Blockcourse on High-Dimensional Computational Geometry	Spring 2013
Lecture 5: Approximating the Volume — July 27, 2013	
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1 Further negative results

After showing that computing the volume of a convex polytope is $\#P-$complete by reducing $\#LEXT$ on VPT we give a further negative result on the approximability of the volume of a convex body in general. Let $K \subset \mathbb{R}^d$ be a convex, bounded, closed, and $\text{vol}(K) > 0$. We assume that K is given by an oracle that gives an answer to the question "Is $q \in K$?".

Theorem 1 ([BF87]). *Suppose a deterministic polynomial-time algorithm that computes numbers V_l and V_u for a convex body $K \subseteq \mathbb{R}$ such that $V_l \leq vol(K) \leq V_u$ using a membership oracle for K. Then there exists a constant c, such that for every dimension d a convex body with $\frac{V_u}{V_l} > c(\frac{d}{\log d})^d$ exists.*

2 A non-deterministic approximation algorithm

In this section the idea of a *FPRAS* for the volume of a convex body of [DFK91] is given.

Definition 2.

- *A* polynomial-time randomised approximation scheme *(PRAS) for a function f is a randomised algorithm $\mathcal{A}$ which computes for an input instance I and $\epsilon > 0$, in time polynomial in $n = |I|$ an output (I) with*

$$Pr[(1-\epsilon)f(I) \leq \mathcal{A}(I) \leq (1+\epsilon)f(I)] \geq \frac{3}{4}.$$

- *A* polynomial-time randomised approximation scheme *((ϵ-)FPRAS) is a PRAS that takes also ϵ as input and has a running time that is still polynomial in n and $\frac{1}{\epsilon}$.*
- *A $\epsilon-\delta-$FPRAS is a ϵ-FPRAS that takes $\delta > 0$ as input, has a running time that is polynomial in $n, \frac{1}{\epsilon}$ and $\log \frac{1}{\delta}$, and guarantees a probability of $1-\delta$ instead of $\frac{3}{4}$.*

The existence of a ϵ-FPRAS implies the existence of a $\epsilon-\delta-$FPRAS.
In following we assume that we know balls B and B' with $B' \subset K \subset B$.

Basic idea: Consider a convex body K with diameter s in the plane and a rectangle R, such that two opposite sides of R are tangent to K in the endpoints of s, see Figure 2. The volume of K can be approximated by randomly (uniformly distributed) sampling N points within the bounding rectangle. If N_K is the number of points contained in K, then the ration $\alpha = \frac{n_k}{N}$ converges almost surely to $\frac{\text{vol}(K)}{\text{vol}(R)}$. In this example we also know that $\text{vol}(R) \leq 2\text{vol}(K)$. For an convex body in arbitrary dimension this simple approach does not work since for the standard simplex S we have $\text{vol}(S) = \frac{1}{d!}$ but the smallest bounding box of S has volume 1, hence the first point contained in S is sampled after $\Theta(d!)$ steps in expectation.

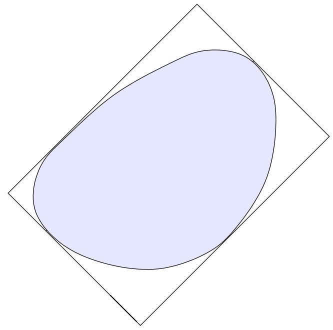

Figure 1: Bounding rectangle.

Idea: We construct a sequence of convex bodies where $B = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_q = K$ with the following properties:

1. There exists a constant c such that $\frac{\text{vol}(K_i)}{\text{vol}(K_{i-1})} \leq c$. This ratio can be approximated for $i = 1$ by the idea before.
2. q is polynomial in α, then $\text{vol}(K) = \text{vol}(K_0)\Pi_{i=1}^{q}\frac{\text{vol}(K_{i-1})}{\text{vol}(K_i)}$.

The problem here is the sampling the uniformly distributed points in K_i.

2.1 Finding random points in a convex body given by a membership oracle

For technical reasons we consider the body $K(\alpha) = K \oplus \alpha B_d$, the Minkowski sum of K and a d-dimensional ball of radius α.

Idea: We lay a grid of width δ over a convex body K. Now we start with one point in the grid and start with a "(natural) random walk" on the cubes. After a sufficiently long walk we take the endpoint:

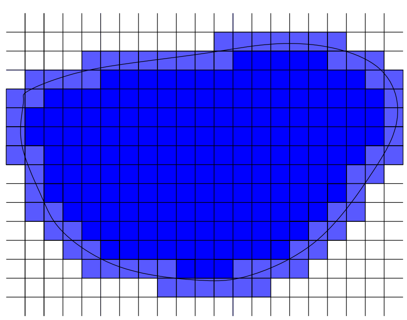

Figure 2: Grid over K.

- start at any cube in $K(\alpha)$.

- at each step within some cube C: stay in C or go to a neighbouring cube C', i.e. a cube that shares a facet by choosing a facet of C (probability $\frac{1}{2d}$ each). If C' intersects $K(\alpha)$ go to C', otherwise stay in C.

Note that it is not really decidable whether C' intersects $K(\alpha)$, therefore only "weak intersection" is calculated by an algorithm by Grötschel, Lovász and Schrijver from [GLS88]. Additionally, the "natural random walk" is modified, such that the probability of staying within the cell C is larger or equal than $\frac{1}{2}$. Hence, pick the cells adjacent to one facet of C with probability $\frac{1}{4d}$ and C with probability $\frac{1}{2}$.

3 Markov chain

It consists of a set S (in our case S is finite) of states and transition probabilities.

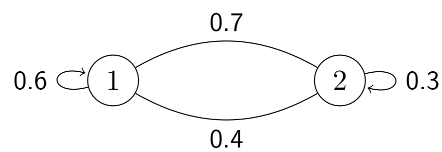

Figure 3: Example Graph

p_{ij} denotes the probability to go to state j in the next step if being in state i.
This gives a $k \times k$ matrix $P = (p_{ij})_{1 \leq i,j \leq k}$ with $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{k} p_{ij} = 1 \forall i$

$$\begin{pmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{pmatrix}$$

Figure 4: Example Matrix

We have discrete time steps $t = 1, 2, 3, ...$, in which the Markov chain may change its state according to transition probabilities, and some start state $X_0 \in \{1, ..., k\}$.
The probability of going from i to j in 2 steps is given by: $p_{ij}^{(2)} = \sum_{l=1}^{k} P_{il} P_{lj} =$ i-j-entry of P^2
in general for t steps: $p_{ij}^{(t)} = (P^t)_{ij}$

We define the following notation:

$r_{ij}^{(t)}$**:** probability that, starting in state i, state j is reached for the first time after t steps

$f_{ij} = \sum_{t=0}^{\infty} r_{ij}^{(t)}$**:** probability that, starting from i ,j will ever be reached

$h_{ij} = \sum_{t=0}^{\infty} t r_{ij}^{(t)}$**:** expected number of steps until j is reached starting from i for the first time

$q^{(t)}$**:** probability distribution after t steps, s.t. starting with probability distribution vector $q^{(0)}$ for X_0, $q^{(t+1)} = q^{(t)} \cdot P \qquad \rightarrow \qquad q^{(t)} = q^{(0)} \cdot P^t$

transient state: A state i with $f_{ii} < 1$

persistent state: A state i with $f_{ii} = 1$

null-persistent state: A state i witch is persistent and $h_{ii} = \infty$

irreducible Markov chain: $\Leftrightarrow$ corresponding graph is strongly connected (all edges have probability > 0)

periodicity of a state i: $\max T \in \mathbb{N}$ such that there exists an initial distribution vector (probability distribution for the states) $q^{(0)} \in \mathbb{R}^k, \exists a \in \mathbb{N}$ such that $\{t | q_i^{(t)} > 0\} \subseteq \{a + T \cdot j | j = 0, 1, \}$ if $T > 1$ the state i is periodic. Otherwise it is aperiodic.

ergodic state aperiodic and non null-persistent.

ergodic Markov chain $\Leftrightarrow$ all states are ergodic

Definition 3. *A* <u>*Stationary distribution*</u> *of a Markov chain with the Matrix P is a distribution $\pi \in \mathbb{R}^k$ with $\pi = \pi P$ π^T is the eigenvector of P^T to eigenvalue 1.*

Theorem 4. *any irreducible finite aperiodic MC has the following properties.*

1. *it is ergodic*
2. *there exists a unique stationary distribution π such that $\pi_i > 0 \forall i$*
3. $f_{ii} = 1, h_{ii} = \frac{1}{\pi_i}$
4. *$N(i,t)$ number of times state i is visited in t steps* $\lim_{t\to\infty} \frac{N(i,t)}{t} = \pi_i$

important property: if $\lim_{t\to\infty} \pi_j \forall i, j$ converges quickly, we call it rapidly mixing Markov chain.

4 Application to random walk

The random walk in the algorithm is a Markov chain.

- The random walk is irreducible (HOMEWORK!),
- aperiodic,
- ergodic,
- P is symmetric (all π_j are equal) and regular $\Rightarrow$ P is time reversible ($P_{ij}\pi_i = P_{ji}\pi_j$)

Theorem 5 (Jerrum/Sinclair 1988)**.** *A Markov chain with the above properties is rapidly mixing. Concretely: $|p_i^{(t)} - \pi_j| \leq (1 - \frac{1}{10^{17} d^{19}})^t$*
e.g. we choose $t = (10^{17} \cdot d^{19}) \cdot k$ then the term becomes $(\frac{1}{e})^k$

References

[BF87] Imre Bárány and Zoltán Füredi. Empty simplices in euclidean space. *Canad. Math. Bull*, 30(4):436–445, 1987.

[DFK91] Martin Dyer, Alan Frieze, and Ravi Kannan. A random polynomial-time algorithm for approximating the volume of convex bodies. *Journal of the ACM (JACM)*, 38(1):1–17, 1991.

[GLS88] Martin Grötschel, Lászlo Lovász, and Alexander Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer, 1988.