \infty$, we obtain the desired contradiction in the DPP.

Boundary condition: To finalize the proof, we must prove that $V^*(x,t,i) = h(x,i)$, for all $x \in \partial I$. In fact, if $X_0 = x \in \partial I$, then $T^I = 0$ $P-$almost surely and, consequently, $V^*(x,t,i) = J(x,t,i,\tau^*\wedge 0) = h(x,i)$. ∎

Until the end of this section, we will introduce some useful auxiliary results to prove the uniqueness result in the next section. From now on, $\hat{h} : (x,t,i) \to D \times [0,\infty) \times \Theta$ is such that $\hat{h}(\cdot,\cdot,i)$ is a continuous function.

Lemma 4.2. *Consider the modified optimal stopping problem*

$$V_\Upsilon(x,t,i) = \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau\wedge\tau_\Upsilon} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_{\tau\wedge\tau_\Upsilon}}\hat{h}(X_{\tau\wedge\tau_\Upsilon},\zeta_{\tau\wedge\tau_\Upsilon},\theta_{\tau\wedge\tau_\Upsilon})1_{\{\tau\wedge\tau_\Upsilon<\infty\}}\right],$$

where $\tau_\Upsilon = \inf\{s \geq 0 : \zeta_s \geq \Upsilon\}$ *and* $\Upsilon > 0$ *is a deterministic and finite time. In this case, the value function* $V_\Upsilon : (x,t,i) \to \overline{I} \times [0,\Upsilon] \times \Theta$ *is a viscosity solution to*

$$\begin{cases} \min\Big\{ -(\tilde{\mathcal{L}}v)(x,t,i) - \Pi(x,i), v(x,t,i) - \hat{h}(x,t,i)\Big\} = 0 \\ v(x,t,i) = \hat{h}(x,t,i), \quad \forall (x,t,i) \in (\partial I \times [0,\Upsilon[\cup I \times \{\Upsilon\}) \end{cases}. \tag{24}$$

This can be easily proven by using similar arguments to the ones used in Proposition 4.3. For future reference, we note that any viscosity solution v to (24) is such that, for $i \in \Theta$, $v(\cdot,\cdot,i)$ satisfies the boundary problem

$$-\tilde{\mathcal{L}}v(x,t,i) - \Pi(x,i) = 0, \quad \text{for all } (x,t) \in A_v^i \tag{25}$$

$$v(x,t,i) + \hat{h}(x,t,i) = 0, \quad \text{for all } (x,t) \in (\overline{I} \times [0,\Upsilon]) \setminus A_v^i, \tag{26}$$

with

$$A_v^i = \{(x,t) \in I \times [0,\Upsilon[\ : \ v(x,t,i) > -\hat{h}(x,t,i)\}. \tag{27}$$

Lemma 4.3. *Let* $v : I \times [0,\infty) \times \Theta \to \mathbb{R}$ *be a viscosity solution to* (24)*. Then,* v *is a viscosity solution to the boundary problem* (25)*-*(26)*-*(27) *and verifies* $v(x,t,i) \geq -\hat{h}(x,t,i)$*.*

Proof. Firstly, we prove that any viscosity supersolution v to (24) verifies $v(x,t,i) \geq -\hat{h}(x,t,i)$, for all $(x,t,i) \in I \times [0,T] \times \Theta$. Let $B_\epsilon(x,t)$ be a ball centered in (x,t) with radius $\epsilon > 0$, such that $\overline{B}_\epsilon(x,t) \subset I \times [0,\Upsilon[$. Since the function $\underline{v}(\cdot,\cdot,i)$, with $i \in \Theta$, is lower semicontinuous, then there is

$$(\overline{x},\overline{t}) = \arg\min\{\underline{v}(x,t,i) \ : (x,t) \in \overline{B}_\epsilon(x,y)\}.$$

Therefore, by choosing $\psi(x,t) = \underline{v}(\overline{x},\overline{t},i)$, for all $(x,t) \in I \times [0,\Upsilon[$, $(\overline{x},\overline{t})$ is a local minimizer of $\underline{v}(x,t,i) - \psi(x,t)$ and $\underline{v}(\overline{x},\overline{t},i) = \psi(\overline{x},\overline{t}) \geq -\hat{h}(\overline{x},\overline{t},i)$. Letting ϵ go to 0 allows us to get that $\underline{v}(x,t,i) \geq -\hat{h}(x,t,i) \Rightarrow v(x,t,i) \geq -\hat{h}(x,t,i)$. Furthermore, $v(x,t,i) = -\hat{h}(x,t,i)$, for all $(t,x) \in I \times [0,\Upsilon] \setminus A_v^i$ and $i \in \Theta$.

To finish this proof, we note that, if $v(x,t,i) > -\hat{h}(x,t,i)$, then $\overline{v}(x,t,i) > -\hat{h}(x,t,i)$. Therefore, since v is a viscosity solution to (24), then v is a viscosity solution to Equation (25). ∎

To introduce the next result, we start by defining the expected value $H(x,t,i)$ in the following way:

$$\begin{cases} H(x,t,i) \equiv E_{x,t,i}\left[\int_0^{\tau_A} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_{\tau_A}}\hat{h}(X_{\tau_A},\zeta_{\tau_A},\theta_{\tau_A})1_{\{\tau_A<\infty\}}\right] \\ \tau_A = \inf\{s \geq 0 : (X_s,\zeta_s,\theta_s) \notin A\}, \quad \text{and} \\ A = \cup_{i\in\Theta} A_i \times \{i\} \quad \text{with} \quad A_i \subset I \times [0,\Upsilon] \text{ an open set.} \end{cases} \tag{28}$$

In the next lemma, we characterize the expected value H as a solution to the boundary problem (28).

Lemma 4.4. *Let* $H : I \times [0,\infty) \times \Theta \to \mathbb{R}$ *be the function defined as in* (28)*. Then* H *is a viscosity solution to the boundary problem* (25)*-*(26)*, replacing* A_v^i *by* A_i *as in* (28)*.*

Proof. First of all, we note that, in light of Assumption (3.2), H is locally bounded. Fixing $i \in \Theta$, by construction,

$$H(x,t,i) = -h(x,i), \quad \text{for all } (x,t) \notin A^i.$$

If $(x,t) \in A^i$ and $\tau \in \mathcal{S}$ is such that $\tau < \tau_A$, then

$$H(x,t,i) = E_{x,t,i}\left[\int_0^\tau e^{-\rho s}\Pi(X_s,\theta_s)ds + e^{-\rho\tau}H(X_\tau,\zeta_\tau,\theta_\tau)\right].$$

Therefore, to finish the proof, one needs to demonstrate that H is a viscosity solution to (25), which is straightforward in light of the arguments used to prove Proposition 4.3. ∎

4.3 The uniqueness result

At this generality, the uniqueness of solutions to the system of HJB equations cannot be guaranteed without further conditions (see Example 4.1). Therefore, in this section, we present some additional conditions that guarantee the uniqueness of the viscosity solution to (17). Under these conditions, such a solution will be V^*.

The next example was inspired by Example 3.1 in Øksendal and Reikvam [21].

Example 4.1. *Consider the following system of HJB equations:*

$$\min\left\{-\frac{1}{2}\sigma^2 v''(x,0), v(x,0) - \frac{x^2}{1+x^2}\right\} = 0, \tag{29}$$

$$\min\left\{-\frac{1}{2}\sigma^2 v''(x,1) - \lambda(v(x,0)-v(x,1)), v(x,1)-1\right\} = 0, \quad \textit{for all } x \in \mathbb{R}. \tag{30}$$

It is straightforward to see that any constant function $v(x,i) = a$ with $a \geq 1$ is a classical solution (and, consequently, a viscosity solution) to (29)-(30).

Uniqueness of viscosity solutions is generally guaranteed through a suitable comparison principle, which, in our case, exists if conditions C.1 and C.2 are satisfied (see Ishii and Koike [26]). Henceforward, $\mathbb{S}_n$ denotes the set of all symmetric matrices of dimension n.

C.1 There is a number $b > 0$ such that if $m = (m_1, \ldots, m_k)$ and $n = (n_1, \ldots, n_k) \in \mathbb{R}^k$, $\max_{u\in\Theta}(m_u - n_u) > 0$ and $(x,t,p,a) \in I \times [0,\infty) \times \mathbb{R}^n \times \mathbb{R}$, then there is a $j = j(m,n,x,t,p,a) \in \Theta$ such that

$$(m_j - n_j) = \max_{u\in\Theta}(m_u - n_u),$$

and, for all $X \in \mathbb{S}_n$,

$$F_j(x,t,\{m_i : i\in\Theta\},a,p,X) - F_j(x,t,\{n_i : i\in\Theta\},a,p,X) \geq b(m_j - n_j).$$

C.2 There is a continuous function $\omega : [0,\infty) \to [0,\infty)$ with $w(0) = 0$ such that if $X, Y \in \mathbb{S}_n$, $b > 1$ and

$$-3b\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3b\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix}$$

then, for all $j \in \Theta$, (x,t), $(y,s) \in I \times [0,\infty)$, and $m \in \mathbb{R}^k$

$$F_j(y,s,\{m_i : i \in \Theta\}, b(t-s), b(x-y), -Y) - \\ - F_j(x,t,\{m_i : i \in \Theta\}, b(t-s), b(x-y), X) \leq \omega\left(a|x-y|^2 + \frac{1}{a}\right).$$

In the next lemma, we prove that conditions C.1 and C.2 are satisfied.

Lemma 4.5. *Consider the system of HJB equations given by* (17) *and assume that Assumption 3.1 holds true. Then, conditions C.1 and C.2 are satisfied in any compact set* $U \subset I \times [0,\infty)$.

Proof. To prove that condition C.1 is verified, we start by introducing the following notation:

$$\begin{aligned} G_j(x,t,\{m_i : i \in \Theta\}, a, p, X) = {} & r(x,j)m_j - a - \alpha(x,j)\cdot p - \frac{1}{2}Tr[\sigma\sigma^T(x,j)X] \\ & - \sum_{i\neq j} \lambda_{j,i}(t)\,(m_i - m_j) - \Pi(x,j) \quad \text{for } i \neq j \in \Theta. \end{aligned}$$

Now, we assume that there is j such that

$$0 < (m_j - n_j) = \max_{u\in\Theta}(m_u - n_u).$$

Therefore,

$$\begin{aligned} G_j(x,t,\{m_i : i \in \Theta\}, a, p, X) &= G_j(x,t,\{n_i : i \in \Theta\}, a, p, X) + \sum_{i\neq j} \lambda_{j,i}(t)\,(m_j - n_j) \\ &\quad - \sum_{i\neq j} \lambda_{j,i}(t)\,(m_i - n_i) + r(x,j)(m_j - n_j) \\ &\geq G_j(x,t,\{n_i : i \in \Theta\}, a, p, X) + r(x,j)(m_j - n_j) \\ m_j + h(x,j) &= n_j + h(x,j) + (m_j - n_j). \end{aligned}$$

Finally, to prove that C.2 is satisfied, we notice that for $j \neq i \in \Theta$, (x,t) and $(y,s) \in U$, $m \in \mathbb{R}^k$, and $b > 1$

$$\begin{aligned} G_j(y,s,&\{m_i : i \in \Theta\}, b(t-s), b(x-y), -Y) - G_j\,(x,t,\{m_i : i \in \Theta\}, b(t-s), b(x-y), X) \\ &= (r(y,j) - r(x,j))m_j - \sum_{i\neq j}(\lambda_{j,i}(s) - \lambda_{j,i}(t))\,(m_i - m_j) - (\Pi(y,j) - \Pi(x,j)) \\ &+ b\,(\alpha(x,j) - \alpha(y,j))\cdot(x-y) + \frac{1}{2}Tr[\sigma\sigma^T(y,j)Y + \sigma\sigma^T(x,j)X] \\ &\leq L|x-y|^2 + \omega\left(|x-y|^2 + |t-s|\right), \end{aligned}$$

for some $L > 0$. The last inequality follows in light of the uniform continuity of $\Pi(\cdot,j)$, $r(\cdot,j)$ and $\lambda_{j,i}(\cdot)$ in U. The calculations involving $Tr[\sigma\sigma^T(y,j)Y + \sigma\sigma^T(x,j)X]$ may be seen in Example 3.6 of Crandall, Ishii and Lions [27]. Finally, the result follows from the uniform continuity of the functions $h(\cdot,i)$ in U, for all $i \in \Theta$. ∎

The next result states that there is a unique solution v to the boundary problem (17)-(18), which is the value function V^*. Furthermore, from the proof of Theorem 4.1, one can observe that

$$V^*(x,t,i) = \begin{cases} u(x,t,i), & (x,t,i) \in A_v \\ -h(x,i), & (x,t,i) \notin A_v \end{cases}, \tag{31}$$

where

$$A_v = \{(x,t,i) \in I \times [0,\infty) \times \Theta : u(x,t,i) > -h(x,i)\} \tag{32}$$

and $u(x,t,i)$ satisfies, in the viscosity sense, the partial differential equation (PDE)

$$-\tilde{\mathcal{L}}v(x,t,i) - \Pi(x,i) = 0. \tag{33}$$

For future reference, we introduce the stopping time

$$\tau_v \equiv \inf\{s \geq 0 : (X_s, \zeta_s, \theta_s) \notin A_v\}. \tag{34}$$

The result presented here follows the idea of Theorem 3.1 of Øksendal and Reikvam [28].

Theorem 4.1. *Suppose that v is a viscosity solution to the system of equations* (17) *and satisfies conditions* (18)*. Additionally, assume that*

$$\{v(X_\tau, \zeta_\tau, \theta_\tau)\}_{\tau\in\mathcal{S}} \text{ is a uniformly integrable family of random variables.} \tag{35}$$

Then, v is the unique solution to (17)-(18) *that satisfies* (35) *and verifies $v = V^*$. Furthermore, $\tau^* = \tau_v$.*

Proof. To prove the result, we introduce the following: (i) an open bounded set $A_N \subset I \times [0,\infty)$ such that $A_N \nearrow I \times [0,\infty)$, as $N \to \infty$, and (ii) the function v_N that verifies $v_N(x,t,i) = v(x,t,i)$, for all $(x,t,i) \in \overline{A}_N \times \Theta$, where v is a viscosity solution to (17), while satisfying conditions (18). By construction, v_N is a solution to (24) with $\hat{h} = v_N$, for all $(x,t,i) \in \overline{A}_N \times \Theta$. Combining Lemma 4.5 with the comparison principle in Ishii and Koike [26], this solution is unique. Consequently, taking into account Proposition 4.3 and Lemma 4.2

$$v_N(x,t,i) = \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau\wedge\tau_N} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau\wedge\tau_N}} v_N\left(X_{\tau\wedge\tau_N}, \zeta_{\tau\wedge\tau_N}, \theta_{\tau\wedge\tau_N}\right) 1_{\{\tau\wedge\tau_N<\infty\}}\right],$$

where $\tau_N \equiv \inf\{s > 0 : (X_s, \zeta_s) \notin A_N\}$. We note that, by construction

$$A_N \nearrow I \times [0,\infty) \quad \text{and} \quad v(x,t,i) = \lim_{N\to\infty} v_N(x,t,i).$$

In additionally, similarly to (15) and (16) we can obtain

$$\lim_{N\to\infty} E_{x,t,i}\left[\int_0^{\tau\wedge\tau_N} e^{-\rho_s}\Pi(X_s,\theta_s)ds\right] = E_{x,t,i}\left[\int_0^{\tau\wedge T^I} e^{-\rho_s}\Pi(X_s,\theta_s)ds\right]$$

and

$$\lim_{N\to\infty} E_{x,i}\left[e^{-\rho_{\tau\wedge\tau_N}} v_N(X_{\tau\wedge\tau_N}, \zeta_{\tau\wedge\tau_N}, \theta_{\tau\wedge\tau_N})\right] = E_{x,i}\left[e^{-\rho_{\tau\wedge T^I}} v(X_\tau, \zeta_\tau, \theta_\tau)\right].$$

Since this holds true for every $\mathcal{F}_s-$stopping time τ, then

$$\begin{aligned}
v(x,i) &= \lim_{N\to+\infty} v_N(x,i)\\
&= \lim_{N\to+\infty} \sup_{\tau\in\mathcal{S}} E_{x,i}\left[\int_0^{\tau\wedge\tau_N} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau\wedge\tau_N}} v_N(X_{\tau\wedge\tau_N},\zeta_{\tau\wedge\tau_N},\theta_{\tau\wedge\tau_N})1_{\{\tau\wedge\tau_N<\infty\}}\right]\\
&= \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_\tau} v(X_\tau,\zeta_\tau,\theta_\tau)1_{\{\tau<\infty\}}\right]\\
&\geq \sup_{\tau\in\mathcal{S}} E_{x,t,i}\left[\int_0^{\tau} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_\tau} h(X_\tau,\theta_\tau)1_{\{\tau<\infty\}}\right] = V^*(x,t,i),
\end{aligned}$$

the last inequality following in light of Lemma 4.3.

In order to obtain the reverse inequality, we note that, by combining Lemma 4.3 with the first part of this proof, v_N is the unique viscosity solution to (25)-(26) in A_N if we replace $\hat{h}$ with v_N and $A_v^i = \{(x,t)\in A_N : v(x,t,i) > h(x,i)\}$. If $A_v^N = \cup_{i\in\Theta} A_v^i = A_v \cap A_N$ and $\tilde{\tau}_N \equiv \inf\{s>0 : (X_s,\zeta_s,\theta_s)\notin A_v^N\} = \tau_v \cap \tau_N$, then, in view of Lemma 4.4, it follows that

$$v_N(x,t,i) = E_{x,t,i}\left[\int_0^{\tilde{\tau}_N} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_{\tilde{\tau}_N}} v_N(X_{\tilde{\tau}_N},\zeta_{\tilde{\tau}_N},\theta_{\tilde{\tau}_N})1_{\tilde{\tau}_N<\infty}\right].$$

With a similar argument to the previous one, we obtain

$$\begin{aligned}
v(x,t,i) &= \lim_{N\to+\infty} v_N(x,t,i)\\
&= E_{x,t,i}\left[\int_0^{\tau_v\wedge T} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau_v\wedge T}} v_N(X_{\tau_v\wedge T},\zeta_{\tau_v\wedge T},\theta_{\tau_v\wedge T})1_{\{\tau_v\wedge T<\infty\}}\right]\\
&= E_{x,t,i}\left[\int_0^{\tau_v\wedge T} e^{-\rho_s}\Pi(X_s,\theta_s)ds + e^{-\rho_{\tau_v\wedge T}} h(X_{\tau_v\wedge T},\theta_{\tau_v\wedge T})1_{\{\tau_v\wedge T<\infty\}}\right]\\
&\leq V^*(x,t,i),
\end{aligned}$$

which concludes the proof. ∎

5 The One-dimensional Case

In this section, we present stronger results concerning the optimal stopping problem, when θ is a homogeneous continuous Markov chain and X is a one dimensional diffusion. To clarify, we make the following assumption:

Assumption 5.1. *The set D is an interval of the form $]a,b[$ with $-\infty\leq a<b\leq\infty$, $n=m=1$ and the Borel measurable functions $\alpha(\cdot,i):I\to\mathbb{R}$ and $\sigma(\cdot,i):D\to\mathbb{R}$ are such that the SDE* (2)*, for each initial condition, has a unique strong solution (W,X) on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_s)_{s\geq0},P)$ that remains in D for all times. Additionally, the process θ is such that, for every $j,m\in\Theta$,*

$$P(\nu_n-\nu_{n-1}\leq s\,|\,\theta_{\nu_{n-1}}=j) = 1-e^{\lambda_j s}, \quad \textit{for all } s\geq 0, \tag{36}$$

$$P(\theta_{\nu_n}=m\,|\,\theta_{\nu_{n-1}}=j) = p_{j,m}, \tag{37}$$

with $\lambda_j > 0$ *and* $p_{j,m} \in [0,1]$ *verifying* $\sum_{j\neq m} p_{j,m} = 1$ *and* $p_{jj} = 0$*. Finally, for every* $n_1, n_2 \in \mathbb{N}$*, the random variables* $(\nu_{n_1} - \nu_{n_1-1})$ *and* $(\nu_{n_2} - \nu_{n_2-1})$ *are independent. Henceforward, we adopt the following notation:* $\lambda_{j,m} = \lambda_j \times p_{j,m}$*.*

As previously mentioned, our main goal is to find the optimal strategy τ^* which maximizes the function defined in (3), in the interval $I \subseteq D$. Under the previous assumption, we prove that V^*, defined in (4) is no longer time dependent, which means that, in the homogeneous case, we simply need to use the process (X, θ). To ensure that the optimal stopping problem is well defined, we make the following assumption:

Assumption 5.2. *The Borel measurable functions* $\Pi(\cdot, i), h(\cdot, i), r(\cdot, i) : D \to \mathbb{R}$*, with* $i \in \Theta$*, are such that:*

(1) Assumption 3.2 is satisfied;

(2) $h(\cdot, i) \in C(D)$*;*

(3) $r(\cdot, i) > 0$*.*

Now, we introduce the differential operator

$$(\tilde{\mathcal{L}}\varphi)(x,i) = -r(x,i)\varphi(x,i) + \alpha(x,i)\varphi'(x,i) + \frac{1}{2}\sigma^2(x)\varphi''(x,i) + \sum_{j\neq i} \lambda_{i,j} \left(\varphi(x,j) - \varphi(x,i)\right),$$

where $\varphi'(\cdot, i)$ and $\varphi''(\cdot, i)$ are respectively the first and the second derivatives of φ in the first argument. Additionally, we consider the system of HJB equations

$$\min\Big\{ - (\tilde{\mathcal{L}}v)(x,i) - \Pi(x,i), v(x,i) + h(x,i)\Big\} = 0, \quad \text{for all } (x,i) \in D \times \Theta, \tag{38}$$

and the boundary condition

$$v(x) = -h(x) \quad \text{for all } x \in \partial I. \tag{39}$$

Before we define the concept regarding the solution we consider throughout this section, we will introduce some notation. We denote by $DC(I)$ the set of functions that are the difference of two convex function in I. Recall that $f \in DC(I)$ if and only if f is absolutely continuous in I ($f \in AC(I)$) and f' is of bounded variation ($f' \in BV(I)$). Furthermore, if $f \in DC(I)$, then the left-hand side derivative of f'_- exists and its second distributional derivative is a measure. From the Lebesgue's Decomposition Theorem, there are two σ-finite signed measures, $f''_{ac}(x)dx$ and $f''_s(dx)$ such that:

- $f''(dx) = f''_{ac}(x)dx + f''_s(dx)$;
- $f''_{ac}(x)dx$ is absolutely continuous with respect to the Lebesgue measure, μ;
- $f''_s(dx)$ and μ are singular.

Definition 5.1. *Let $v : \overline{I} \times \Theta \to \mathbb{R}$ be a function such that $v(\cdot, i) \in DC(I)$, for each $i \in \Theta$, and $\tilde{\mathcal{L}}^{ac}$ be the differential operator defined by*

$$(\tilde{\mathcal{L}}^{ac} v)(x,i) = -r(x,i)v(x,i) + \alpha(x)v'_{-}(x,i) + \frac{1}{2}\sigma^2(x)v''_{ac}(x,i) + \sum_{j \neq i} \lambda_{i,j} \left(\varphi(x,j) - \varphi(x,i)\right).$$

v is a solution to the system of HJB equations (38) *if it satisfies, for every $i \in \Theta$,*

$$\min\Big\{ -(\tilde{\mathcal{L}}^{ac} v)(x,i) - \Pi(x,i), v(x,i) + h(x,i) \Big\} = 0, \quad \mu - \textit{almost everywhere in } D. \tag{40}$$

Remark 5.1. *Our analysis will rely on solutions v satisfying Definition 5.1 such that, for each $i \in \Theta$:*

1) $-v''_s(dx,i)$ is a positive measure;

2) $\operatorname{supp} v''_s(dx,i) \subseteq \{x \in I : v(x,i) = -h(x,i)\}$.

Note that this definition was already used in the literature of optimal stopping and optimal switching as one can see, for instance, in Lamberton and Zervos [29] and Zervos [30]). Henceforward, we will adopt a similar argumentation to the one present at the first aforementioned reference.

Theorem 5.1 provides a general verification result for the optimal stopping problem when θ is a homogeneous Markov chain and X is a one-dimensional switching diffusion. To prove such a result, we need an appropriate Itô formula. For one-dimensional semimartingales, the Meyer-Itô formula is a well-known generalization of the classical Itô formula that relies on the concept of *Local Time*, which is valid for any function $f \in DC(I)$ (see Protter [23]). In Lemma 5.1, we provide an appropriate Meyer-Itô formula for the process (X, θ). From now on, we will denote by L^c the local time associated with the process X at level c, by A^i the process

$$A^i_t = \frac{1}{2}\int_D \varphi''_s(dc,i)L^c_t, \quad \text{for } i = 0, 1,$$

and by $\mathcal{A}^{ac}$ the operator

$$(\mathcal{A}^{ac}\varphi)(x,i) \equiv -r(x)\varphi(x,i) + \alpha(x)\varphi'_{-}(x,i) + \frac{1}{2}\sigma^2(x)\varphi''_{ac}(x,i),$$

where φ is such that $\varphi(\cdot, i) \in DC(I)$, for each $i = 0, 1$.

Lemma 5.1. *Let (X, θ) be the $(1+1)$-dimensional process defined by Equations* (1) *and* (2)*, taking into account Assumptions 5.1. Furthermore, assume that $\varphi : D \times \Theta \to \mathbb{R}$ is such that $\varphi(\cdot, i) \in DC(D)$, with $i \in \Theta$. Then, for every $t \in [0, T^D]$*

$$E_{x,i}\left[e^{-\rho_t}\varphi(X_t,\theta_t)\right] = \varphi(x,i) + E_{x,i}\left[\int_0^t e^{-\rho_s}(\tilde{\mathcal{L}}^{ac})\varphi(X_s,\theta_s)ds\right] + E_{x,i}\left[\sum_{j=1}^k \int_0^t e^{-\rho_s} 1_{\{\theta_s = j\}} dA^j_s\right]$$
$$+ E_{x,i}\left[\int_0^t \varphi'_{-}(X_s,i)\sigma(X_s,i)dW_s\right]. \tag{41}$$

Proof. Let ν_n, with $n \in \mathbb{N}$, be defined as in Section 2, and assume that $t \in [\nu_n, \nu_{n+1}[$, then φ admits the following decomposition

$$\begin{aligned} e^{-\rho_t}\varphi(X_t,\theta_t) =& e^{-\rho_t}\varphi(X_t,\theta_t) - e^{-\rho_{\nu_n}}\varphi(X_{\nu_n},\theta_{\nu_n}) + \sum_{j=1}^{n}\left(e^{-\rho_{\nu_j}}\varphi(X_{\nu_j},\theta_{\nu_j}) - e^{-\rho_{\nu_j^-}}\varphi(X_{\nu_j^-},\theta_{\nu_j^-})\right) \\ &+ \sum_{j=2}^{n}\left(e^{-\rho_{\nu_j^-}}\varphi(X_{\nu_j^-},\theta_{\nu_j^-}) - e^{-\rho_{\nu_{j-1}}}\varphi(X_{\nu_{j-1}},\theta_{\nu_{j-1}})\right) + e^{-\rho_{\nu_1^-}}\varphi(X_{\nu_1^-},\theta_{\nu_1^-}). \end{aligned}$$

Therefore, for any $t \geq 0$,

$$\begin{aligned} e^{-\rho}\varphi(X_t,\theta_t) =& e^{-\rho_{t\wedge\nu_1^-}}\varphi(X_{t\wedge\nu_1^-},i) + \sum_{j=1}^{\infty}\left(e^{-\rho_{\nu_j}}\varphi(X_{\nu_j},\theta_{\nu_j}) - e^{-\rho_{t\wedge\nu_j^-}}\varphi(X_{t\wedge\nu_j^-},\theta_{t\wedge\nu_j^-})\right)1_{\{t\geq\nu_j\}} \\ &+ \sum_{j=2}^{\infty}\left(e^{-\rho_{t\wedge\nu_j^-}}\varphi(X_{t\wedge\nu_j^-},\theta_{t\wedge\nu_j^-}) - e^{-\rho_{\nu_{j-1}}}\varphi(X_{\nu_{j-1}},\theta_{t\wedge\nu_{j-1}})\right)1_{\{t\geq\nu_{j-1}\}}. \end{aligned}$$

Since θ_t is constant for $\nu_n \leq t < \nu_{n+1}$, we can say, without loss of generality, that $\theta_t = i$. Then, from the Meyer-Itô Formula (see Theorem IV.70 in Protter [23]), we get

$$\varphi(X_t,i) - \varphi(X_{\nu_n},i) = \int_{\nu_n}^{t}\alpha(X_s,i)\varphi'_-(X_s,i)ds + \frac{1}{2}\int_D \varphi''(dc,i)L_t^c + \int_0^t \varphi'_-(X_s,i)\sigma(X_s,i)dW_s,$$

where φ is such that $\varphi(\cdot,i) \in DC(D)$. From the Occupation Times Formula we obtain

$$\int_D \varphi''_{ac}(dc,i)L_t^c = \int_{\nu_n}^{t}\sigma^2(X_s,i)\varphi''_{ac}(X_s,i)ds,$$

which allows us to write

$$\begin{aligned} \varphi(X_t,i) - \varphi(X_{\nu_n},i) =& \int_{\nu_n}^{t}\alpha(X_s)\varphi'_-(X_s,i) + \frac{1}{2}\sigma^2(X_s,i)\varphi''_{ac}(X_s,i)ds + \frac{1}{2}\int_D \varphi''_s(dc,i)L_t^c \\ &+ \int_{\nu_n}^{t}\varphi'_-(X_s,i)\sigma(X_s,i)dW_s. \end{aligned}$$

By using the integration by parts, we have

$$\begin{aligned} e^{-\rho_t}\varphi(X_t,i) - e^{-\rho_{\nu_n}}\varphi(X_{\nu_n},i) =& \int_{\nu_n}^{t}e^{-\rho_s}(\mathcal{A}^{ac}\varphi)(X_s,i)ds + \int_{\nu_n}^{t}e^{-\rho_s}dA_s^i \\ &+ \int_{\nu_n}^{t}e^{-\rho_s}\varphi'_-(X_s,i)\sigma(X_s)dW_s. \end{aligned} \tag{42}$$

Taking into account that this argument remains valid in any interval $[\nu_n, \nu_{n+1}[$, with $n \in \mathbb{N}$, we obtain that, for every $0 < t \leq T^D$

$$\begin{aligned}e^{-\rho_t}\varphi(X_t,\theta_t) =& \varphi(x,i) + \int_0^t e^{-\rho_t}(\mathcal{A}^{ac}\varphi)(X_s,\theta_s)ds + \sum_{j=1}^{k}\int_0^t e^{-\rho_s}1_{\{\theta_t=j\}}dA_s^j \\ &+ \sum_{j=1}^{\infty}\left(e^{-\rho_{\nu_j}}\varphi(X_{\nu_j},\theta_{\nu_j}) - e^{-\rho_{t\wedge\nu_j^-}}\varphi(X_{t\wedge\nu_j^-},\theta_{t\wedge\nu_j^-})\right)1_{\{s\geq\nu_j\}} \\ &+ \int_0^t e^{-\rho_s}\varphi'_-(X_s,\theta_s)\sigma(X_s,\theta_s)dW_s.\end{aligned}$$

Now, by using a similar argument to the one used in (9), we obtain the result. ∎

Theorem 5.1. *Let V^* be the value function defined as in* (4)*, taking into account Assumptions 5.1 and 5.2. Assume that there is a function $v : \overline{I} \times \Theta \to \mathbb{R}$ such that $v(\cdot, i) \in DC(I)$ and v is a solution to the system of HJB equations* (38) *in the sense of Definition 5.1 and the process*

$$\left\{\int_0^t \varphi'_-(X_s,i)\sigma(X_s,i)dW_s\right\}_{t\geq 0} \quad \text{is a martingale.} \tag{43}$$

Furthermore, assume that v is such that 1) in Remark 5.1 is fulfilled. The following statements are true:

1) *$v(x,i) \geq J(x,i,\tau)$, for all $\tau \in \mathcal{S}$;*
2) *additionally, if v is such that statement 2) in Remark 5.1 holds true, the boundary condition* (39) *is satisfied and*

$$\{v(X_\tau,\theta_\tau)\}_{\tau\in\mathcal{S}} \text{ is a uniformly integrable family of random variables.} \tag{44}$$

Then, $V^* = v$, $\tau^* = \inf\{s \geq 0 : v(X_s,\theta_s) \leq -h(X_s,\theta_s)\}$ is the optimal strategy.

Proof. We start by proving statement 1) of Theorem 5.1. Fix $\tau \in \mathcal{S}$ and let $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ be an increasing sequence, such that $\tau_n \nearrow \tau$. Then, by using Lemma 5.1 and condition (43), we obtain

$$\begin{aligned}J(x,t,i,\tau_n) =& E_{x,t,i}\left[\int_0^{\tau_n} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_{\tau_n}}h(X_{\tau_n},\theta_{\tau_n})\right] \\ =& v(x,i) + E_{x,t,i}\left[\int_0^{\tau_n} e^{-\rho_s}\left(\Pi(X_s,\theta_s) + (\tilde{\mathcal{L}}^{ac})v(X_s,\theta_s)\right)ds\right] \\ &- E_{x,t,i}\left[e^{-\rho_{\tau_n}}\left(v(X_{\tau_n},\theta_{\tau_n}) + h(X_{\tau_n},\theta_{\tau_n})\right)\right] + E_{x,t,i}\left[\sum_{j=1}^{k}\int_0^{\tau_n} e^{-\rho_s}1_{\{\theta_s=j\}}dA_s^j\right].\end{aligned}$$

In light of Definition 5.1, we get

$$J(x,t,i,\tau_n) \leq v(x,i) + E_{x,t,i}\left[\sum_{j=1}^{k}\int_0^{t} e^{-\rho_s}1_{\{\theta_s=j\}}dA_s^j\right].$$

To proceed, we note that the local time associated with the process X at level c, L^c, is increasing and càdlag. Therefore,

$$dA_s^i = \frac{1}{2} d \int_D v_s''(dc, i) L_t^c = \frac{1}{2} d \int_{\text{supp}_i} v_s''(dc, i) L_t^c \leq 0, \tag{45}$$

and, consequently, $J(x, t, i, \tau_n) \leq v(x, i)$. With a similar argument to (15) and (16), we have that

$$\lim_{n \to \infty} J(x, t, i, \tau_n) = J(x, t, i, \tau) \leq v(x, i), \quad \text{for all } \tau \in \mathcal{S}.$$

To prove statement 2), we consider the stopping time $\tau_0 \in \mathcal{S}$ given by $\tau_0 \equiv \inf\{s \geq 0 : v(X_s, \theta_s) \leq -h(X_s, \theta_s)\}$ and $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$, which is an increasing sequence verifying $\tau_n \nearrow T^I$. Then, we obtain from Lemma 5.1 that

$$\begin{aligned} e^{-\rho_{\tau_0 \wedge \tau_n}} v(X_{\tau_0 \wedge \tau_n}, \theta_{\tau_0 \wedge \tau_n}) &= v(x, i) + \int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} (\tilde{\mathcal{L}}^{ac}) v(X_s, \theta_s) ds \\ &+ \sum_{j=1}^k \int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} 1_{\{\theta_s = j\}} dA_s^j + \int_0^{\tau_0 \wedge \tau_n} v'_-(X_s, i) \sigma(X_s, i) dW_s. \end{aligned}$$

Consequently, taking into account that $\tau_0 = \tau_0 \wedge T^I$ and the boundary problem (38)-(39) is satisfied (in the sense of Definition 5.1)

$$\begin{aligned} \int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} \Pi(X_s, \theta_s) ds - e^{-\rho_{\tau_0}} h(X_{\tau_0}, \theta_{\tau_0}) 1_{\{\tau_0 \leq \tau_n\}} &= v(x, i) \\ - e^{-\rho_{\tau_n}} v(X_{\tau_n}, \theta_{\tau_n}) 1_{\{\tau_0 > \tau_n\}} &+ \sum_{j=1}^k \int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} 1_{\{\theta_s = j\}} dA_s^j \\ &+ \int_0^{\tau_0 \wedge \tau_n} v'_-(X_s, i) \sigma(X_s, i) dW_s. \end{aligned} \tag{46}$$

Assuming that $\theta_s = j$, with $j \in \Theta$ and for every $s \in [\nu_n \wedge \tau_0, \nu_{n+1} \wedge \tau_0[$, then, by combining the definition of τ_0, Equation (45), and statement 2) of Remark 5.1, we get

$$dA_s^i = 0.$$

Thus, from condition (43), we obtain

$$\begin{aligned} E_{x,t,i} \left[\int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} \Pi(X_s, \theta_s) ds - e^{-\rho_{\tau_0}} h(X_{\tau_0}, \theta_{\tau_0}) 1_{\{\tau_0 \leq \tau_n\}} \right] &= v(x, i) \\ &- E_{x,t,i} \left[e^{-\rho_{\tau_n}} v(X_{\tau_n}, \theta_{\tau_n}) 1_{\{\tau_0 > \tau_n\}} \right]. \end{aligned}$$

Consequently, with a similar argument to the one used in (15), we obtain

$$\lim_{n \to \infty} E_{x,t,i} \left[\int_0^{\tau_0 \wedge \tau_n} e^{-\rho_s} \Pi(X_s, \theta_s) ds \right] = E_{x,t,i} \left[\int_0^{\tau_0 \wedge T^I} e^{-\rho_s} \Pi(X_s, \theta_s) ds \right].$$

Additionally, $\{e^{-\rho\tau_0}h(X_{\tau_0},\theta_{\tau_0})1_{\{\tau_0\leq\tau\}}\}_{\tau\in\mathcal{S}}$ is a uniformly integrable family of random variables and, consequently,

$$\lim_{n\to\infty} E_{x,t,i}\left[e^{-\rho\tau_0}h(X_{\tau_0},\theta_{\tau_0})1_{\{\tau_0\leq\tau_n\}}\right] = E_{x,t,i}\left[e^{-\rho\tau_0}h(X_{\tau_0},\theta_{\tau_0})1_{\{\tau_0\leq T^I\wedge\infty\}}\right]. \quad (47)$$

From Assumption 5.2, $0 \leq e^{-\rho\tau} < 1$ for every $\tau \in \mathcal{S}$, and, accordingly, $\{e^{-\rho\tau}v(X_\tau,\theta_\tau)1_{\tau_0>\tau}\}_{\tau\in\mathcal{S}}$ is a uniformly integrable family of random variables. If $T^I = \infty$, then $e^{-\rho\tau_n} \to 0$ and, consequently, $e^{-\rho\tau_n}v(X_{\tau_n},\theta_{\tau_n})1_{\tau_0>\tau_n} \to 0$, $P-$almost surely. Additionally, if $T^I < \infty$, then $e^{-\rho\tau_n}v(X_{\tau_n},\theta_{\tau_n})1_{\tau_0>\tau_n} \to -e^{-\rho T^I}h(X_{T^I},\theta_{T^I})1_{\tau_0>T^I} = 0$, $P-$almost surely. Therefore, if condition (44) holds true, we get

$$J(x,t,i,\tau^*) = V^*(x,i) = v(x,i).$$

∎

Before we finish this section, we note that, if we relax the assumption $r(.,i) > 0$ for every $i \in \Theta$, the condition (44) may not be, in general, sufficient to keep the result true. In this case, a condition like

$$\lim_n E_{x,i}[e^{-\rho\tau_n}|v(X_{\tau_n},\theta_{\tau_n})|] = 0, \text{ with } \tau_n \nearrow T^I,$$

would be required.

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