Blockcourse on High-Dimensional Computational Geometry	Spring 2013

Lecture 1: Introduction — July 22, 2013

Wolfgang Mulzer *Scribe: Sebastian Stugk and Rémy Thomasse*

1 Overview

Classic computational geometry:

- Convex hulls
- Voronoi diagrams
- Point location
- Range searching
- Triangulations

2 or 3 dimensions. Why?

- easier to visualize & understand
- planarity constraints input & bounds complexity

Why do we need higher dimensions?

- Machine learning & computer vision produce high dimensional data to process.
- LP/SUP-solvers produce high dimensional vectors to be processed.

What do we do?

- Some solutions just generalize to higher dimensions?
- But: Dependence on dimensions often is exponential. (Curse of dimensionality)
- Some problems become NP-hard.
- Solution: Approximation & randomization

More specifically:

1. High-dimensional polytomes: basics & complexity, volume
2. Nearest-neighbor searching: approximation
3. Metric embeddings & dimension reduction

2 Polytopes in high dimensions

Definition 1. *(V-polytope) Let $P \subseteq \mathbb{R}^d$, $|P| = n$. Then $\mathcal{P} := \mathrm{conv}(P)$ is called V-polytope.*

Definition 2. *(H-polytope) Let H be a set of n hyperplanes in $\mathbb{R}^d$. Write $\{h_1, \ldots, h_n\}$ and h^- for the closed half-space to the left of h. If $\bigcap_{h \in H} h$ is bounded, then it is called H-polytope.*

Examples:

1. Simplex
   Suppose $|P| = d + 1$ in $\mathbb{R}^d$ in general position. Then $\mathrm{conv}(P)$ is called simplex. (The simplex is the smallest possible d-dimensional polytope.)

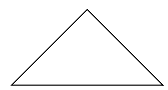


Figure 1: Simplex for $d = 2$.

2. Cross-polytope
   The d-dimensional cross-polytope is defined as $\mathrm{conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d\}$ where e_i is the i-th unit vector.

3. Hypercube
   Intersection of halfspaces

$$x_i \geq -1 \text{ for } i = 1, \ldots, d$$
$$x_i \leq 1 \text{ for } i = 1, \ldots, d$$

Fact: Every H-polytope is a V-polytope and every V-polytope is an H-polytope.

3 Duality

Definition 3. *(Dual hyperplane / dual point) Let $a \in \mathbb{R}^d \setminus \{0\}$ be a point. The dual hyperplane of a is defined as $a^* = \{x \in \mathbb{R}^d \mid \langle a, x \rangle = a_1x_1 + a_2x_2 + \cdots + a_dx_d = 1\}$ Let h be a hyperplane with $0 \notin h$. Write $h := \{x \in \mathbb{R}^d \mid \langle a, x \rangle = 1\}$. The dual point h^* is defined as $h* = a$*

Fact: Duality preserves incidence and containment.

- $a \in h \leftrightarrow h^* \in a^*$
- $a \in h^- \leftrightarrow h^* \in (a^*)^- E$

Definition 4. *(Dual polytope) Let $\mathcal{P} = \mathrm{conv}\{p_1, p_2, \ldots, p_n\}$ a V-polytope such that 0 is in the interior of $\mathcal{P}$. The dual polytope $\mathcal{P}^*$ is defined as*

$$\mathcal{P}^* = \bigcap_{i=1}^{n} \underbrace{(p_i^*)^-}_{\substack{\text{half-space} \\ \text{contains} \\ \text{the origin}}}$$

Similarly for H-polytope.

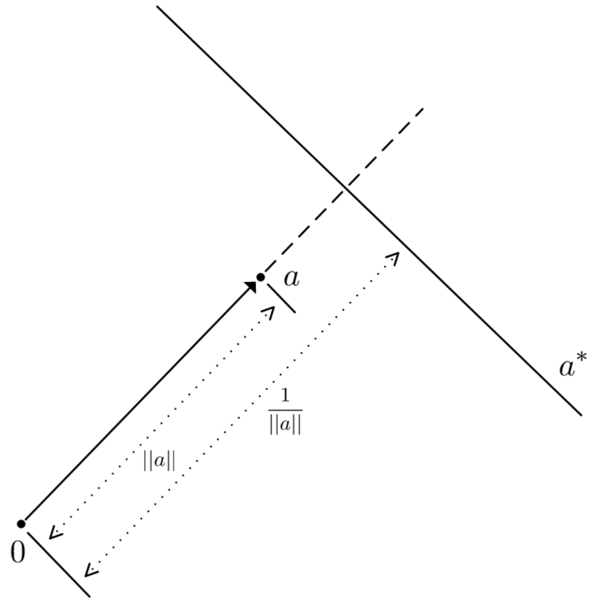


Figure 2: Dual hyperplane a^*

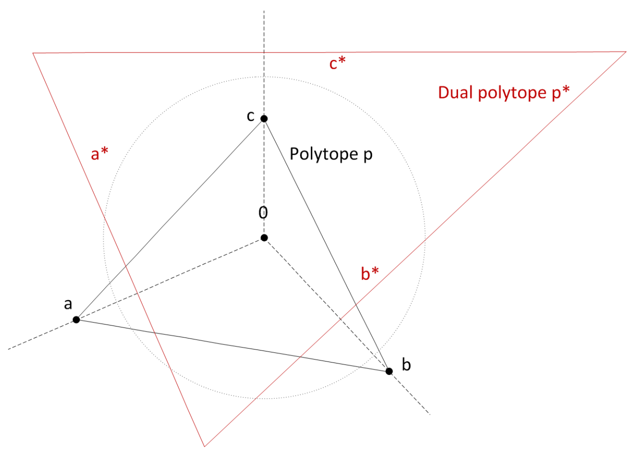


Figure 3: Dual polytope p^* w.r.t unit circle

Examples:

- The dual of a simplex: a simplex.
- The dual of the cross-polytope is a hypercube and vice-versa.

Fact: $(\mathcal{P}^*)^* = \mathcal{P}$

4 Faces

Definition 5. *Let $\mathcal{P}$ be a polytope. A face of $\mathcal{P}$ is either*

i) $\mathcal{P}$ itself

ii) a set of the form $\mathcal{P} \cap h$, where h is a hyperplan with $\mathcal{P} \subseteq h^+$ or $\mathcal{P} \subseteq h^-$.

A $j-$face is a face of dimension j.

Remark 6.

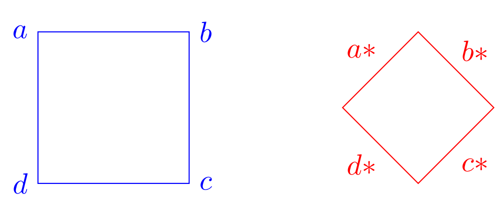
- *There is exactly one $-1-$face: $\emptyset$*
- *There is exactly one $d-$face: $\mathcal{P}$*
- *The $0-$faces are the vertices*
- *The $1-$faces are the edges*
- *The $(d-2)-$faces are the ridges*
- *The $(d-1)-$faces are the facets.*

Examples

- Simplex : Every subset of j vertices defines a $(j-1)$-face. We get $\binom{d+1}{j}$ $(j-1)-$faces for $0 \leq j \leq d-1$.
- Cross Polytope: for $0 \leq j \leq d$, a $j-$subset $A = \{e_1, -e_2, e_3, \dots\}$ spans a $(j-1)-$face if A does not contain e_i and $-e_i$ for some i. We get $\binom{d}{j} 2^j$ $(j-1)$-faces.
- Hypercube: Similar to Cross Polytope.

Properties 7.

1. *For $V-$polytopes, vertices are extremal points in defining set.*
   For H-polytopes, facets are contained in non-redondant defining hyperplanes.
2. *Let f be a $j-$face of $\mathcal{P}$. Then, f is a $j-$dimensional polytope. The vertices and faces of f are exactly the vertices and faces of $\mathcal{P}$ that are contained in f.*
3. *There is a bijection between $j-$faces of $\mathcal{P}$ and $(d-j-1)-$faces of $\mathcal{P}^*$ and the incidences are reversed.*



Definition 8 (Simplicial Polytope)**.** *A simplicial polytope is a polytope where all the facets are simplices. It holds for $V-$polytopes zith general position defining set.*

Definition 9 (Simple Polytope)**.** *A Simple polytope is a polytope where every vertex is incident to d edges. It holds for general position H−polytopes.*

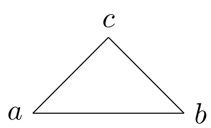
So we have

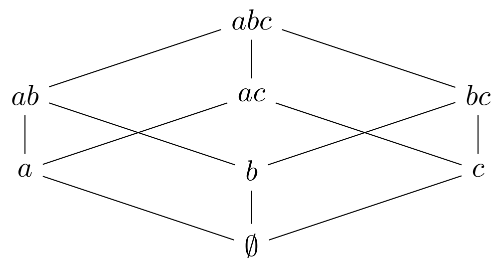
$$\text{Simplicial} \underset{\text{dual}}{\longleftrightarrow} \text{Simple}$$

The dual of a simplicial polytope is a simple polytope and the dual of a simple polytope is a simplicial polytope.
For $d > 2$, the only polytope that is simple and simplicial is the simplex.

Definition 10 (Face Lattice)**.** *The face lattice of a polytope is a graph with one node for each face, and there is an edge between a $(j-1)-$ face f and a $j-$face g if and only if $f \subseteq g$.*

Example:





Fact 11 (for lattice experts)**.** *A face lattice is atomic, co-atomic and graduated.*

To compute a polytope means to compute its face lattice.
Polytopes are represented in a computer by their face lattices.

4.1 How bad can polytopes become?

Given $\langle p_1, p_2, \ldots, p_n \rangle =: P$, find $\text{conv}(P)$.
How large can a face lattice become? A trivial upper bound is $\mathcal{O}(n^d)$.

A bad example: The cyclic polytope

Definition 12 (Moment Curve)**.** *The function $\gamma : \mathbb{R} \to \mathbb{R}^d$ with $\gamma(t) = (t, t^2, t^3, \ldots, t^d)$ is called the Moment Curve.*

For $n \in \mathbb{N}$, define $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = \gamma(i)$ and $C_n^d := \text{Conv}(P)$. C_n^d is called cyclic polytope with n vertices.

Lemma 13.

i) C_n^d is simplicial.

ii) Let $f = \{p_{i_1}, p_{i_2}, \ldots, p_{i_d}\}$ be d points defining C_n^d. Then, $\text{Conv}(f)$ is a facet of C_n^d if and only if for every p_k, $p_l \in P \setminus F$, the number of indices with $k < i_j < l$ is even

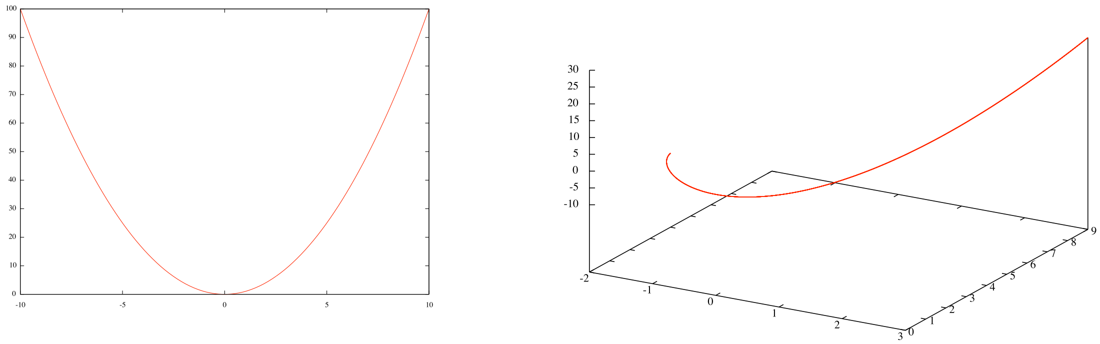


Figure 4: The moment curve for $d = 2$ and $d = 3$.

Proof. i) We want to show that each facet contains exactly d vertices.

Let f be a facet with vertices $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$. All vertices lie in a hyperplane

$$h: \quad a_1 x_1 + a_2 x_2 = \cdots + a_d x_d = 1$$

This means:

$$\begin{aligned} a_1 i_1 + a_2 i_1^2 + \cdots + a_d i_1^d - 1 &= 0 \\ a_1 i_2 + a_2 i_2^2 + \cdots + a_d i_2^d - 1 &= 0 \\ &\vdots \\ a_1 i_k + a_2 i_k^2 + \cdots + a_d i_k^d - 1 &= 0 \end{aligned}$$

i.e $i_1, i_2, \ldots, i_k$ are zeros at polynomial

$$\sum_{i=1}^{d} a_i X^i - 1$$

This means $k \leq d$.

ii) Gale's evenness criterion:

□

Corollary 14. *C_n^d has $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$ facets.*