2nd Problem Set for the MDS Block Course on

High-Dimensional Computational Geometry

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The problems will be discussed in the afternoon session.

Problem 1 Seidel's Algorithm — Details

Work out the implementation details of Seidel's convex hull algorithm. Which data structures are needed? How are they organized? How is the conflict location implemented? How is the convex hull updated in each step?

Problem 2 Moments of the Binomial Distribution

For $p \in [0,1]$ and $n \in \mathbb{N}$, let $B(n,p)$ denote the binomial distribution with n trials and success probability p. Let X be distributed according to $B(n,p)$ and let $c \in \mathbb{N}_0$. Show that $\mathrm{E}[X^c] \leq (c+np)^c$. For this, prove the stronger bound $\mathrm{E}[(X+j)^c] \leq (c+np+j)^c$, for all $j \in \mathbb{N}_0$ by induction on c.

(a) Check that the claim holds for $c = 0$ and $c = 1$.

(b) Show that for $c \geq 1$, we have

$$\mathrm{E}[(X+j)^{c+1}] = j\mathrm{E}[(X+j)^c] + \sum_{i=1}^{n} p\,\mathrm{E}[(X+j)^c \mid X_i = 1].$$

Here, we put $X = \sum_{i=1}^n X_i$, where X_i is an indicator random variable that is 1 with probability p.

(c) Use (b) to finish the induction on c.

Remark: This bound is due to Brönnimann, Chazelle, and Matoušek.

Problem 3 Chazelle-Friedman Bound

Let H be a set of n hyperplanes in $\mathbb{R}^d$ whose intersection is non-empty and bounded. Let $p \in [0,1]$ and let $S \subseteq H$ be a subset that contains each hyperplane in H independently with probability p. Let $t \in \mathbb{N}$, and let $S_{\geq t}$ be the number of simplices σ in the canonical triangulation of $S^\cap$ with conflict size $n_\sigma \geq t/p$. Then, the Chazelle-Friedman bound states that

$$\mathrm{E}[S_{\geq t}] = O\big(t^{d^2} 2^{-t} (pn)^{\lfloor d/2 \rfloor}\big).$$

(a) Let T be sample that contains each hyperplane with probability p/t. Argue that the expected number of simplices in the canonical triangulation of T is $O\big((pn/t)^{\lfloor d/2 \rfloor}\big)$.

(b) Let σ by a possible simplex with $n_\sigma \geq t/p$. Denote by $p_S(\sigma)$ the probability that σ occurs in the canonical triangulation of $S^\cap$ and by p_T the probability that σ occurs in the canonical triangulation of $T^\cap$. Show that

$$\frac{p_T(\sigma)}{p_S(\sigma)} \geq t^{-d^2} e^{t-2}.$$

Hint: Use the bounds $1 - x \leq e^{-x}$ for all x and $1 - x \geq e^{-2x}$ for $0 \leq x \leq 1/2$.

(c) Using (b), show that

$$\mathrm{E}[S_{\geq t}] \leq t^{d^2} e^{2-t}\, \mathrm{E}[|T^\cap|],$$

where $|T^\cap|$ is the number of simplices in the canonical triangulation of $T^\cap$.

(d) Conclude the Chazelle-Friedman bound.