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11. Wavelets

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11 Wavelets

11.0 Introduction

Wavelets are families of basis functions that are represented by $f(x) = \sum_{k=0}^{d-1} c_k f(2x - k)$. The Fourier Transformation allow construction of waves with sines and cosines, wavelets allow contruction of waves with a solution of $\phi(x)$ and coefficients c_k.
The big difference is that wavelets good with dealing with signals (e.g. images) with local structures, because of this wavelets are best suited to be used in image compression and de-noising.

Figure 1: Every edge of the right image is a abrupt change in contrast in the signal of the image

11.1 Dilation

A Dilation equation is function that consists of linear combination of recursive shifted or scaled versions of itself.
(1)

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k) \quad (1)$$

Lemma 11.1. *If a dilation equation in which all the dilations are a factor of two reduction has a solution, then either the coeffcients on the right hand side of the equation sum to two or the integral* $\int_{-\infty}^{\infty} f(x)dx$ *of the solution is zero.*

1-norm also known as L_1-distance (DE: Summennorm) is the sum of absolute values of a vector components. $||x||_1 := \sum_{i=1}^{x} |x_i|$

2-norm also known as Euclidean norm is the Euclidean distance.
$||x||_2 := \sqrt{\sum_{i=1}^{x} |x_i|^2} = \sqrt{x_1^2 + \cdots + x_n^2}$

$$\sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(2x)dx = \frac{1}{2} \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(x)dx \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)dx \neq 0 \rightarrow \textbf{dividing both sides by } \int_{-\infty}^{\infty} f(x)dx \rightarrow \sum_{k=0}^{d-1} c_k = 2 \tag{3}$$

11.2 The Haar Wavelet

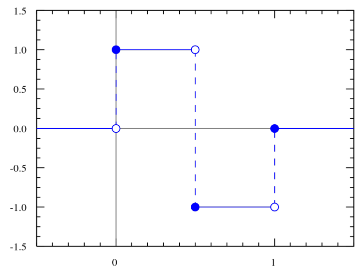

Figure 2: Haar Wavelet

The Haar Wavelet is a simple to understand wavelet with an easy to implement transformation. Main problems are that due to its rectangle form its *not continuous*, and therefore *not differentiable*.

$\phi(x)$ is called a *Scale function* or *Scale vector* that generates through its non-negative indexes j, k a 2 dimensional family of functions.

$$\begin{matrix} \phi_{00} & \phi_{01} & \phi_{02} & \phi_{03} \\ \phi_{10} & \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{20} & \phi_{21} & \phi_{22} & \phi_{23} \end{matrix}$$

Haar Wavelet function familiy from $j = [0, 3], k = [0, 2]$.

$\forall_j$, functions of $\phi_{jk} \mid k \geq 0$ form a basis for a vector space V_j and are orthogonal.
Basis vectors ϕ_{jk} form an *overcomplete*[1] basis. ϕ_{jk}, $\phi_{j+1,2k}$ and $\phi_{j+1,2k+1}$ are linear dependent.

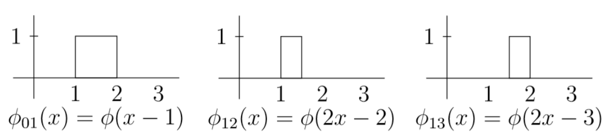

Figure 3: 3 linear dependent $\phi(x)$ functions: ϕ_{01}, ϕ_{12} and ϕ_{13}

j scaling factor

k shiftung factor

[1] every vector in space can be aproximated by a vector subset

For ortogonal set of Basis vectors:

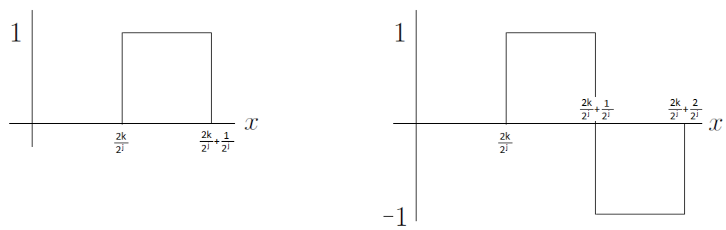

Figure 4: Replacment of $\phi_{j,2k} \rightarrow \ \psi_{j+1,2k}$

11.3 Wavelet Systems

General Wavelet systems are build from $\phi(x)$ which comes from dilation equation(1). Likewise the Haar-Wavelet, the scale function $\phi(x)$ is used to build a 2 dimensional function set.
To condensing the function family through removal of the unnecessary linear combination functions, which span the vector spaces V_j:$V_0 \cdots V_{d-1}$, one have to solve the dilation Equation.

11.4 Solving the Dilation Equation

11.4.1 Approach 1

For the solving of the dilation equation, using a *Daubechies scale function* to approximate the coefficients c_k. The more coefficients, the higher the resolution. (More coefficients mean one can better characterize oscillatory behavior in a signal).

11.4.2 Approach 2

Compute $\phi(\frac{i}{2^j})$ with chosen j for the dilation equation $\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2)$ with desired resolution.

11.5 Conditions on the Dilation Equation

Lemma 11.2. *Let*

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k) \tag{4}$$

If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k \neq 0$ and $\phi(x)$ has been normalized so that $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$, then $\sum_{i=0}^{d-1} c_i c_{i-2k} = 2\delta(k)$.

Lemma 11.2 does not cover all necessary conditions for the coefficients to be orthogonal. Lemma 11.2 conditions are not true for *triangular* or *piecewise quadratic* functions.

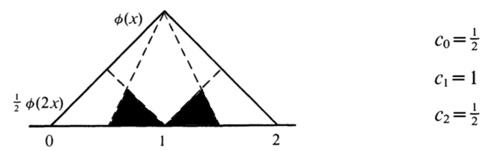

Figure 5: Black marked areas of $\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2)$ overlap

Lemma 11.3. *If $0 \leq x \leq d$ is the support of $\phi(x)$, and the set of integer shifts, $\{\phi(x-k) | k \geq 0\}$, are linearly independent, then $c_k = 0$ unless $0 \leq k \leq d-1$.*

11.6 Derivation of the Wavelets from the Scaling Function

Lemma 11.4. *(Orthogonality of $\phi(x)$ and $\phi(x-k)$) Let $\phi(x) = \sum_{k=0}^{d-1} b_k \phi(2x-k)$. If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k \neq 0$ and $\phi(x)$ has been normalized so that $\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(x)$, then*

$$\sum_{i=0}^{d-1} (-1)^k b_i b_{i-2k} = 2\delta(k) \tag{5}$$

Lemma 11.5. *(Orthogonality of $\phi(x)$ and $\phi(x-k)$) Let $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$ and $\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$. If $\int_{x=-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$ and $\int_{x=-\infty}^{\infty} \phi(x)\phi(x-k)dx = 0$ For all k, then $\sum_{i=0}^{d-1} c_i b_{i-2k} = 0$ for all k.*

Lemma 11.6. *Let the scale function $\phi(x)$ equal $\sum_{k=0}^{d-1} c_k \phi(2x-k)$ and let the wavelet function $\phi(x)$ equal $\sum_{k=0}^{d-1} b_k \phi(2x-k)$. If the scale functions are orthogonal.*

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k) \tag{6}$$

and the wavelet functions are orthogonal with the scale function

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k) \tag{7}$$

for all k, then $b_k = (-1)^k c_{d-1-k}$.