1st Problem Set for the MDS Block Course on

High-Dimensional Computational Geometry

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The problems will be discussed in the afternoon session.

Problem 1 Examples of Polytopes

(a) Draw the face lattice for the 4-dimensional simplex, the crosspolytope, and the hypercube. Try to visualize these polytopes, and describe your intuition to the group.

(b) Verify the claims that we made in class about the number and structure of faces of the d-dimensional simplex, the crosspolytope, and the hypercube.

Problem 2 The Cyclic Polytope

(a) Use Gale's evenness criterion to prove that the number of facets of the d-dimensional cyclic polytope with n vertices is

$$2\binom{n - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor}$$

for d odd, and

$$\binom{n - \lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{n - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1}$$

for d even. Show that for constant d, the number of facets is $\Theta(n^{\lfloor d/2 \rfloor})$.

(b) Show that the cyclic polytope is *neighborly*: for $0 \leq j \leq \lfloor d/2 \rfloor$, every j-subset of vertices spans a $(j-1)$-face of the polytope.

Problem 3 The Dehn-Sommerville Relations

Let $\mathcal{P}$ be a simple d-dimensional polytope with n facets. Suppose that no two vertices of $\mathcal{P}$ have the same x_d-coordinate. Direct the edges of $\mathcal{P}$ according to the x_d-direction. For $j = 0, \ldots, d$, let h_j be the number of vertices in $\mathcal{P}$ with outdegree j. We call $(h_0, h_1, \ldots, h_d)$ the *h-vector* of $\mathcal{P}$.

(a) For $i = 0, \ldots, d$, let f_i be the number of i-faces in $\mathcal{P}$. Show that

$$f_i = \sum_{j=i}^{d} \binom{j}{i} h_j,$$

for $i = 0, \ldots, d$.

Hint: Observe that each i-face of $\mathcal{P}$ has a unique vertex with minimum x_d-coordinate. Argue that a vertex of outdegree j is the lowest point for $\binom{j}{i}$ i-faces.

(b) Define $f(t) = \sum_{i=0}^{d} f_i t^i$ and $h(t) = \sum_{i=0}^{d} h_j t^j$. Show that $f(t) = h(t+1)$. Conclude that $h(t) = f(t-1)$. Derive from this an expression for each h_j in terms of the f_i.

(c) Conclude from (b) that the h-vector is (almost) independent of the choice of the x_d-axis. From this, derive the *Dehn-Sommerville relations*: for $0 \leq i \leq d$, we have $h_i = h_{d-i}$. Write the Dehn-Sommerville relations in terms of the f_i.

(d) Argue that $h_i \leq f_i \leq \binom{n}{d-i}$, for $i = 0, \ldots, d$. Use the Dehn-Sommervile relations to conclude

$$h_i \leq \min\left\{\binom{n}{i}, \binom{n}{d-i}\right\}.$$

Using the expression for f_i in terms of the h_j, conclude that $f_i = O(n^{\lfloor d/2 \rfloor})$, for $i = 0, \ldots, d$.