PATHWISE SUPER-REPLICATION VIA VOVK'S OUTER MEASURE

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Abstract. Since Hobson's seminal paper [17] the connection between model-independent pricing and the Skorokhod embedding problem has been a driving force in robust finance. We establish a general pricing-hedging duality for financial derivatives which are susceptible to the Skorokhod approach.

Using Vovk's approach to mathematical finance we derive a model-independent super-replication theorem in continuous time, given information on finitely many marginals. Our result covers a broad range of exotic derivatives, including lookback options, discretely monitored Asian options, and options on realized variance.

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1. Introduction

Starting with Hobson [17], the theory of model-independent pricing has received substantial attention from the mathematical finance community, we refer to the survey [18]. Starting with [4, 15], the Skorokhod embedding approach has been complemented through optimal transport techniques. In particular, first versions of a robust super-replication theorem have been established: In discrete time we mention [1] and the important contribution of Bouchard and Nutz [5]; for related work in a quasi-sure framework in continuous time we refer to the work of Neufeld and Nutz [25] and Possamai, Royer, and Touzi [28]. Our results are more closely related to the continuous time super-replication theorem of Dolinsky and Soner [12], which we recall here: Given a centered probability measure μ on $\mathbb{R}$ they study the primal maximization problem

$$P = \sup\{\mathbb{E}_{\mathbb{P}}[G(S)]\}$$

where S denotes the canonical process on $C[0,1]$, the supremum is taken over all martingale measures $\mathbb{P}$ on $C[0,1]$ with $S_1(\mathbb{P}) = \mu$ and G denotes a functional on the path space satisfying appropriate continuity assumptions. The main result of [12] is a super-replication theorem that appeals to this setup: they show that for each $p > P$ there exists a hedging strategy H and a "European payoff function" ψ with $\int \psi \, \mathrm{d}\mu = 0$ such that

$$p + (H \cdot S)_1 + \psi(S_1) \geq G(S).$$

This is in principle quite satisfying, however, a drawback is that the option G needs to satisfy rather strong continuity assumptions, which in particular excludes all exotic option payoffs involving volatility. Given the practical importance of volatility derivatives it is desirable to give a version of the Dolinsky-Soner theorem that appeals also to this case. More recently Dolinsky and Soner [13] have extended the original results of [12] to include càdlàg price processes, multiple maturities and price processes in higher dimensions; Hou

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and Obłój [22] have also recently extended these results to incorporate investor beliefs via a 'prediction set' of possible outcomes.

Subsequently, we shall establish a super-replication theorem that applies to G which is invariant under time changes in an appropriate sense. Opposed to the result of [12] this excludes the case of continuously monitored Asian options but covers other practically relevant derivatives such as options on volatility or realized variance, lookback options and discretely monitored Asian options. In particular, it constitutes a general duality result appealing to the rich literature on the connection of model-independent finance and Skorokhod embedding. In a series of impressive achievements, Brown, Cox, Davis, Hobson, Klimmek, Neuberger, Obłój, Pedersen, Raval, Rogers, Wang, and others [29, 17, 6, 21, 7, 11, 9, 8, 10, 20, 19] were able to determine the values of related primal and dual problems for a number of exotic derivatives/market data, proving that they are equal. Here we establish the duality relation for generic derivatives, in particular recovering duality for the specific cases mentioned above.

After the completion of this work, we learned that Guo, Tan, and Touzi [16] derived a duality result similar in spirit to Theorem 5.5. Their approach relies on different methods, and includes an interesting application to the optimal Skorokhod embedding problem.

Organization of the paper: In Section 2 we state our main result. In Section 3 Vovk's approach to mathematical finance is introduced and preliminary results are given. Section 4 is devoted to the statement and proof of our main result in its simplest form, a super-replication theorem for time-invariant payoffs for one period. In Section 5 we present an extension to finitely many marginals with "zero up to full information", in particular we will then obtain our most general super-replication result, Theorem 5.8.

2. Formulation of the super-replication theorem

Let $C[0,n]$ be the space of continuous function $\omega\colon [0,n] \to \mathbb{R}$ with $\omega(0) = 0$ and consider $G\colon C[0,n] \to \mathbb{R}$ of the form

$$G(\omega) = \gamma(\mathfrak{t}(\omega)_{\restriction[0,\langle\omega\rangle_n]}, \langle\omega\rangle_1, \dots, \langle\omega\rangle_n),$$

where $\mathfrak{t}(\omega)$ stands for a version of the path ω which is rescaled in time so that its quadratic variation up to time t equals precisely t. Under appropriate regularity conditions on γ (see Theorems 4.1 and 5.8 below) we obtain the following robust super-hedging result:

Theorem 2.1. *Let $I \subseteq \{1,\dots,n\}$, $n \in I$, and consider*

$P_n := \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P}$ *is a Martingale measure on* $C[0,n]$, $S_0 = 0$, $S_i \sim \mu_i$ *for all* $i \in I\}$

and

$$D_n := \inf\left\{ a \ : \begin{array}{c} \text{there exist } H \text{ and } (\psi_j)_{j\in I}, \int \psi_j \, d\mu_j = 0, \\ a + \sum_{j\in I} \psi_j(S_j) + (H\cdot S)_n \geq G((S_t)_{t\leq n}) \end{array} \right\}.$$

Then one has $P_n = D_n$.

Of course the present statement of our main result is imprecise in that neither the pathwise stochastic integral appearing in the formulation of D_n, nor the pathwise quadratic variation in the definition of G are properly introduced. We will address this in the following sections.

3. Super-hedging and outer measure

Very recently, Vovk [33, 34, 35], see also [32], developed a new hedging based, model free approach to mathematical finance. Without presuming any probabilistic structure, Vovk considers the space of real-valued continuous functions as possible price paths and introduces an outer measure on this space which is based on a minimal super-hedging price.

More precisely, the set of price paths is given by the space $\Omega := C(\mathbb{R}_+)$ of all continuous functions $\omega\colon [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$. The coordinate process on Ω is denoted by $B_t(\omega) := \omega(t)$ and we introduce the natural filtration $(\mathcal{F}_t)_{t\geq 0} := (\sigma(B_s : s \leq t))_{t\geq 0}$ and set $\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t$. Stopping times τ and the associated σ-algebras $\mathcal{F}_\tau$ are defined as usual.

A process $H\colon \Omega\times\mathbb{R}_+ \to \mathbb{R}$ is called a *simple strategy* if it is of the form

$$H_t(\omega) = \sum_{n\geq 0} F_n(\omega)\mathbf{1}_{(\tau_n(\omega),\tau_{n+1}(\omega)]}(t), \quad (\omega,t)\in\Omega\times\mathbb{R}_+,$$

where $0 = \tau_0(\omega) < \tau_1(\omega) < \dots$ are stopping times such that for every $\omega\in\Omega$ one has $\lim_{n\to\infty}\tau_n(\omega) = \infty$, and $F_n\colon \Omega \to \mathbb{R}$ are $\mathcal{F}_{\tau_n}$-measurable bounded functions for $n\in\mathbb{N}$. For such a simple strategy H the corresponding capital process

$$(H\cdot B)_t(\omega) = \sum_{n=0}^{\infty} F_n(\omega)(B_{\tau_{n+1}(\omega)\wedge t} - B_{\tau_n(\omega)\wedge t})$$

is well-defined for every $\omega\in\Omega$ and every $t\in\mathbb{R}_+$. A simple strategy H is called λ-*admissible* for $\lambda > 0$ if $(H\cdot B)_t(\omega) \geq -\lambda$ for all $t\in[0,\infty)$ and all $\omega\in\Omega$. We write $\mathcal{H}_\lambda$ for the set of λ-admissible simple strategies.

To recall Vovk's outer measure as introduced in [35], let us define the set of processes

$$\mathcal{V}_\lambda := \left\{\sum_{k=0}^{\infty} H^k : H^k\in\mathcal{H}_{\lambda_k}, \lambda_k > 0, \sum_{k=0}^{\infty}\lambda_k = \lambda\right\}$$

for an initial capital $\lambda\in(0,\infty)$. Note that for every $G = \sum_{k\geq 0} H^k \in \mathcal{V}_\lambda$, all $\omega\in\Omega$, and all $t\in\mathbb{R}_+$, the corresponding capital process

$$(G\cdot B)_t(\omega) := \sum_{k\geq 0}(H^k\cdot B)_t(\omega) = \sum_{k\geq 0}(\lambda_k + (H^k\cdot B)_t(\omega)) - \lambda$$

is well-defined and takes values in $[-\lambda,\infty]$. Then, Vovk's outer measure on Ω is given by

$$\overline{Q}(A) := \inf\Big\{\lambda > 0 : \exists\, G\in\mathcal{V}_\lambda \text{ s.t. } \lambda + \liminf_{t\to\infty}(G\cdot B)_t(\omega) \geq \mathbf{1}_A(\omega)\ \forall\omega\in\Omega\Big\}, \quad A\subseteq\Omega.$$

A slight modification of the outer measure $\overline{Q}$ was introduced in [26, 27], which seems more in the spirit of the classical definition of super-hedging prices in semimartingale models. In this context one works with general admissible strategies and the Itô integral against a general strategy is given as limit of integrals against simple strategies. So in that sense the next definition seems to be more analogous to the classical one.

Definition 3.1. *The* outer measure $\overline{P}$ *of* $A\subseteq\Omega$ *is defined as the minimal super-hedging price, that is*

$$\overline{P}(A) := \inf\Big\{\lambda > 0 : \exists\,(H^n)_{n\in\mathbb{N}}\subseteq\mathcal{H}_\lambda \text{ s.t. } \liminf_{t\to\infty}\liminf_{n\to\infty}(\lambda + (H^n\cdot B)_t(\omega)) \geq \mathbf{1}_A(\omega)\ \forall\omega\in\Omega\Big\}.$$

A set $A\subseteq\Omega$ *is said to be a* null set *if it has outer measure zero. A property (P) holds for* typical price paths *if the set A where (P) is violated is a null set.*

Of course, for both definitions of outer measures it would be convenient to just minimize over simple strategies rather than over the limit (inferior) along sequences of simple strategies. However, this would destroy the very much appreciated countable subadditivity of both outer measures.

Remark 3.2. It is conjectured that the outer measure $\overline{P}$ coincides with $\overline{Q}$. However, up to now it is only known that $\overline{P}(A) \leq \overline{Q}(A)$ for a general set $A\subseteq\Omega$, see Section 2.4 of [26], and that they coincide for time-superinvariant sets, see Definition 3.5 and Theorem 3.6 below. Therefore, the outer measures $\overline{P}$ and $\overline{Q}$ are basically the same in the present paper since we focus on time-invariant financial derivatives.

Perhaps the most interesting feature of $\overline{P}$ is that is comes with the following arbitrage interpretation for null sets.

Lemma 3.3 ([26, Lemma 2.4]). *A set $A \subseteq \Omega$ is a null set if and only if there exists a sequence of 1-admissible simple strategies $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_1$, such that*

$$1 + \liminf_{t\to\infty} \liminf_{n\to\infty} (H^n \cdot B)_t(\omega) \geq \infty \cdot \mathbf{1}_A(\omega),$$

where we use the convention $\infty \cdot 0 := 0$ and $\infty \cdot 1 := \infty$.

A null set is essentially a model free arbitrage opportunity of the first kind. Recall that B satisfies *(NA1)* (no arbitrage opportunities of the first kind) under a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ if the set $\mathcal{W}_1^\infty := \{1 + \int_0^\infty H_s \,\mathrm{d}B_s \,:\, H \in \mathcal{H}_1\}$ is bounded in probability, that is if $\lim_{n\to\infty} \sup_{X\in\mathcal{W}_1^\infty} \mathbb{P}(X \geq n) = 0$. The notion (NA1) has gained a lot of interest in recent years since it is the minimal condition which has to be satisfied by any reasonable asset price model; see for example [24, 30, 23].

The next proposition briefly collects further properties of $\overline{P}$.

Proposition 3.4 ([27, Proposition 3.3]).
1. *$\overline{P}$ is an outer measure with $\overline{P}(\Omega) = 1$, i.e. $\overline{P}$ is nondecreasing, countably subadditive, and $\overline{P}(\emptyset) = 0$.*
2. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process B is a $\mathbb{P}$-local martingale, and let $A \in \mathcal{F}$. Then $\mathbb{P}(A) \leq \overline{P}(A)$.*
3. *Let $A \in \mathcal{F}$ be a null set, and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ such that the coordinate process B satisfies (NA1) under $\mathbb{P}$. Then $\mathbb{P}(A) = 0$.*

Especially, the last statement is of interest in robust mathematical finance because it says that every property which is satisfied by typical price paths holds quasi-surely for all probability measures fulfilling (NA1).

An essential ingredient to obtain our super-replication theorem for time-invariant derivatives is a very remarkable pathwise Dambis Dubins-Schwarz theorem as presented in [35]. In order to give its precise statement here, we recall the definition of time-superinvariant sets, cf. Section 3 in [35].

Definition 3.5. *A continuous non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ satisfying $f(0) = 0$ is said to be a* time-change. *The set of all time-changes will be denoted by $\mathcal{G}_0$, the group of all time-changes that are strictly increasing and unbounded will be denoted by $\mathcal{G}$. Given $f \in \mathcal{G}_0$ we define $T_f(\omega) := \omega \circ f$. A subset $A \subseteq \Omega$ is called* time-superinvariant *if for all $f \in \mathcal{G}_0$ it holds that*

$$T_f^{-1}(A) \subseteq A. \tag{3.1}$$

A subset $A \subseteq \Omega$ is called time-invariant *if* (3.1) *holds true for all $f \in \mathcal{G}$.*

We denote by $\mathbb{W}$ the Wiener measure on $(\Omega, \mathcal{F})$ and recall Vovk's pathwise Dambis Dubins-Schwarz theorem.

Theorem 3.6 ([35, Theorem 3.1]). *Each time-superinvariant set $A \subseteq \Omega$ satisfies $\overline{P}(A) = \overline{Q}(A) = \mathbb{W}(A)$.*

Proof. For every $A \subseteq \Omega$ Proposition 3.4 and Remark 3.2 imply $\mathbb{W}(A) \leq \overline{P}(A) \leq \overline{Q}(A)$. If A is additionally time-superinvariant, Theorem 3.1 in [35] says $\overline{Q}(A) = \mathbb{W}(A)$, which immediately gives the desired result. □

Let us now introduce the *normalizing time transformation* $\mathfrak{t}$ in the sense of [35]. For this purpose denote $\mathbb{D}_n := \{k2^{-n} \,:\, k \in \mathbb{Z}\}$ for $n \in \mathbb{N}$ and define the dyadic stopping times

$$\tau_0^n := \inf\{t \geq 0 \,:\, \omega(t) \in \mathbb{D}_n\} \quad \text{and} \quad \tau_k^n := \inf\{t \geq \tau_{k-1}^n \,:\, \omega(t) \in \mathbb{D}_n \text{ and } \omega(t) \neq \omega(\tau_{k-1}^n)\},$$

for $k \in \mathbb{N}$. For $n \in \mathbb{N}$ and each $\omega \in \Omega$ the discrete quadratic variation along these stopping times is given by

$$V_t^n(\omega) := \sum_{k=0}^\infty (B_{\tau_{k+1}^n \wedge t} - B_{\tau_k^n \wedge t})^2, \quad t \in \mathbb{R}_+.$$

For typical price paths $\omega \in \Omega$ the sequence $(V^n_\cdot(\omega))$ converges uniformly on compacts, see [35, Theorem 5.1]. We write

$$\overline{A}_t(\omega) := \limsup_n V^n_t(\omega), \qquad \underline{A}_t(\omega) := \liminf_n V^n_t(\omega), \qquad \langle B \rangle_t(\omega) := \underline{A}_t(\omega),$$

and follow [35] in defining the sequence of stopping times

$$\tau_t(\omega) := \inf\left\{ s \geq 0 : \overline{A}_{\restriction[0,s)}(\omega) = \underline{A}_{\restriction[0,s)}(\omega) \in C[0,s) \text{ and } \sup_{u<s} \overline{A}_u(\omega) = \sup_{u<s} \underline{A}_u(\omega) \geq t \right\}$$

for $t \in \mathbb{R}_+$ and $\tau_\infty := \sup_n \tau_n$. The *normalizing time transformation* $\mathfrak{t}\colon C(\mathbb{R}_+) \to \mathbb{R}^{\mathbb{R}_+}$ is given by

$$\mathfrak{t}(\omega)_t := \omega(\tau_t), \quad t \in \mathbb{R}_+, \tag{3.2}$$

where we set $\omega(\infty) := 0$ for all $\omega \in \Omega$. Note that $\mathfrak{t}(\omega)_\cdot$ stays constant from time $\langle B \rangle_\infty(\omega)$ on (which is of course only relevant if that time is finite). Below we shall also use $\mathfrak{t}\colon C[0,1] \to \mathbb{R}^{\mathbb{R}_+}$ which is defined analogously. On the product space $\Omega \times \mathbb{R}_+ = C(\mathbb{R}_+) \times [0,\infty)$ we further introduce

$$\bar{\mathfrak{t}}(\omega, t) := (\mathfrak{t}(\omega), \langle B \rangle_t(\omega)).$$

Lemma 3.7. *If A is a predictable subset of $\Omega \times [0,\infty)$, then $\bar{\mathfrak{t}}^{-1}(A)$ is predictable as well.*

Proof. It suffices to verify the statement for $A = B \times (s,t]$, where $0 \leq s < t < \infty$ and $B \in \mathcal{F}_s$, and in that case the predictability of $\bar{\mathfrak{t}}^{-1}(A)$ is an easy consequence of Proposition 4.4 in [3]. □

Finally, we also need to keep track of the time where the quadratic variation ceases to exist:

$$\sigma_\infty(\omega) := \inf\left\{ t \geq 0 : \overline{A}_t(\omega) \neq \underline{A}_t(\omega) \text{ or } \overline{A}_t(\omega) = \infty \text{ or } \overline{A}_{t+}(\omega) \neq \overline{A}_{t-}(\omega) \right\}.$$

Note that due to the fact that $(\mathcal{F}_t)$ is not right-continuous, σ_∞ is not a stopping time but only an optional time (that is $\{\sigma_\infty < t\} \in \mathcal{F}_t$ but not $\{\sigma_\infty \leq t\} \in \mathcal{F}_t$). We also define σ_∞ on $C[0,1]$ in the same way.

We are now ready to state the main result of [35]:

Theorem 3.8 ([35, Theorem 6.4])**.** *For any bounded and nonnegative Borel measurable function $F\colon \Omega \to \mathbb{R}$, one has*

$$\overline{\mathbb{E}}[F \circ \mathfrak{t},\, \sigma_\infty = \infty, \langle B \rangle_\infty = \infty] = \int_\Omega F \, \mathrm{d}\mathbb{W},$$

where $\overline{\mathbb{E}}$ is the obvious extension of $\overline{P}$ from sets to nonnegative functions, $F(\mathfrak{t}(\omega)) := 0$ for all $\omega \notin \mathfrak{t}^{-1}(\Omega)$, and $\langle B \rangle_\infty := \sup_{t \geq 0} \langle B \rangle_t$.

4. Duality for one period

Here we are interested in a one period duality result for derivatives G on $C[0,1]$ of the form $\omega \mapsto G(\omega, \langle \omega \rangle_1)$, which are invariant under suitable time changes of ω. Typical examples for such derivatives are the running maximum up to time 1 or functions of the quadratic variation. Formally, this amounts to

$$G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$$

for some predictable process $(\tilde{G}_t)_{t \geq 0}$ on $(\Omega, (\mathcal{F}_t)_{t \geq 0})$, and more specifically we will focus on processes $\tilde{G}$ which are of the form $\tilde{G}_t(\omega) = \gamma(\omega_{\restriction[0,t]}, t)$, where $\gamma\colon \Upsilon \to \mathbb{R}$ is an upper semi-continuous functional which is bounded from above. Here we wrote Υ for the space of *stopped paths*

$$\Upsilon = \{(f,s) : f \in C[0,s], s \in \mathbb{R}_+\},$$

equipped with the distance d_Υ which is defined for $s < t$ by

$$d_\Upsilon((f,s),(g,t)) = \max\Big(t - s, \sup_{0 \leq u \leq s} |f(u) - g(u)|, \sup_{s \leq u \leq t} |g(u) - g(s)| \Big), \tag{4.1}$$

and which turns Υ into a Polish space. Given $\omega \in \Omega = C(\mathbb{R}_+)$ we will often identify it with an element of Υ and abuse notation by writing $(\omega, t) := (\omega_{\upharpoonright[0,t]}, t)$.

From now we make the following convention: For $\gamma\colon \Upsilon \to \mathbb{R}$ we write

$$\gamma \circ \bar{\mathfrak{t}}(\omega, t) := \mathbf{1}_{\Omega\times\mathbb{R}_+}(\bar{\mathfrak{t}}(\omega, t)) \cdot (\gamma \circ \bar{\mathfrak{t}})(\omega, t), \qquad (\omega, t) \in C[0, 1] \times [0, 1],$$

where we interpret the first component of $\bar{\mathfrak{t}}(\omega, t)$ as an element of $\mathbb{R}^{\mathbb{R}_+}$ by keeping it constant from time $\langle B\rangle_t(\omega)$ on. Similarly for $G\colon \Omega \to \mathbb{R}$ we set $G \circ \mathfrak{t}(\omega) := \mathbf{1}_\Omega(\mathfrak{t}(\omega))(G \circ \mathfrak{t})(\omega)$.

Given a centered probability measure μ on $\mathbb{R}$, we want to solve the primal maximization problem

$$P := \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \text{ is a martingale measure on } C[0, 1] \text{ s.t. } S_1(\mathbb{P}) = \mu]\}, \tag{4.2}$$

where S denotes the canonical process on $C[0, 1]$.

Since μ satisfies $\int |x| \,\mathrm{d}\mu(x) < \infty$, there exists a smooth convex function $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ with $\varphi(0) = 0$, $\lim_{x\to\pm\infty} \varphi(x)/|x| = \infty$, and such that $\int \varphi(x) \,\mathrm{d}\mu(x) < \infty$ (apply for example the de la Vallée-Poussin Theorem). From now on we fix such a function φ and we define

$$\zeta_t(\omega) := \mathbf{1}_{\{\sigma_\infty\geq t\}}(\omega)\frac{1}{2}\int_0^t \varphi''(S_s(\omega)) \,\mathrm{d}\langle S\rangle_s(\omega) + \mathbf{1}_{\{\sigma_\infty<t\}}(\omega) \cdot \infty, \qquad (\omega, t) \in C[0, 1] \times [0, 1],$$

where we make the convention $0 \cdot \infty := 0$. We then consider for $\alpha, c > 0$ the set of simple strategies

$$Q_{\alpha,c} := \{H : H \text{ is a simple strategy s.t. } (H \cdot S)_t(\omega) \geq -c - \alpha\zeta_t(\omega)\ \forall(\omega, t) \in C[0, 1]\times[0, 1]\}.$$

We also define the set of "European options available at price 0":

$$\mathcal{E}^0 := \Big\{\psi \in C(\mathbb{R}) : \frac{|\psi|}{1 + \varphi} \text{is bounded, } \int \psi(x) \,\mathrm{d}\mu(x) = 0\Big\}.$$

Theorem 4.1. *Let $\gamma\colon \Upsilon \to \mathbb{R}$ be Υ- upper semi-continuous and bounded from above and let $\tilde{G}_t = \gamma(\omega, t)$, $G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$. Put*

$$D := \inf\left\{p : \begin{array}{l} \exists c, \alpha > 0, (H^n) \subseteq Q_{\alpha,c}, \psi \in \mathcal{E}^0 \text{ s.t. } \forall\omega \in C[0, 1] \\ p + \liminf_n(H^n \cdot S)_1(\omega) + \psi(S_1(\omega)) \geq G(\omega) \end{array}\right\}.$$

Then we have the duality relation

$$P = D. \tag{4.3}$$

The inequality $P \leq D$ is easy: If $p > P$, then there exists a sequence $(H^n) \subseteq Q_{\alpha,c}$ and a $\psi \in C(\mathbb{R})$ with $\int \psi(x) \,\mathrm{d}\mu(x) = 0$ such that $p + \liminf_n(H^n \cdot S)_1(\omega) + \psi(S_1(\omega)) \geq G(\omega)$. In particular, for all martingale measures $\mathbb{P}$ on $C[0, 1]$ with $S_1(\mathbb{P}) = \mu$ we have

$$\mathbb{E}_{\mathbb{P}}[G] \leq \mathbb{E}_{\mathbb{P}}[p + \liminf_n(H^n \cdot S)_1 + \psi(S_1)] \leq p + \liminf_n \mathbb{E}_{\mathbb{P}}[(H^n \cdot S)_1] + \mathbb{E}_{\mathbb{P}}[\psi(S_1)] \leq p,$$

where in the second step we used Fatou's lemma, which is justified because $(H^n \cdot S)_1$ is uniformly bounded from below by $-c - \alpha\zeta_1$ and from Itô's formula we get $\mathbb{P}$-almost surely

$$\varphi(S_t) = \int_0^t \varphi'(S_s) \,\mathrm{d}S_s + \zeta_t,$$

which shows that ζ is the compensator of the $\mathbb{P}$-submartingale $\varphi(S)$ and therefore $\mathbb{E}_{\mathbb{P}}[\zeta_1] < \infty$.

In the following we concentrate on the inequality $P \geq D$. The idea, going back to Hobson [17], is to translate the primal problem to that of finding a solution to the Skorokhod embedding problem which is in a certain sense optimal. Let us start by observing that if $\mathbb{P}$ is a martingale measure for S, then by the Dambis-Dubins-Schwarz theorem the process $(\mathfrak{t}(S)_{t\wedge\langle S\rangle_1})_{t\geq0}$ is a stopped Brownian motion under $\mathbb{P}$ in the filtration $(\mathcal{F}^S_{\tau_t})_{t\geq0}$, where $(\mathcal{F}^S_t)_{t\in[0,1]}$ is the canonical filtration generated by S and where $(\tau_t)_{t\geq0}$ are the stopping times defined in (3.2). It is then straightforward to verify that $\langle S\rangle_1 = \tau(\mathfrak{t}(S))$, where τ is a stopping time with respect to the filtration $(\mathcal{F}^S_{\tau_t})$. Moreover,

$$\mathbb{E}_{\mathbb{P}}[\mathfrak{t}(S)_{\langle S\rangle_1}|\mathcal{F}_{\tau_t}] = \mathbb{E}_{\mathbb{P}}[S_1|\mathcal{F}_{\tau_t}] = S_{\tau_t} = \mathfrak{t}(S)_{t\wedge\langle S\rangle_1},$$

where we used that S and $\langle S\rangle$ have $\mathbb{P}$-almost surely the same intervals of constancy. To conclude, we arrive at the following observation:

Lemma 4.2. *The value P defined in* (4.2) *is given by*

$$P = P^* := \sup\left\{\mathbb{E}_{\mathbb{W}}[\gamma((B_s)_{s\le\tau})] : \begin{array}{l}(\bar{\mathcal{F}}_t)_{t\ge0}\in\mathfrak{F}, \mathbb{W}\in\mathfrak{W}(\bar{\mathcal{F}}_t), \tau\in\mathfrak{T}((\bar{\mathcal{F}}_t)),\\ B_\tau\sim\mu, B_{\cdot\wedge\tau} \text{ is a u.i. martingale}\end{array}\right\}, \tag{4.4}$$

where $\mathfrak{F}$ denotes the set of all filtrations of $(\Omega,\mathcal{F})$ to which B is adapted, $\mathfrak{W}(\bar{\mathcal{F}}_t)$ is the set of all probability measures on $(\Omega, \bigvee_{t\ge0}\bar{\mathcal{F}}_t)$ for which B is a $(\bar{\mathcal{F}}_t)$ Brownian motion, and $\mathfrak{T}((\bar{\mathcal{F}}_t))$ denotes the set of $(\bar{\mathcal{F}}_t)$-stopping times.

For what follows it will be convenient to fix a nice version of the conditional expectation with respect to the Wiener measure.

Definition 4.3. *Let $X\colon C(\mathbb{R}_+)\to\mathbb{R}$ be a measurable function which is bounded or positive. Then we define $\mathbb{E}_{\mathbb{W}}[X|\mathcal{F}_t]$ to be the unique $\mathcal{F}_t$-measurable function satisfying*

$$\mathbb{E}_{\mathbb{W}}[X|\mathcal{F}_t](\omega) = \int X((\omega_{\restriction[0,t]})\oplus\tilde\omega)\,\mathbb{W}(\mathrm{d}\tilde\omega),$$

where $\omega_{\restriction[0,t]}$ denotes the restriction of ω to the interval $[0,t]$, and where $(\omega_{\restriction[0,t]})\oplus\tilde\omega$ is the concatenation of $\omega_{\restriction[0,t]}$ and $\tilde\omega$, that is $(\omega_{\restriction[0,t]})\oplus\tilde\omega(r) := \mathbf{1}_{r\le t}\omega(r) + \mathbf{1}_{r>t}(\omega(t)+\tilde\omega(r-t))$. Then $\mathbb{E}_{\mathbb{W}}[X|\mathcal{F}_t](\omega)$ depends only on $\omega_{\restriction[0,t]}$, and in particular we can (and will) interpret the conditional expectation also as a function on $C[0,t]$.

Proposition 4.4 ([3, Proposition 4.11])**.** *Let $X\in C_b(C(\mathbb{R}_+))$. Then $X_t(\omega) := \mathbb{E}_{\mathbb{W}}[X|\mathcal{F}_t](\omega)$ defines a Υ-continuous martingale on $(\Omega,(\mathcal{F}_t),\mathbb{W})$.*

We need the following result from [3]:

Theorem 4.5. *Let $\gamma\colon\Upsilon\to\mathbb{R}$ be Υ-upper semi-continuous and bounded from above. Put*

$$D^* := \inf\left\{p : \begin{array}{l}\exists\alpha\ge0, \psi\in\mathcal{E}^0, m\in C_b(\Omega) \text{ s.t. } \mathbb{E}_{\mathbb{W}}[m]=0 \text{ and } \forall(\omega,t)\in\Omega\times\mathbb{R}_+\\ p+\mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_t](\omega)+\alpha Q(\omega,t)+\psi(B_t(\omega))\ge\gamma(\omega,t)\end{array}\right\},$$

where we wrote $Q(\omega,t) := (\varphi(B_t(\omega)) - 1/2\int_0^t\varphi''(B_s(\omega))\,\mathrm{d}s)$. Let P^* be as defined in (4.4)*. Then one has*

$$P^* = D^*.$$

Let now $p>P=P^*$. Then the previous theorem gives us a function $\psi\in\mathcal{E}^0$, a constant $\alpha\ge0$, and a continuous bounded function $m\colon\Omega\to\mathbb{R}$ with $\mathbb{E}_{\mathbb{W}}[m]=0$ such that for all $(\omega,t)\in\Omega\times\mathbb{R}_+$

$$M_t(\omega) := \mathbb{E}_{\mathbb{W}}[m|\mathcal{F}_t](\omega) \ge -p-\psi(B_t(\omega))-\alpha Q(\omega,t)+\gamma(\omega,t). \tag{4.5}$$

Consider now the functional $\tilde m\colon\Omega\to\mathbb{R}$ given by

$$\tilde m := m\circ\mathrm{t} := \mathbf{1}_{\mathrm{t}^{-1}(\Omega)}m\circ\mathrm{t},$$

which is $\mathcal{G}$-invariant, i.e. invariant under all strictly increasing time changes, and satisfies $\mathbb{E}_{\mathbb{W}}[\tilde m] = \mathbb{E}_{\mathbb{W}}[m] = 0$. Denote by m_0 the supremum of $|m(\omega)|$ over all $\omega\in\Omega$. Then $m_0+m\ge0$, and if we fix $\varepsilon>0$ and apply Theorem 3.8 in conjunction with Remark 3.2, we obtain a sequence of simple strategies $(\tilde H^n)\subseteq\mathcal{H}_{m_0+\varepsilon}$ such that

$$\liminf_{t\to\infty}\liminf_{n\to\infty}\varepsilon+(\tilde H^n\cdot B)_t(\omega)\ge\tilde m(\omega)\mathbf{1}_{\{\sigma_\infty=\infty,\langle B\rangle_\infty=\infty\}}(\omega),\qquad\omega\in\Omega.$$

By stopping we may suppose without loss of generality that $(\tilde H^n\cdot B)_t(\omega)\le m_0$ for all (ω,t). Set

$$\tilde M_t(\omega) := (M\circ\bar{\mathrm{t}})(\omega,t) := \mathbf{1}_{\Omega\times\mathbb{R}_+}(\bar{\mathrm{t}}(\omega,t))(M\circ\bar{\mathrm{t}})(\omega,t),\qquad(\omega,t)\in\Omega\times\mathbb{R}_+.$$

Lemma 4.6. *For all $(\omega,t)\in\Omega\times\mathbb{R}_+$ we have*

$$\varepsilon+\liminf_{n\to\infty}(\tilde H^n\cdot B)_t(\omega)\ge\mathbf{1}_{\{\sigma_\infty\ge t,\tau_\infty>t\}}(\omega)\tilde M_t(\omega).$$

Proof. We claim that $(M \circ \bar{\mathrm{t}})_t \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}} = \mathbb{E}[\mathbf{1}_{\{\sigma_\infty = \infty, \langle B \rangle_\infty = \infty\}} m \circ \mathrm{t} \,|\mathcal{F}_t]$. Indeed we have

$$[(M \circ \bar{\mathrm{t}})\mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}](\omega_{\restriction[0,t]} \oplus \tilde{\omega}, t) = \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega_{\restriction[0,t]} \oplus \tilde{\omega}) M_{\langle B \rangle_t}(\,\mathrm{t}(\omega_{\restriction[0,t]} \oplus \tilde{\omega})),$$

where the latter quantity actually does not depend on $\tilde{\omega}$, i.e. with a slight abuse of notation we may write it as $[\mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}} M_{\langle B \rangle_t} \circ \mathrm{t}](\omega_{\restriction[0,t]})$. Also, we have

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\sigma_\infty = \infty, \langle B \rangle_\infty = \infty\}} m \circ \mathrm{t} \,|\mathcal{F}_t](\omega_{\restriction[0,t]}) \\
&= \int \mathbf{1}_{\{\sigma_\infty = \infty, \langle B \rangle_\infty = \infty\}}(\omega_{\restriction[0,t]} \oplus \tilde{\omega}) \mathbf{1}_\Omega(\mathrm{t}(\omega_{\restriction[0,t]} \oplus \tilde{\omega}))(m \circ \mathrm{t})(\omega_{\restriction[0,t]} \oplus \tilde{\omega}) \, \mathbb{W}(\mathrm{d}\tilde{\omega}) \\
&= \int \mathbf{1}_{\{\sigma_\infty = \infty, \langle B \rangle_\infty = \infty\}}(\tilde{\omega}) \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega_{\restriction[0,t]}) \mathbf{1}_\Omega(\mathrm{t}(\omega_{\restriction[0,t]})) \mathbf{1}_\Omega(\mathrm{t}(\tilde{\omega})) m(\,\mathrm{t}(\omega_{\restriction[0,t]}) \oplus \mathrm{t}(\tilde{\omega})) \, \mathbb{W}(\mathrm{d}\tilde{\omega}) \\
&= \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega_{\restriction[0,t]}) \mathbf{1}_\Omega(\mathrm{t}(\omega_{\restriction[0,t]})) \int m(\,\mathrm{t}(\omega_{\restriction[0,t]}) \oplus \tilde{\omega}) \, \mathbb{W}(\mathrm{d}\tilde{\omega}) \\
&= \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega_{\restriction[0,t]}) M_{\langle B \rangle_t}(\,\mathrm{t}(\omega_{\restriction[0,t]})).
\end{aligned}$$

Writing $(\tilde{H}^n \cdot B)_t^\infty = (\tilde{H}^n \cdot B)_\infty - (\tilde{H}^n \cdot B)_t$, we thus find

$$\begin{aligned}
\mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}} \tilde{M}_t &= \mathbb{E}[\mathbf{1}_{\{\sigma_\infty = \infty, \langle B \rangle_\infty = \infty\}} \tilde{m} | \mathcal{F}_t] \leq \varepsilon + \mathbb{E}[\liminf_n (\tilde{H}^n \cdot B)_s | \mathcal{F}_t] \\
&= \varepsilon + \mathbb{E}[\liminf_s \liminf_n ((\tilde{H}^n \cdot B)_t + (\tilde{H}^n \cdot B)_t^s) | \mathcal{F}_t] \\
&= \varepsilon + \liminf_n (\tilde{H}^n \cdot B)_t + \mathbb{E}[\liminf_s \liminf_n (\tilde{H}^n \cdot B)_t^s | \mathcal{F}_t].
\end{aligned}$$

Now it is easily verified that $(\liminf_n (\tilde{H}^n \cdot B)_t^s)_{s \geq t}$ is a bounded $\mathbb{W}$-supermartingale started in 0 (recall that $-m_0 - \varepsilon \leq (\tilde{H}^n \cdot B)_s \leq m_0$ for all s, which yields $|(\tilde{H}^n \cdot B)_t^s| \leq 2m_0 + \varepsilon$ for all s), and therefore the conditional expectation on the right hand side is nonpositive, which concludes the proof. □

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. Lemma 4.6 and (4.5) show that

$$\varepsilon + \liminf_n (\tilde{H}^n \cdot B)_t(\omega) \geq \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega)(\,-p - \psi((\bar{\mathrm{t}}B)(\omega,t)) - \alpha Q \circ \bar{\mathrm{t}}(\omega,t) + \gamma \circ \bar{\mathrm{t}}(\omega,t))$$

for all $(\omega,t) \in \Omega \times \mathbb{R}_+$. Noting that $\psi(\bar{\mathrm{t}}B(\omega,t)) = \mathbf{1}_{\Omega \times [0,\infty)}(\bar{\mathrm{t}}(\omega,t))\psi(B_t(\omega))$ and

$$Q \circ \bar{\mathrm{t}}(\omega,t) = \mathbf{1}_{\Omega \times [0,\infty)}(\bar{\mathrm{t}}(\omega,t))(\varphi(B_t(\omega)) - \zeta_t(\omega)),$$

we get

$$\begin{aligned}
&p + \varepsilon + \liminf_n (\tilde{H}^n \cdot B)_t(\omega) \\
&\qquad \geq \mathbf{1}_{\{\sigma_\infty \geq t, \tau_\infty > t\}}(\omega) \mathbf{1}_{\Omega \times [0,\infty)}(\bar{\mathrm{t}}(\omega,t))[\,-\psi(B_t(\omega)) - \alpha(\varphi(B_t(\omega)) - \zeta_t(\omega))) + \gamma \circ \bar{\mathrm{t}}(\omega,t)].
\end{aligned}$$

Theorem 5.1 in [35] shows that the complement of the set in the indicator function on the right hand side has outer measure 0, and therefore we obtain a new sequence of simple strategies $(G^n) \subseteq \mathcal{H}_{m_0+2\varepsilon}$ such that

$$p + 2\varepsilon + \liminf_n (G^n \cdot B)_t(\omega) + \psi(B_t(\omega)) + \alpha(\varphi(B_t(\omega)) - \zeta_t(\omega)) \geq \gamma \circ \bar{\mathrm{t}}(\omega,t)$$

for all $(\omega,t) \in \Omega \times \mathbb{R}_+$. It now suffices to apply Föllmer's pathwise Itô formula [14] along the dyadic Lebesgue partition defined in Section 3 to obtain a sequence of simple strategies $(G^n) \subseteq \mathcal{Q}_{1,\alpha}$ such that $\liminf_n(\varepsilon + (G^n \cdot B)_t(\omega)) \geq \alpha(\varphi(B_t(\omega)) - \zeta_t(\omega))$ for all (ω,t), and, hence, we have established that there exist $(H^n) \subseteq \mathcal{Q}_{m_0+2,\alpha}$ and $\psi \in \mathcal{E}^0$ such that

$$p + 3\varepsilon + \liminf_n (H^n \cdot B)_t(\omega) + \psi(B_t(\omega)) \geq \gamma \circ \bar{\mathrm{t}}(\omega,t)$$

for all $(\omega,t) \in C(\mathbb{R}_+) \times \mathbb{R}_+$. Now the functionals on both sides are adapted (see Lemma 3.7), so for fixed time t we can consider them as functionals on $C[0,t]$, and thus the inequality holds in particular for all $(\omega,t) \in C[0,1] \times [0,1]$. Since $p > P$ and $\varepsilon > 0$ are arbitrarily small, we deduce that $D \leq P$ and thus that $D = P$. □

Remark 4.7. We observe that the requirement that $G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$ in Theorem 4.1 can now easily be weakened to require only $G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$ outside of a $\overline{P}$-null set, since by Lemma 3.3 we can find a sequence of simple strategies with arbitrarily small cost which superhedge the payoff on the set where $G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$. One particular case where this difference is important is on the class of paths with smooth sections. If a path contains an interval on which the path is smooth, starts and ends at the same point, and is not constant, then the normalising time transformation will simply cut out this section of the path. In some cases (e.g. when the payoff depends on the running maximum), then $G = \tilde{G} \circ \bar{\mathfrak{t}}(\cdot, 1)$ may not hold for such paths. However it is easily checked that these paths form an atypical set, and hence may be ignored in computing the super-hedging price.

5. Duality in the multi-marginal case

In this section, we will show a general duality result for the multi marginal Skorokhod embedding problem and moreover, for a slightly more general problem.

To do this we first recall some notions and results from the one-marginal case covered in [3, Section 4], in fact we shall also be interested in certain (relatively straightforward) extensions.

5.1. Revision of one-marginal duality.

We denote by $\mathcal{F}$ the natural and by $\mathcal{F}^a$ the augmented filtration on $C(\mathbb{R}_+)$. A process $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}$-predictable iff $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}$-optional iff $X_t(\omega)$ can be calculated from the restriction $\omega_{\restriction[0,t]}$, [3, Proposition 4.4]. We introduce the mapping

$$r\colon C(\mathbb{R}_+) \times \mathbb{R}_+ \to \Upsilon, \quad r(\omega, t) = (\omega_{\restriction[0,t]}, t). \tag{5.1}$$

Note that the topology on Υ introduced in (4.1) coincides with the final topology induced by the mapping r; in particular r is a continuous open mapping. A function $X\colon C(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$ is called Υ- (upper/lower semi-) continuous iff there exists a (upper/lower semi-) continuous function $H\colon \Upsilon \to \mathbb{R}$ such that $X = H \circ r$.

Our principle interest is in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C(\mathbb{R}_+)$ and $\mathbb{P} = \mathbb{W}$. In what follows, we will also use a natural extension of the filtered probability space denoted by $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0}, \bar{\mathbb{P}})$, where we take $\bar{\Omega} = \Omega\times[0,1]$, $\bar{\mathcal{F}} = \mathcal{F}\otimes\mathcal{B}([0,1])$, $\bar{\mathbb{P}}(A_1\times A_2) = \mathbb{P}(A_1)\mathcal{L}(A_2)$, and set $\bar{\mathcal{F}}_t = \mathcal{F}_t^a \otimes \sigma([0,1])$. Here, $\mathcal{L}$ denotes Lebesgue measure. We will write $\bar{B} = (\bar{B}_t)_{0\leq t}$ for the canonical process on $\bar{\Omega}$, i.e. $\bar{B}_t(\omega, u) = \omega_t$.

Given random times τ, τ' on $\bar{\Omega}$ and a bounded continuous function $f : C(\mathbb{R}_+)\times\mathbb{R}_+ \to \mathbb{R}$ we define

$$d_f(\tau, \tau') := \left|\bar{\mathbb{E}}[f(\omega, \tau(\omega, x)) - f(\omega, \tau'(\omega, x))]\right|.$$

We then identify τ and τ' if $d_f(\tau, \tau') = 0$ for all continuous bounded f. On the resulting space of equivalence classes denoted by RT, the family of semi-norms $(d_f)_f$ gives rise to a Polish topology. It follows from the characterization given in [3, Corollary 4.19] that for a stopping time τ on $\bar{\Omega}$ all elements of the respective equivalence class are stopping times. We will call this equivalence class, as well as (by abuse of notation) its representatives *randomized stopping times* (in formulae: RST).

Given a martingale M on $(C(\mathbb{R}_+), \mathbb{W})$ we write $\bar{M}$ for its extension to $\bar{\Omega}$, that is $\bar{M}_t(\omega, u) = M_t(\omega)$. A random time τ on $\bar{\Omega}$ is a *pseudo randomized stopping time* if

$$\bar{\mathbb{E}}\bar{M}_\tau = M_0$$

for all Υ-continuous bounded martingales. (As before, random times which are equivalent to a pseudo randomized stopping time are in turn pseudo randomized stopping times.)

A random time τ is a pseudo randomized stopping time iff its dual optional projection τ^o is a randomized stopping time. Moreover the set of pseudo randomized stopping times is a closed subset of RT, we refer to [3, Section 4.4] for details.

Fix a centered probability measure μ with finite first moment. As above, there exists a convex function φ such that $\varphi(0) = 0, \lim_{x\to\pm\infty} \varphi(x)/|x| = \infty$, and $\int \varphi(x)\,\mathrm{d}\mu(x) < \infty$.

Moreover, there exists a continuous compensating process $\zeta_t : \Upsilon \to \mathbb{R}$ such that $\varphi(B_t) - \zeta_t$ is a martingale under $\mathbb{W}$ (see [3, Proposition 8.3]). We write $\mathsf{RST}(\mu)$ (resp. $\mathsf{PRST}(\mu)$) for the sets of (pseudo) randomized stopping times which embed a given measure μ, and such that $\mathbb{E}[\zeta_\tau] < \infty$, this last condition also being equivalent to $\mathbb{E}[\zeta_\tau] = V$, for $V = \int \varphi(x)\,\mu(dx)$. It it is then not hard to show that these sets are *compact*, see [3, Theorem 4.28, Corollary 4.29 and Proposition 8.3].

Given an upper semi-continuous function $\gamma : \Upsilon \to \mathbb{R}$, in the previous section we were interested in the primal problem

$$P := \sup_{\tau \in \mathsf{RST}(\mu)} \bar{\mathbb{E}}\gamma_\tau,$$

where $\gamma_\tau = \gamma(\omega_{\restriction[0,\tau]}, \tau)$. Recall that we use the abbreviation $\gamma(\omega, t) = \gamma(\omega_{\restriction[0,t]}, t)$. Theorem 4.5 gave a duality result for this problem.

5.2. **Multi-marginal duality.** We now go on to extend these notions and results to the multi-marginal case. (In fact, in many cases the results follow exactly as in the one-marginal case.) In particular, we establish an extension of the duality result Theorem 4.5 to multiple marginals which will then imply the main result of this paper by a straightforward modification of the proof of the one-period result. To this end, we introduce the set of all randomized multi stopping times or n-tuples of randomized stopping times. As before we consider the space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$. We consider all n-tuples $\bar{\tau} = (\tau_1, \dots, \tau_n)$ with $\tau_1 \leq \ldots \leq \tau_n$ and $\tau_i \in \mathsf{RT}$ for all i. We identify two such tuples if

$$d_f(\bar{\tau}, \bar{\tau}') := \left|\bar{\mathbb{E}} f(\omega, \tau_1(\omega, x), \dots, \tau_n(\omega, x)) - \bar{\mathbb{E}} f(\omega, \tau'_1(\omega, x), \dots, \tau'_n(\omega, x))\right| \tag{5.2}$$

vanishes for all continuous, bounded $f : C(\mathbb{R}_+) \times \mathbb{R}^n_+ \to \mathbb{R}$ and denote the resulting space by RT_n. Moreover we consider RT_n as a topological space by testing against all continuous bounded functions as in (5.2). As above we need to consider pseudo randomized stopping times.

Analogously to the result of [3] recalled above we get

Proposition 5.1. *Let $k \leq n$.*

(1) If $\tau_1 \leq \cdots \leq \tau_k$ are stopping times, and $\tau_{k+1} \leq \cdots \leq \tau_n$ are pseudo stopping times on $\bar{\Omega}$ w.r.t. $\bar{\mathcal{F}}$, and $\tau'_1 \leq \ldots \leq \tau'_n$ get identified with them by (5.2), then $\tau'_1, \ldots \tau'_k$ are stopping times as well, and $\tau'_{k+1}, \dots, \tau'_n$ are pseudo-stopping times. We then say that $(\tau_1, \dots, \tau_n)$ is a (k, n)-pseudo multi-stopping time. If $k = n$, we simply say that $(\tau_1, \dots, \tau_n)$ is a multi-stopping time. We write $\mathsf{PRST}_{k,n}$ for the set of (k, n)-pseudo multi-stopping times, and $\mathsf{RST}_n = \mathsf{PRST}_{n,n}$.

(2) The set $\mathsf{PRST}_{k,n}$ of all (k, n)-pseudo multi-stopping times $(\tau_1 \leq \ldots \leq \tau_n)$ is closed for all $k \leq n$. In particular, the set RST_n of multi-stopping times is closed.

Below we will only be interested in the sets $\mathsf{PRST}_{k,n}$ for $k = n$, and $k = n - 1$.

Fix $I \subseteq \{1, \dots, n\}$ with $n \in I$ and $|I| \leq n$ measures $(\mu_i)_{i \in I} = \mu$ in convex order with finite first moment. If $i \in \{1, \dots, n\} \setminus I$, write $i+$ for the smallest element of $\{j \in I : j \geq i\}$. By an iterative application of the de la Vallée-Poussin Theorem, there is an increasing family of smooth, non-negative, strictly convex functions $(\varphi_i)_{i=1,\dots,n}$ (increasing in the sense that $\varphi_i \leq \varphi_j$ for $i \leq j$) such that $\varphi_{i+1}/\varphi_i \to \infty$ as $x \to \pm\infty$, and $\int \varphi_i d\mu_{i+} < \infty$ for all $i = 1, \dots, n$. Denote the corresponding compensating processes by ζ^i_t such that $Q_i(\omega, t) := \varphi_i(B_t(\omega)) - \zeta^i_t(\omega)$ is a martingale. We also write $\mathcal{E}_i := \left\{\psi \in C(\mathbb{R}) : \frac{|\psi|}{1+\varphi_i} \text{ is bounded}\right\}$.

Then, we define $\mathsf{RST}_n(\mu)$ (resp. $\mathsf{PRST}_{k,n}(\mu)$) to be the subset of RST_n (resp. $\mathsf{PRST}_{n,k}$) consisting of all tuples $(\tau_1 \leq \ldots \leq \tau_n)$ such that $B_{\tau_i} \sim \mu_i$ for all $i \in I$ and $\bar{\mathbb{E}}[\zeta^n_{\tau_n}] < \infty$. As a consequence of Proposition 5.1 and the compactness of $\mathsf{RST}(\mu_n)$ (resp. $\mathsf{PRST}(\mu_n)$) we get

Lemma 5.2. *For any $I \subseteq \{1, \dots, n\}$ with $n \in I$ and any family of measures $(\mu_i)_{i \in I} = \mu$ in convex order the set $\mathsf{RST}_n(\mu)$ (resp. the "pseudo" version $\mathsf{PRST}_{k,n}(\mu)$) is compact.*

We introduce the space of paths where we have stopped n times:

$$\Upsilon_n := \{(f, s_1, \ldots, s_n) \,:\, (f, s_n) \in \Upsilon, 0 \le s_1 \le \ldots \le s_n\},$$

equipped with the topology generated by the obvious analogue of (4.1). As a natural extension of an optional process, we say that a process $\gamma\colon \Omega \times \mathbb{R}_+^n$ is optional if for any family of stopping times $\tau_1 \le \cdots \le \tau_n$, the map $\gamma(\tau_1, \ldots, \tau_n)$ is $\mathcal{F}_{\tau_n}$-measurable. On Υ_n, the optional processes are functions $\gamma : \Upsilon_n \to \mathbb{R}$. Indeed, it is easy to show that a Borel function, $\gamma : \Upsilon_n \to \mathbb{R}$ is an optional process in the sense given above, and moreover, any optional process can be written in this way (see [3] for the case with one stopping time; the general case is then immediate).

The critical property (for our purposes) of pseudo multi-stopping times is the following result:

Lemma 5.3. *Let $(\tau_1, \ldots, \tau_n) \in \mathsf{PRST}_{n-1,n}(\mu)$. Then:*

(1) There exists a stopping time τ_n^o (on a possibly extended probability space) such that for all Borel measurable $\gamma : \Upsilon_n \to \mathbb{R}$

$$\bar{\mathbb{E}}\left[\gamma(\tau_1, \ldots, \tau_n)\right] = \bar{\mathbb{E}}\left[\gamma(\tau_1, \ldots, \tau_n^o)\right].$$

Moreover, $(\tau_1, \ldots, \tau_n^o) \in \mathsf{RST}_n(\mu)$.

(2) We have $(\tau_1, \ldots, \tau_n) \in \mathsf{PRST}_{n-1,n}$ if and only if $(\tau_1, \ldots, \tau_{n-1}) \in \mathsf{RST}_{n-1}$ and

$$\tau_n \in \left\{\tau \in \mathsf{RT} : \tau \ge \tau_{n-1}, \bar{\mathbb{E}}\left[\bar{M}_\tau\right] = \bar{M}_0 \text{ for all bounded } \Upsilon\text{-continuous martingales } (M_t)\right\}.$$

Proof. The first part of the proof follows once we take the dual optional projection of τ_n. Specifically, we get an increasing, optional process A_t^n such that $A^n_{\tau_{n-1}-} = 0$, and, since γ is optional,

$$\bar{\mathbb{E}}\left[\gamma(\tau_1, \ldots, \tau_n)\right] = \bar{\mathbb{E}}\left[\int_0^\infty \gamma(\tau_1, \ldots, \tau_{n-1}, s)\, \mathrm{d}A_s^n\right].$$

Identifying a randomized stopping time with its distribution function (for more details we refer to Section 4.3 of [3]) A_s^n is the law of a stopping time τ_n^o on an extended probability space, on which space the above equality holds also. By taking the special choice $\gamma(t_1, \ldots, t_n) = f(B_{t_n})$, we conclude that $B_{\tau_n} \sim B_{\tau_n^o}$, and since ζ_t^n is optional, we also have $\bar{\mathbb{E}}[\zeta^n_{\tau_n}] = \bar{\mathbb{E}}[\zeta^n_{\tau_n^o}]$.

The second half of the statement follows immediately from the definition of a pseudo-stopping time. □

Given $\gamma : \Upsilon_n \to \mathbb{R}$, we are interested in the following n-step primal problem

$$P := \sup\left\{\bar{\mathbb{E}}[\gamma(\omega, \tau_1, \ldots, \tau_n)] : (\tau_i)_{i=1}^n \in \mathsf{RST}_n(\mu)\right\} \tag{5.3}$$

and its relation to the dual problem

$$D := \inf\left\{a : \begin{array}{c} \text{there exist } (\psi_j)_{j\in I}, \text{ martingales } (M_i)_{i=1}^n, \bar{\mathbb{E}}[M_i] = 0, \int \psi_j d\mu_j = 0, \\ a + \sum_{j\in I} \psi_j(B_{t_j}(\omega)) + \sum_{i=1}^n M_i(\omega, t_i) \ge \gamma(\omega, t_1, \ldots, t_n) \end{array}\right\}. \tag{5.4}$$

Remark 5.4. **Important convention:** In the formulation of D in (5.4) and in the rest of this section $M_1, \ldots, M_n$ will range over Υ-continuous martingales such that $M_k(\omega, t) = \bar{\mathbb{E}}[m|\mathcal{F}_t^0] + Q(\omega, t)$ for some $m \in C_b(\Omega)$ and $Q(\omega, t) = f(B_t(\omega)) - \zeta_t^f(\omega)$ where f is a smooth function such that $|f|/(1+\varphi_k)$ is bounded, and ζ^f is the corresponding compensating process ($\zeta_t^f = \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s$). In addition, we assume that the functions $\psi_i \in \mathcal{E}_i$ for all $i \le n$.

Theorem 5.5. *Let $\gamma : \Upsilon_n \to \mathbb{R}$ be upper semicontinuous and bounded from above. Under the above assumptions we have $P = D$.*

As usual the inequality $P \le D$ is not hard to see. The proof of the opposite inequality is based on the following minmax theorem.

Theorem 5.6 (see e.g. [31, Thm. 45.8] or [2, Thm. 2.4.1])**.** *Let K, L be convex subsets of vector spaces H_1 resp. H_2, where H_1 is locally convex and let $F : K \times L \to \mathbb{R}$ be given. If*

(1) K is compact,
(2) $F(\cdot, y)$ is continuous and convex on K for every $y \in L$,
(3) $F(x, \cdot)$ is concave on L for every $x \in K$

then

$$\sup_{y\in L} \inf_{x\in K} F(x, y) = \inf_{x\in K} \sup_{y\in L} F(x, y).$$

The inequality $P \geq D$ will be proved inductively on n. To this end, we need the following preliminary result.

Theorem 5.7. *Let $c : \Upsilon_2 \to \mathbb{R}$ be upper semi-continuous and bounded from above. Let $V_2 := \int \varphi_2(x)\, d\mu_2(x)$. Put*

$$P^{V_2} := \sup\{\bar{\mathbb{E}}[c(\omega, \tau_1, \tau_2)] \ : \ \tau_1 \in \mathsf{RST}_1(\mu_1), \bar{\mathbb{E}}[\zeta^2_{\tau_2}] \leq V_2, (\tau_1, \tau_2) \in \mathsf{RT}_2\}$$

and

$$D^{V_2} := \inf\left\{\mu_1(\psi_1) \ : \ \begin{array}{l} \psi_1 \in \mathcal{E}_1, \text{ there exists a } \Upsilon\text{-continuous martingale } M_1, \alpha \geq 0 \\ M_1(\cdot, 0) = 0, \psi_1(\omega(t_1)) + M_1(\omega, t_1) - \alpha(V_2 - \zeta^2_{t_2}) \geq c(\omega, t_1, t_2) \end{array}\right\}.$$

Then, we have

$$P^{V_2} = D^{V_2}.$$

Proof. The inequality $P^{V_2} \leq D^{V_2}$ follows easily, so we are left to show the other inequality. Using standard approximation procedures (cf. [3, Lemma 5.6]), we can assume that c is continuous and bounded, bounded from above by 0 and satisfies for some L

$$\operatorname{supp}(c) \subseteq \{(f, s_1, s_2) \in \Upsilon^{\oplus 2}, s_2 \leq L\}.$$

Then, it follows that

$$\begin{aligned}
\sup_{\substack{\tau_1\in\mathsf{RST}_1(\mu_1)\\ \bar{\mathbb{E}}[\zeta^2_{\tau_2}]\leq V_2\\ (\tau_1,\tau_2)\in\mathsf{RT}_2}} \bar{\mathbb{E}}[c(\omega, \tau_1, \tau_2)] &= \sup_{\substack{\tau_1\in\mathsf{RST}_1(\mu_1)\\ \tau_2\leq\max\{L,\tau_1\}\\ (\tau_1,\tau_2)\in\mathsf{RT}_2}} \inf_{\alpha\geq 0} \bar{\mathbb{E}}[c(\omega, \tau_1, \tau_2) + \alpha(V_2 - \zeta^2_{\tau_2})] \\
&= \inf_{\alpha\geq 0} \sup_{\substack{\tau_1\in\mathsf{RST}_1(\mu_1)\\ \tau_2\leq\max\{L,\tau_1\}\\ (\tau_1,\tau_2)\in\mathsf{RT}_2}} \bar{\mathbb{E}}[c(\omega, \tau_1, \tau_2) + \alpha(V_2 - \zeta^2_{\tau_2})] \\
&= \inf_{\alpha\geq 0} \sup_{\tau_1\in\mathsf{RST}_1(\mu_1)} \bar{\mathbb{E}}[\bar{c}_\alpha(\omega, \tau_1)],
\end{aligned}$$

where

$$\bar{c}_\alpha(\omega, t_1) = \sup_{t_1\leq t_2\leq\max\{L,t_1\}} c(\omega, t_1, t_2) + \alpha(V_2 - \zeta^2_{t_2})$$

which is a continuous and bounded function since c is bounded, ζ^2 is Υ-continuous and increasing, and $\{t_2 \ : \ t_1 \leq t_2 \leq \max\{L, t_1\}\}$ is closed, and we used Theorem 5.6. Using Theorem 4.5 we get

$$\sup_{\substack{\tau_1\in\mathsf{RST}_1(\mu_1)\\ \bar{\mathbb{E}}[\zeta^2_{\tau_2}]\leq V_2\\ (\tau_1,\tau_2)\in\mathsf{RT}_2}} \bar{\mathbb{E}}[c(\omega, \tau_1, \tau_2)] = \inf_{\alpha\geq 0} \inf_{\psi_1+M_1\geq\bar{c}_\alpha} \int \psi_1\, d\mu_1 = D^{V_2}$$

where the final infimum is taken over terms of the form described in Remark 5.4. □

Proof of Theorem 5.5. By [3, Lemma 5.6] we can assume that γ is continuous and bounded. We will show the result inductively by including more and more constraints (resp. Lagrange multipliers) in the duality result Theorem 4.5. In fact, we will only show the result for the two cases $n = 2, I = \{2\}$ and $n = |I| = 2$. The general claim follows then by an iterative application of the arguments that lead to Theorem 5.7 and the arguments below. We first consider the case where $n = |I| = 2$.

Observe that, by Lemma 5.3, we have

$$\sup_{(\tau_1,\tau_2)\in \mathsf{RST}_2(\mu_1,\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)] = \sup_{(\tau_1,\tau_2)\in \mathsf{PRST}_{1,2}(\mu_1,\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)].$$

Moreover, by the characterising property of pseudo-random times, we can write this second term as an infimum over *random times* which satisfy a martingale constraint, and a distributional constraint. Hence, we have using Theorem 5.6 and the notation $X_t^M(\omega) := \bar{\mathbb{E}}[X|\mathcal{F}_t^0](\omega)$

$$\begin{aligned}
&\sup_{(\tau_1,\tau_2)\in \mathsf{RST}_2(\mu_1,\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)] \\
&\quad = \sup_{\substack{\tau_1\in \mathsf{RST}(\mu_1)\\ (\tau_1,\tau_2)\in \mathsf{RT}_2\\ \bar{\mathbb{E}}[\zeta_{\tau_2}^2]\le V_2}} \inf_{\substack{\psi_2\in C_b(\mathbb{R})\\ m_2\in C_b(\Omega)}} \bar{\mathbb{E}}\left[\gamma(\omega,\tau_1,\tau_2) - m_2^M(\omega,\tau_2) + \bar{\mathbb{E}}[m_2] - \psi_2(\omega(\tau_2)) + \int \psi_2\, d\mu_2\right] \\
&\quad = \inf_{\substack{\psi_2\in C_b(\mathbb{R})\\ m_2\in C_b(\Omega)}} \sup_{\substack{\tau_1\in \mathsf{RST}(\mu_1)\\ (\tau_1,\tau_2)\in \mathsf{RT}_2\\ \bar{\mathbb{E}}[\zeta_{\tau_2}^2]\le V_2}} \bar{\mathbb{E}}\left[\gamma_{\psi_2,m_2}(\omega,\tau_1,\tau_2)\right],
\end{aligned}$$

where $\gamma_{\psi_2,m_2}(\omega,t_1,t_2) := \gamma(\omega,t_1,t_2) - m_2^M(\omega,t_2) + \bar{\mathbb{E}}[m_2] - \psi_2(\omega(t_2)) + \int \psi_2\, d\mu_2 \in C_b(\Upsilon_2)$. Applying Theorem 5.7, we get

$$\begin{aligned}
&\sup_{(\tau_1,\tau_2)\in \mathsf{RST}_2(\mu_1,\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)] \\
&= \inf_{\substack{\psi_2\in C_b(\mathbb{R})\\ m_2\in C_b(\Omega)}} \inf \left\{ \int \psi_1 d\mu_1 : \begin{array}{l} \text{there exist a } \Upsilon\text{- continuous martingale } M_1, \\ M_1(\cdot,0)=0, \psi_1\in\mathcal{E}_1, \alpha\ge 0 \text{ such that:} \\ \psi_1(\omega(t_1)) + M_1(\omega,t_1) - \alpha(V_2-\zeta_{t_2}^2) \ge \gamma_{\psi_2,m_2}(\omega,t_1,t_2) \end{array} \right\} \\
&= \inf_{\psi_2\in C_b(\mathbb{R})} \inf \left\{ \int \psi_1 d\mu_1 + \int \psi_2 d\mu_2 : \begin{array}{l} \text{there exist two } \Upsilon\text{- continuous martingales } M_i, \\ M_i(\cdot,0)=0, \psi_1\in\mathcal{E}_1 \text{and } \alpha\ge 0 \text{ such that:} \\ \sum_{i=1}^2(\psi_i(\omega(t_i)) + M_i(\omega,t_i)) \\ -\alpha(V_2 - \varphi_2(\omega(t_2)) + \varphi_2(\omega(t_2)) - \zeta_{t_2}^2) \\ \qquad \ge \gamma(\omega,t_1,t_2) \end{array} \right\} \\
&= \inf_{\psi_1,\psi_2\in\mathcal{E}_1\times\mathcal{E}_2} \left\{ \int \psi_1 d\mu_1 + \int \psi_2 d\mu_2 : \begin{array}{l} \text{there exist two } \Upsilon\text{- continuous martingales } M_i, \\ M_i(\cdot,0)=0, \text{ such that:} \\ \sum_{i=1}^2(\psi_i(\omega(t_i)) + M_i(\omega,t_i)) \ge \gamma(\omega,t_1,t_2) \end{array} \right\} \\
&= D,
\end{aligned}$$

where in the final step we used the fact that $\bar{\mathbb{E}}[\varphi_2(B_{\tau_2})] = \bar{\mathbb{E}}[\zeta_{\tau_2}^2] = V_2$ and that $\varphi_2(B_t) - \zeta_t^2$ is a martingale.

For later use, we write:

$$D(\gamma) := \left\{ (\psi_1,\psi_2)\in\mathcal{E}_1\times\mathcal{E}_2 : \begin{array}{l} \text{there exist two } \Upsilon\text{-continuous martingales } M_i, M_i(\cdot,0)=0 \\ \text{such that: } \sum_{i=1}^2(\psi_i(\omega(t_i)) + M_i(\omega,t_i)) \ge \gamma(\omega,t_1,t_2) \end{array} \right\}$$

We now consider the case where $n=2, |I|=1$ and $I=\{2\}$, so we are prescribing μ_2 but not μ_1. Writing $\rho\preceq\nu$ to denote that ρ precedes ν in convex order, we use the result of the case where $|I|=2$ to see that:

$$\begin{aligned}
P = \sup_{(\tau_1,\tau_2)\in \mathsf{RST}_2(\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)] &= \sup_{\mu_1\preceq\mu_2} \sup_{(\tau_1,\tau_2)\in \mathsf{RST}_2(\mu_1,\mu_2)} \bar{\mathbb{E}}[\gamma(\omega,\tau_1,\tau_2)] \\
&= \sup_{\mu_1\preceq\mu_2} \inf_{(\psi_1,\psi_2)\in D(\gamma)} \left\{ \int \psi_1\, d\mu_1 + \int \psi_2\, d\mu_2 \right\}
\end{aligned}$$

We now need to introduce some additional compactness. Recall from the definitions of φ_i that $\varphi_2/\varphi_1 \to \infty$ as $x \to \pm\infty$. Now let $\varepsilon > 0$ and write

$$D^\varepsilon(\gamma^\varepsilon) := \left\{ (\psi_1^\varepsilon, \psi_2) : \begin{array}{l} \psi_1^\varepsilon + \varepsilon\varphi_2 \in \mathcal{E}_1, \psi_2 \in \mathcal{E}_2, \text{ and there exist two } \Upsilon\text{-continuous} \\ \text{martingales } M_i, M_i(\cdot, 0) = 0 \text{ such that:} \\ \psi_1^\varepsilon(\omega(t_i)) + \psi_2(\omega) + \sum_{i=1}^2 M_i(\omega, t_i)) \geq \gamma^\varepsilon(\omega, t_1, t_2) \end{array} \right\}.$$

In particular, we have $(\psi_1, \psi_2) \in D(\gamma) \iff (\psi_1 - \varepsilon\varphi_2, \psi_2) \in D^\varepsilon(\gamma - \varepsilon\varphi_2(\omega(t_1)))$ and so (with $\psi_1^\varepsilon = \psi_1 - \varepsilon\varphi_2, \gamma^\varepsilon = \gamma - \varepsilon\varphi_2(\omega(t_1))$)

$$\begin{aligned} \inf_{(\psi_1,\psi_2)\in D(\gamma)} \left\{ \int \psi_1 \, d\mu_1 + \int \psi_2 \, d\mu_2 \right\} &= \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \left\{ \int (\psi_1^\varepsilon + \varepsilon\varphi_2) \, d\mu_1 + \int \psi_2 \, d\mu_2 \right\} \\ &= \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \left\{ \int \psi_1^\varepsilon \, d\mu_1 + \int \psi_2 \, d\mu_2 \right\} + \varepsilon \int \varphi_2 \, \mu_1(dx). \end{aligned}$$

In particular, the final integral can be bounded over the set of $\mu_1 \leq \mu_2$, and so by taking $\varepsilon > 0$ small, this term can be made arbitrarily small. Moreover, by neglecting it we get a quantity that is smaller than P.

If we introduce the set

$$\mathsf{CV} = \{c : \mathbb{R} \to \mathbb{R} : c \text{ convex}, c(x) \geq 0, c \text{ smooth}, c(x) \leq L(1 + |x|), \text{ some } L \geq 0\},$$

then we may test the convex ordering property by penalising against CV. In particular, we can write after another application of Theorem 5.6

$$\begin{aligned} P &\geq \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \sup_{\mu_1\leq\mu_2} \left\{ \int \psi_1^\varepsilon \, d\mu_1 + \int \psi_2 \, d\mu_2 \right\} \\ &= \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \sup_{\mu_1} \inf_{c\in\mathsf{CV}} \left\{ \int (\psi_1^\varepsilon - c) \, d\mu_1 + \int (\psi_2 + c) \, d\mu_2 \right\} \end{aligned}$$

In addition, for fixed $\psi_1^\varepsilon \in D^\varepsilon(\gamma^\varepsilon)$, we observe that, by the fact that $\psi_1^\varepsilon + \varepsilon\varphi_2 \in \mathcal{E}_1$, we must have $\psi_1^\varepsilon(x) \to -\infty$ as $x \to \pm\infty$. Hence we can find a constant K, which may depend on ψ_1^ε, so that $\psi_1^\varepsilon(x) < \psi_1^\varepsilon(0)$ for all $x \notin [-K, K]$. In particular, we may restrict the supremum over measures μ_1 above to the set of probability measures $\mathcal{P}_K := \{\mu : \mu(\complement[-K, K]) = 0\}$, where $\complement A$ denotes the complement of the set A. Note that this set is compact, so we can then apply Theorem 5.6 to get:

$$\begin{aligned} &\inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \sup_{\mu_1\leq\mu_2} \left\{ \int \psi_1^\varepsilon \, d\mu_1 + \int \psi_2 \, d\mu_2 \right\} \\ &\qquad = \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \inf_{c\in\mathsf{CV}} \sup_{\mu_1\in\mathcal{P}_K} \left\{ \int (\psi_1^\varepsilon - c) \, d\mu_1 + \int (\psi_2 + c) \, d\mu_2 \right\} \\ &\qquad = \inf_{(\psi_1^\varepsilon,\psi_2)\in D^\varepsilon(\gamma^\varepsilon)} \inf_{c\in\mathsf{CV}} \left\{ \sup_{x\in[-K,K]} \left[\psi_1^\varepsilon(x) - c(x) \right] + \int (\psi_2 + c) \, d\mu_2 \right\} \end{aligned}$$

In particular, for any $\delta > 0$, we can find $(\psi_1^\varepsilon, \psi_2) \in D^\varepsilon(\gamma^\varepsilon)$ and $c \in \mathsf{CV}$ such that

$$P \geq \sup_{x\in\mathbb{R}} \left[\psi_1^\varepsilon(x) - c(x) \right] + \int (\psi_2 + c) \, d\mu_2 - \delta.$$

Take $\psi_2^\varepsilon(\omega(t_2)) := \sup_{x\in\mathbb{R}}\left[\psi_1^\varepsilon(x) - c(x)\right] + \psi_2(\omega(t_2)) + c(\omega(t_2)) + \varepsilon\varphi_2(\omega(t_2))$. Then there exist M_1, M_2 such that

$$\begin{aligned}
\gamma^\varepsilon(\omega, t_1, t_2) &\leq \psi_1^\varepsilon(\omega(t_1)) + \psi_2(\omega(t_2)) + \sum_{i=1}^{2} M_i(\omega, t_i) \\
&= \psi_2^\varepsilon(\omega(t_2)) + \sum_{i=1}^{2} M_i(\omega, t_i) - \varepsilon\varphi_2(\omega(t_2)) - c(\omega(t_2)) + c(\omega(t_1)) \\
&\quad + \left[\psi_1^\varepsilon(\omega(t_1)) - c(\omega(t_1))\right] - \sup_{x\in\mathbb{R}}\left[\psi_1^\varepsilon(x) - c(x)\right] \\
&\leq \psi_2^\varepsilon(\omega(t_2)) + \sum_{i=1}^{2} M_i(\omega, t_i) + \varepsilon(\varphi_2(\omega(t_1)) - \varphi^*(\omega(t_2))) \\
&\quad - c(\omega(t_2)) + c(\omega(t_1)) \\
&= \psi_2^\varepsilon(\omega(t_2)) + \sum_{i=1}^{2} M_i(\omega, t_i) \\
&\quad + \varepsilon\left[(\varphi_2(\omega(t_1)) - \zeta_{t_1}^2) - (\varphi_2(\omega(t_2)) - \zeta_{t_2}^2)\right] + \varepsilon(\zeta_{t_1}^2 - \zeta_{t_2}^2) \\
&\quad + \left[(c(\omega(t_1)) - \zeta_{t_1}^c) - (c(\omega(t_2)) - \zeta_{t_2}^c)\right] + (\zeta_{t_1}^c - \zeta_{t_2}^c)
\end{aligned}$$

Since ζ_t^2 is an increasing process, compensating φ_2, then $\zeta_{t_2} - \zeta_{t_1} \geq 0$ whenever $t_1 \leq t_2$. Similarly, ζ_t^c is the increasing process compensating c, and the same argument as above holds. Note that ζ^c is Υ-continuous since c is assumed smooth. It follows that $(\psi_1^\varepsilon, \psi_2) \in D^\varepsilon(\gamma^\varepsilon)$ implies $\psi_2^\varepsilon \in D'(\gamma)$, where

$$D'(\gamma) := \left\{\psi_2 \in \mathcal{E}_2 : \begin{array}{l} \text{there exist two } \Upsilon\text{-continuous martingales } M_i, M_i(\cdot, 0) = 0 \\ \text{such that } \psi_2(\omega(t_2)) + \sum_{i=1}^2 M_i(\omega, t_i) \geq \gamma(\omega, t_1, t_2) \end{array}\right\}.$$

It follows by making ε, δ small that

$$P \geq \inf_{\psi_2 \in D'(\gamma)} \int \psi_2 \, d\mu_2(x),$$

and as usual, the inequality in the other direction is easy.

To establish the claim in the general case we can now successively introduce more and more constraints accounting for more and more Lagrange multipliers and use either only the first or the first and the second argument to prove the full claim. □

To conclude, we can follow the reasoning of Section 4 and obtain for all $G\colon C[0, n] \to \mathbb{R}$ of the form

$$G(\omega) = \gamma(\mathfrak{t}(\omega)_{\restriction[0,\langle\omega\rangle_n]}, \langle\omega\rangle_1, \ldots, \langle\omega\rangle_n), \tag{5.5}$$

where γ is Υ_n-upper semi-continuous and bounded from above the following robust super-hedging result:

Theorem 5.8. *Let $I \subseteq \{1, \ldots, n\}$, $n \in I$, and consider*

$$P_n := \sup\{\mathbb{E}_{\mathbb{P}}[G] : \mathbb{P} \text{ is a Martingale measure on } C[0, n],\ S_0 = 0,\ S_i \sim \mu_i \text{ for all } i \in I\}$$

and

$$D_n := \inf\left\{a : \begin{array}{l} \text{there exist } H \text{ and } (\psi_j)_{j\in I}, \int \psi_j \, d\mu_j = 0 \text{ such that:} \\ a + \sum_{j\in I} \psi_j(S_j(\omega)) + (H \cdot S)_n \geq G((S_t)_{t\leq n}) \end{array}\right\}.$$

Under the above assumptions we have $P = D$.

Finally, we note that Theorem 5.8 could be further extended based on the above arguments. For example, we could include additional market information on prices of further options of the invariant form (5.5).

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