3rd Problem Set for the MDS Block Course on

High-Dimensional Computational Geometry

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The problems will be discussed in the afternoon session.

Problem 1 Exact Nearest-Neighbor Search in d Dimensions

Let P be a set of points in $\mathbb{R}^d$. We want to obtain a data structure with space $O(n^{\lceil d/2 \rceil + \varepsilon})$ that can answer the following query in $O(\log n)$ time: given a point $q \in \mathbb{R}^d$, find the point $p \in P$ that is closest to q.

Let H be the hyperplanes obtained by lifting P to the $(d+1)$-dimensional unit paraboloid and taking tangents. As we saw in class, to answer a nearest neighbor query, we need to find the intersection of the vertical ray through q in $\mathbb{R}^{d+1}$ with the upper envelope of H. Fix some constant $t \in \mathbb{N}$.

(a) Let $S \subseteq H$ be obtained by taking each hyperplane with probability t/n. Let $\mathcal{T}$ be the canonical triangulation of the intersection of the halfspaces above the hyperplanes in S. Show that with probability at least $1/2$, the triangulation $\mathcal{T}$ has $O(t^{\lceil d/2 \rceil})$ simplices and each simplex in $\mathcal{T}$ is intersected by $O((n/t)\log t)$ hyperplanes from H.

Hint: Use the Chazelle-Friedman bound.

(b) Use (a) recursively to obtain a data structure that can answer nearest neighbor queries in time $O(t^{\lceil d/2 \rceil} \log n)$ and whose space requirement fulfills the recursion

$$S(n) = O(t^{\lceil d/2 \rceil}) + O(t^{\lceil d/2 \rceil})\, S(O((n/t)\log t)).$$

(c) Show that the recursion from (b) solves to $O(n^{\lceil d/2 \rceil + \varepsilon})$, where $\varepsilon > 0$ is some constant that gets smaller as t gets larger.

Problem 2 #P

(a) Let FP be the set of functions from $\{0,1\}^*$ to $\mathbb{N}$ that are computable in polynomial time. Show that $FP \subseteq \#P$.

(b) Is the standard reduction from SAT to 3SAT parsimonious? Why is #3SAT #P-complete provided that #SAT is #P-complete, nevertheless?

Problem 3 number of linear extensions

Show that the following partial order has $\frac{((a+1)b+c)!}{(a+1)^b}$ linear extensions.
Hint: probabilistic argument.