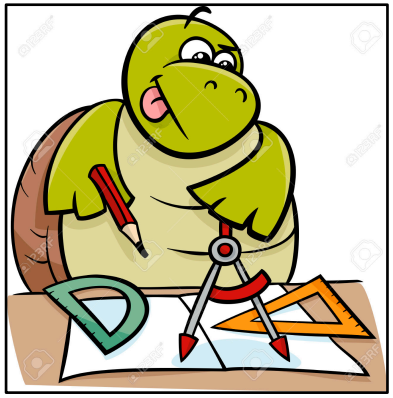
Decomposition Theorems in UDGs and Applications in Param. Complexity

1. Excluded Grid Minors and Efficient Polynomial-Time Approximation Schemes. Fomin, Lokshtanov, Saurabh. J. ACM 2018.

2. Finding, Hitting and Packing Cycles in Subexponential Time on Unit Disk Graphs. Fomin, Lokshtanov, Panolan, Saurabh, Zehavi. Discrete & Computational Geometry 2019.

3. Contraction Decomposition in Unit Disk Graphs and Algorithmic Applications in Parameterized Complexity. Panolan, Saurabh, Zehavi. SODA 2019.

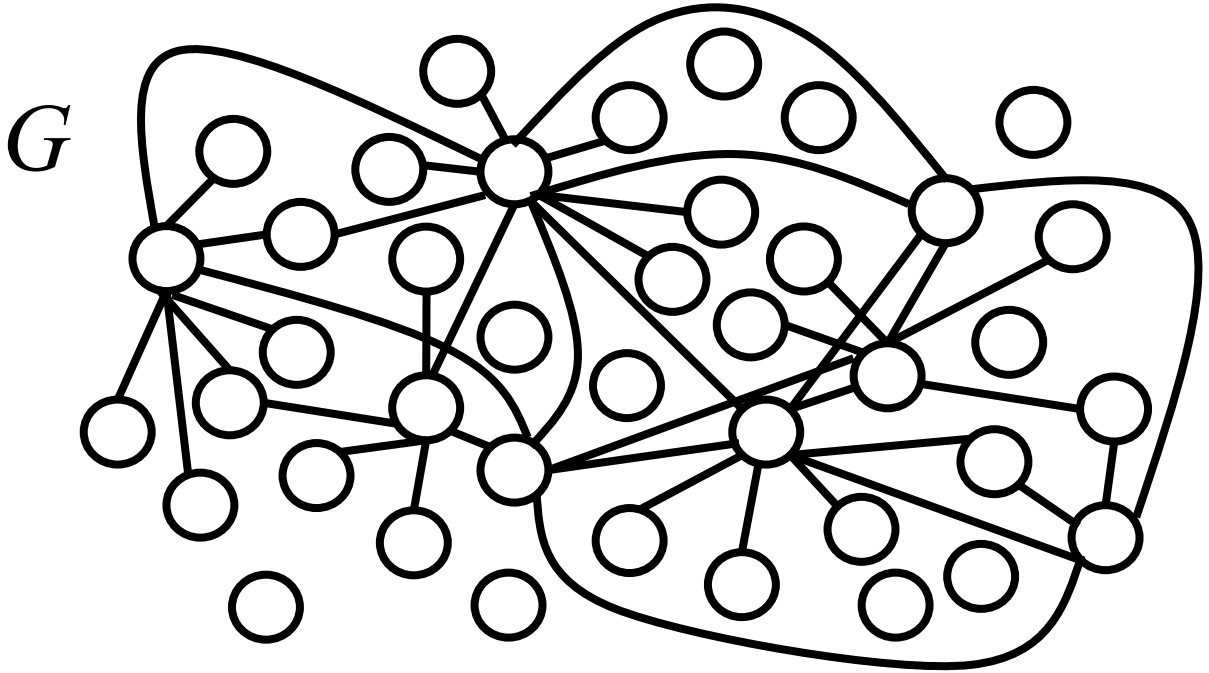


Parameterized Complexity

Vertex Cover.

Input: Graph G; non-negative integer k.

Question: Does G have a vertex cover of size at most k?



$n = m = 40$

$k = 5$

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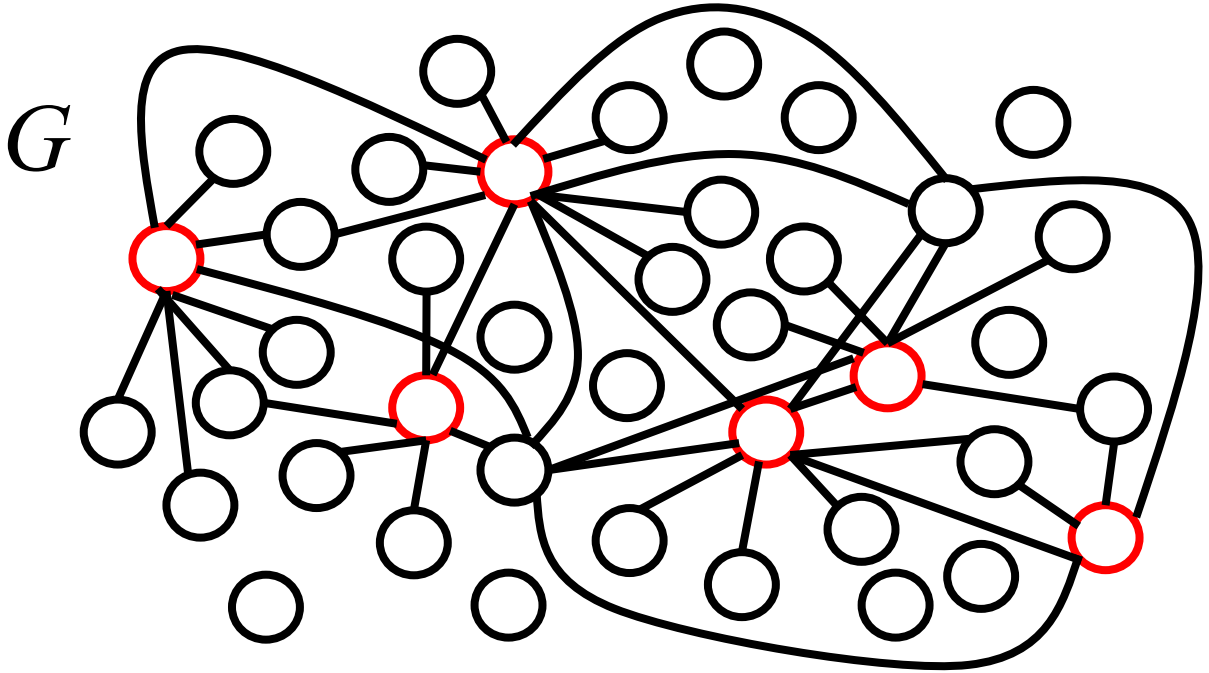


No

$n = m = 40$

$k = 5$

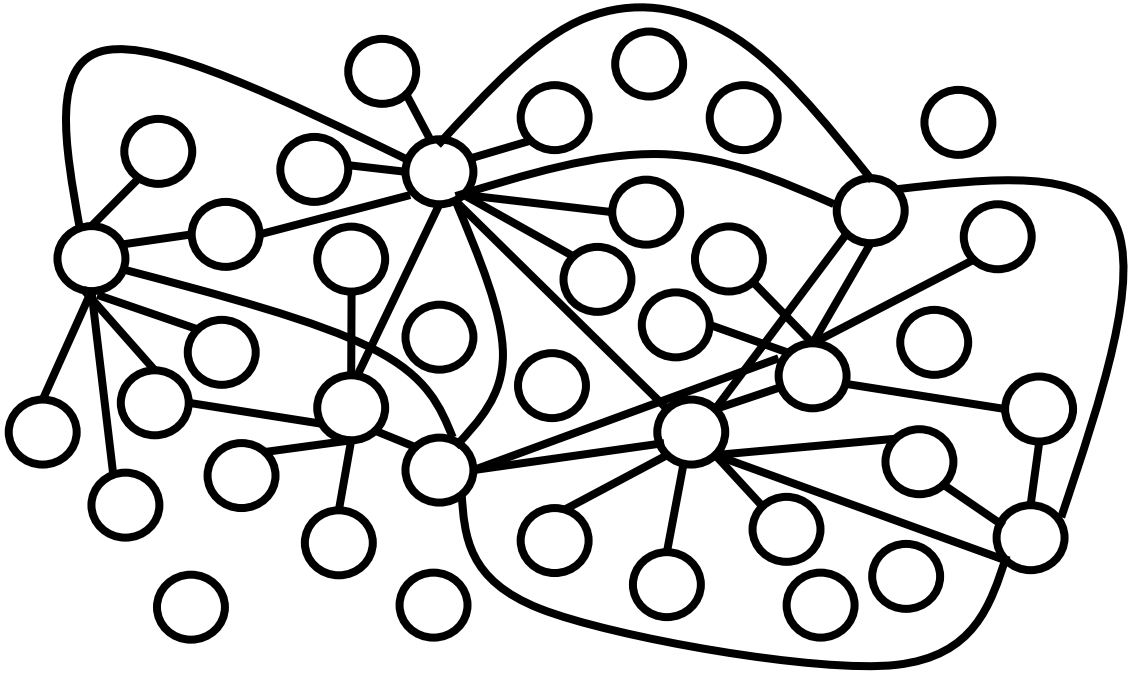
Parameterized Complexity

Brute-Force. $\binom{n}{k}(n+m) = \binom{40}{5}(40+40)$

$= 658{,}008 \cdot 80$

$= 52{,}640{,}640$

G



$n = m = 40$

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Parameterized Complexity

Brute-Force. $\binom{n}{k}(n+m) = 658{,}008 \cdot 80 = 52{,}640{,}640$

Magic 1. $2^k(n+m) = 32 \cdot 80 = 2{,}560$

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Magic 2. $(n+m+k^3)$ = 205 → Equivalent instance:
$n' = 2k = 10$; $m' = k^2 = 25$.

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With Brute-Force. 205 + $\binom{10}{5}(10+25)$ = 9,025

With Magic 1. 205 + $2^5(10+25)$ = 1,245

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What if *k* = 10, *n* = 100 and *m* = 300?

Parameterized Complexity

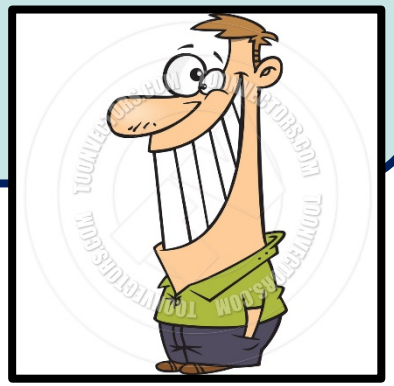
Brute-Force. $\binom{n}{k}(n+m) \sim 7 \cdot 10^{15}$

Magic 1. $2^k(n+m) = 409,600$

Magic 2. $(n + m + k^3)$ = 1,400 → Equivalent instance:
n' = $2k$ = **20**; m'=k^2 = **100**.

With Brute-Force. 1,400 + $\binom{20}{10} \cdot 120$ = **22,172,120**

With Magic 1. 1,400 + $2^{10} \cdot 120$ = **124,280**

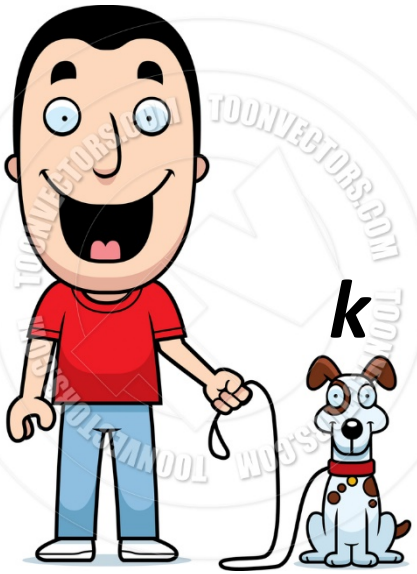


What if *k* = 10, *n* = 100 and *m* = 300?

Parameterized Complexity

Parameterized problem.

Each instance is associated with a parameter k.



Fixed-Parameter Tractability

$k << n$

$f(k) \cdot n^{O(1)} = O^*(f(k))$

$\binom{n}{k}$ … or worse!

Fixed-Parameter Tractability

$k << n$

$f(k) \cdot n^{O(1)} = O^*(f(k))$ $\binom{n}{k}$ *… or worse!*

Unfortunately, not all problems are FPT!

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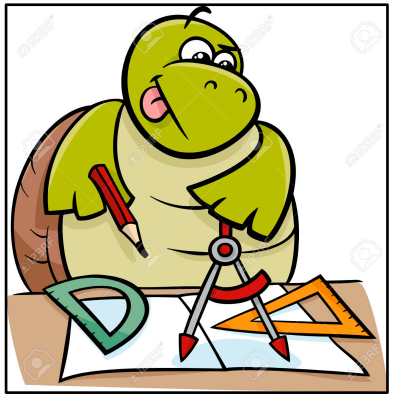
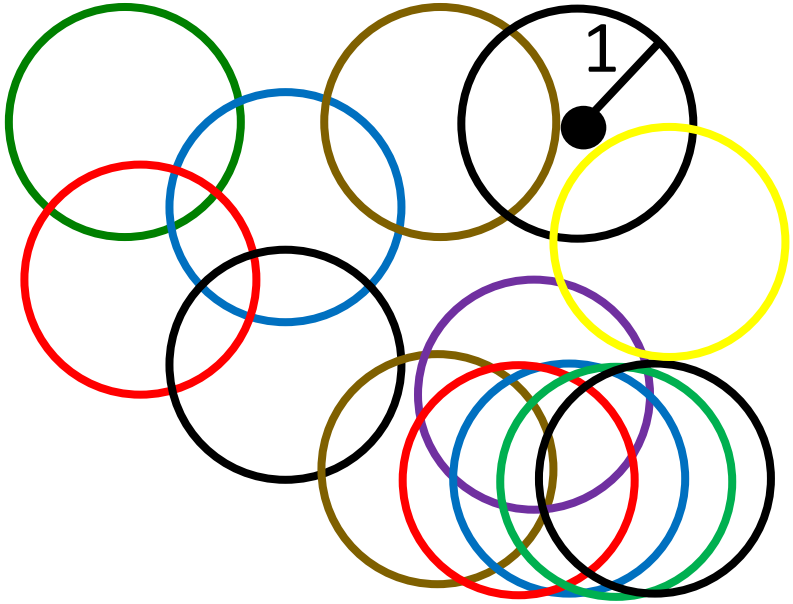


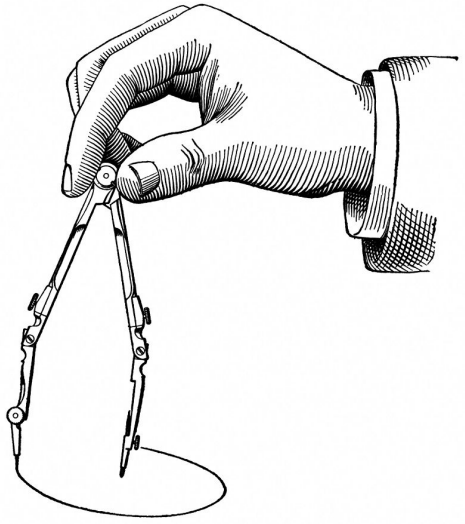
Outline

- **Introduction:**
  - UDGs: Definition
  - Decomposition I: Branch on Cliques
- Decomposition II: General Decomposition
- Decomposition III: Contraction Decomposition

Introduction: Unit Disk Graphs

Unit Disk Graph.
Intersection graph of unit circles in the plane.





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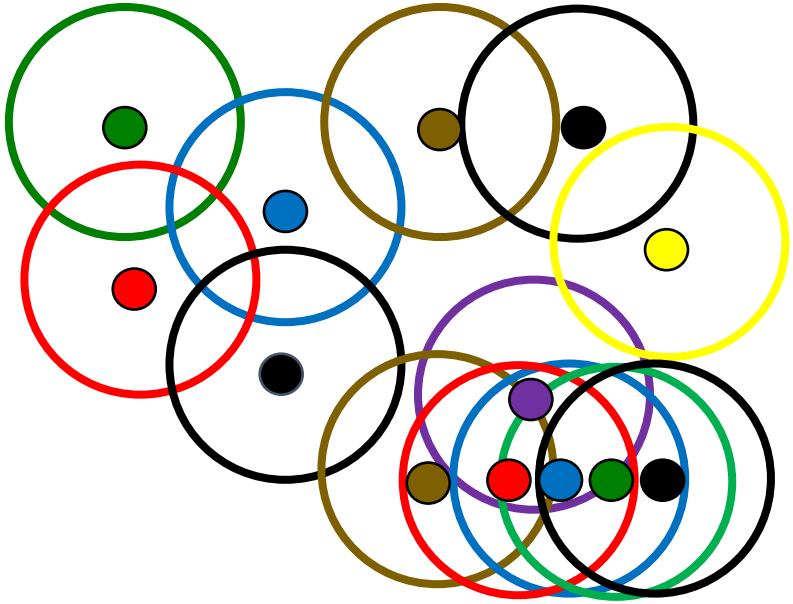

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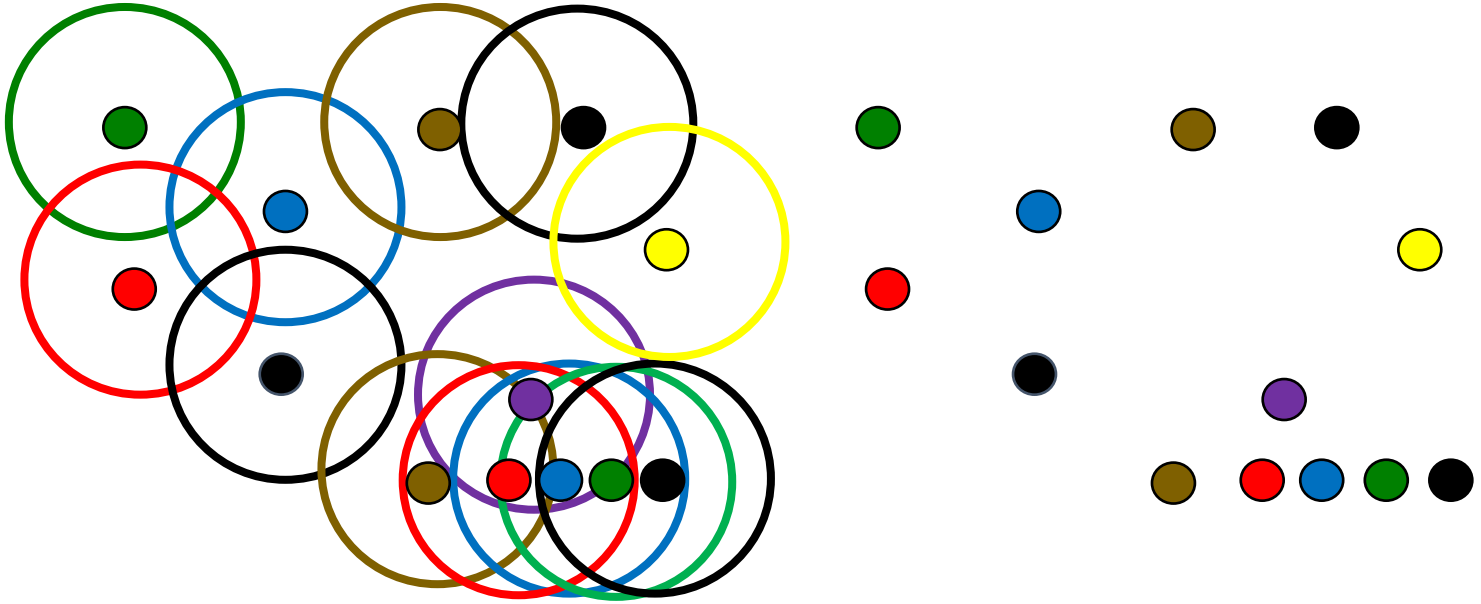

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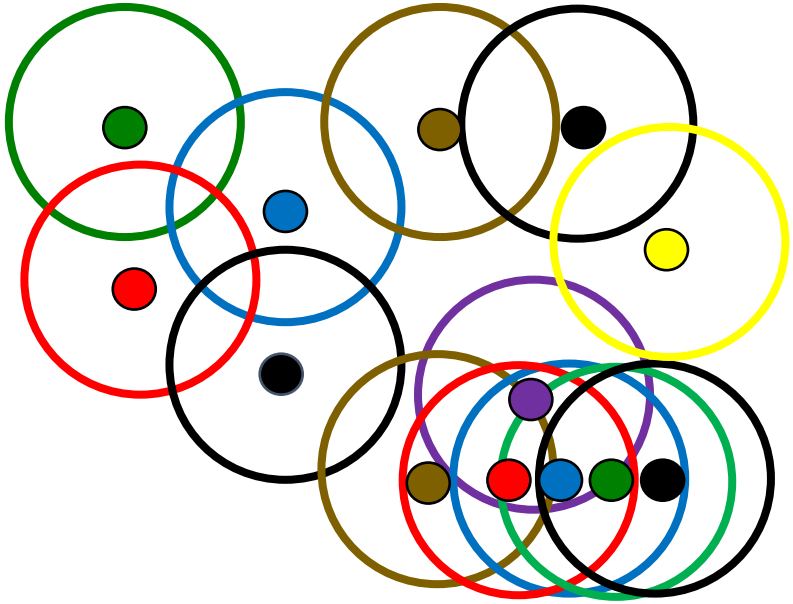

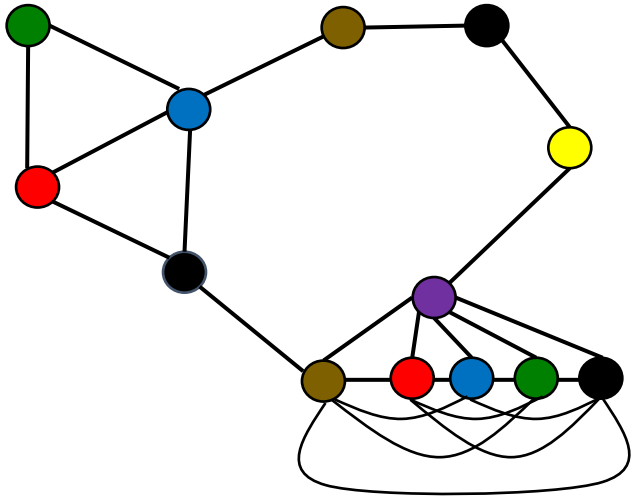

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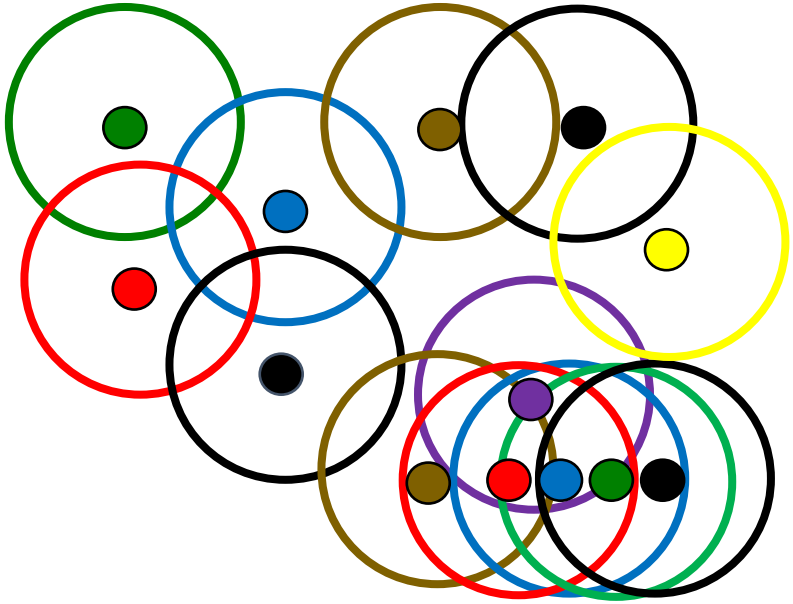
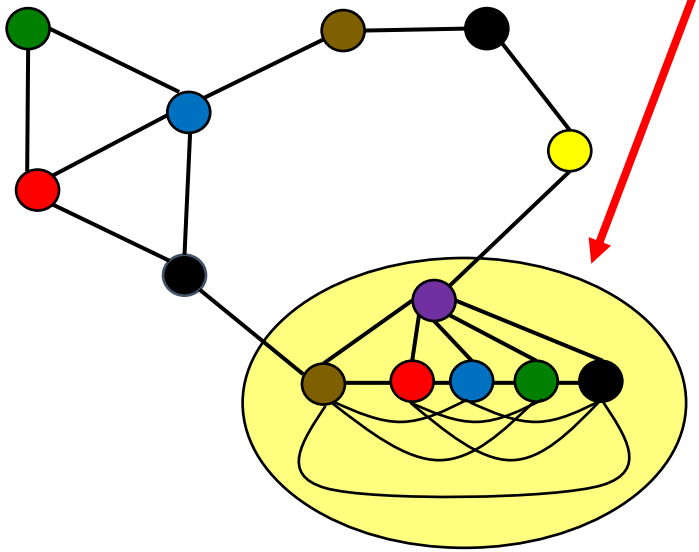


Large cliques! (Not planar.)



Introduction: Tree Decomposition

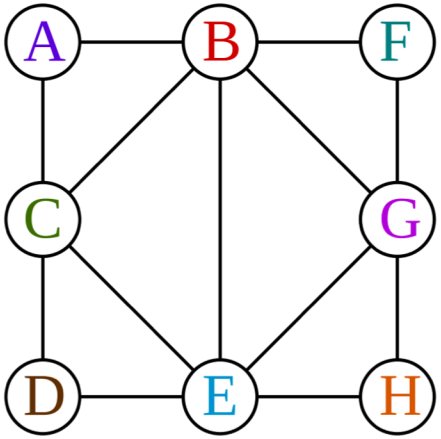
A **tree decomposition** of a graph G is a pair (T,β) where T is a tree and $\beta : V(T) \to 2^{V(G)}$, such that

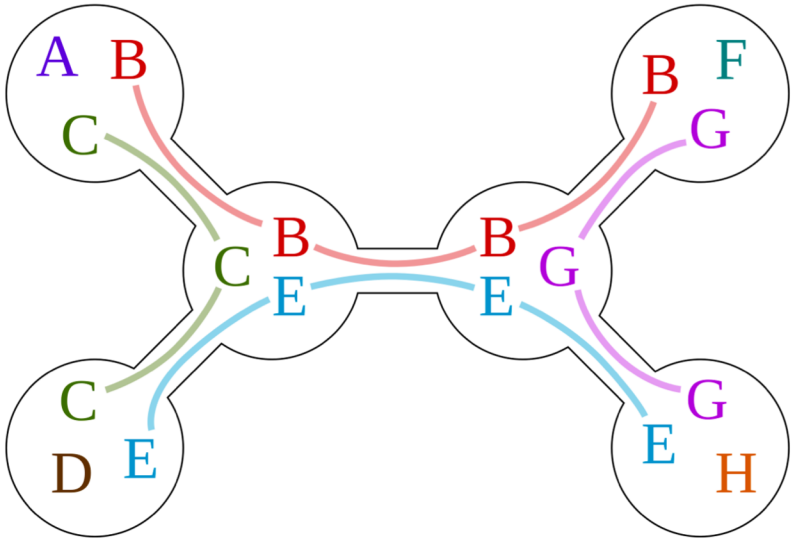
(i) For each $v \in V(G)$,

$$T[\{x \in V(T) : v \in \beta(x)\}]$$

is a non-empty tree.

(ii) For each $\{u,v\} \in E(G)$, there is $x \in V(T)$ such that $u,v \in \beta(x)$.





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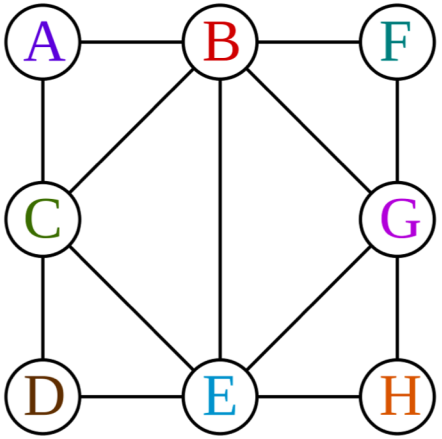

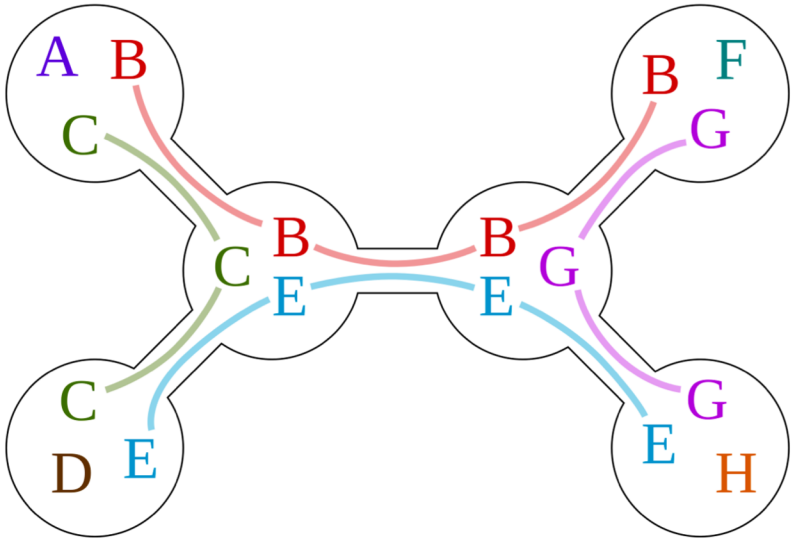

Width of (T,β). Size of the largest ***bag*** + 1.

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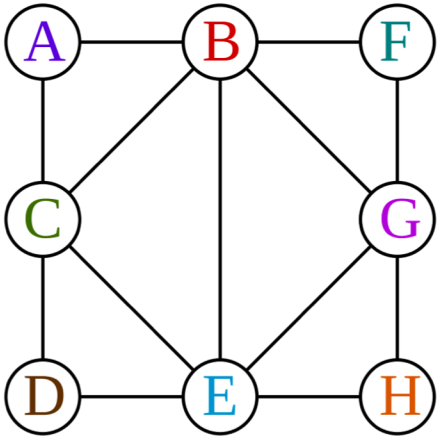

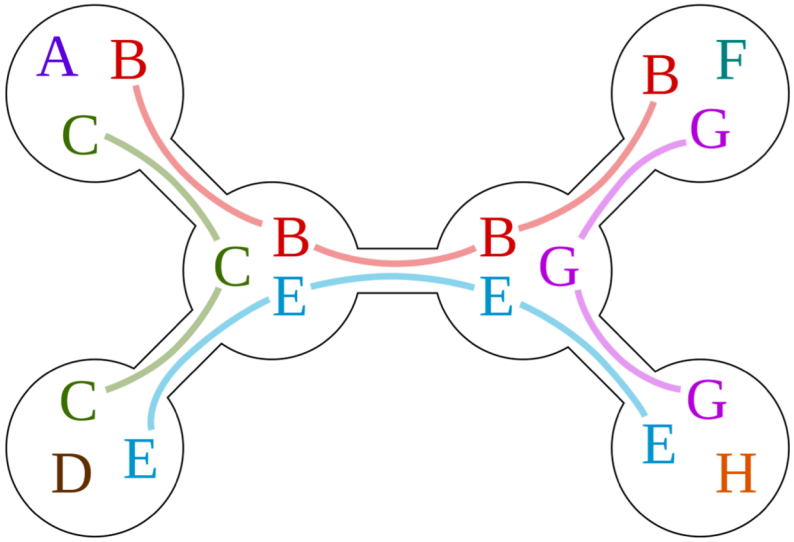

Treewidth of *G*. Minimum width of a tree decomposition of G.

Introduction: Branch on Cliques

Decomposition I(a). Given w and a UDG G of max. degree Δ, in time $O^*(2^{O(w\cdot\Delta)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\Delta)$.

Introduction: Branch on Cliques

Decomposition I(b). Given w and a UDG G of max. clique size γ, in time $O^*(2^{O(w \cdot \gamma)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w \cdot \gamma)$.

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Decomposition I(b). Given w and a UDG G of max. clique size γ, in time $O^*(2^{O(w\cdot\gamma)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\gamma)$.

We would like to utilize this theorem to reach the case where the treewidth is small.

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But the max. clique size γ can be large!

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But the max. clique size γ can be large!

(Unlike real life,) cliques are not a major issue!

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(Connected) Vertex Cover, Feedback Vertex Set, Cycle Packing. Branch to shrink the max. clique size!

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Find a max. size clique C (polynomial time for UDGs).

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Find a max. size clique C (polynomial time for UDGs).

In VC (FVS), at most 1 (resp. 2) vertices of C can be unchosen. (**Branch!**)

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(Connected) Vertex Cover, Feedback Vertex Set, Cycle Packing. Branch to shrink the max. clique size!

Find a max. size clique C (polynomial time for UDGs).

In Cycle Packing, there exists an optimal solution where at most $O(1)$ vertices of C are not used to pack local cycles. (**Branch!**)

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(Connected) Vertex Cover, Feedback Vertex Set, Cycle Packing. Branch to shrink the max. clique size!

Find a max. size clique C (polynomial time for UDGs).

Number of branches: $\binom{|V(C)|}{c}$.

Parameter decrease: $|V(C)|$-c.

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But we can have large grids as minors!

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But we can have large grids as minors!

(Unlike real life,)
grids are great!

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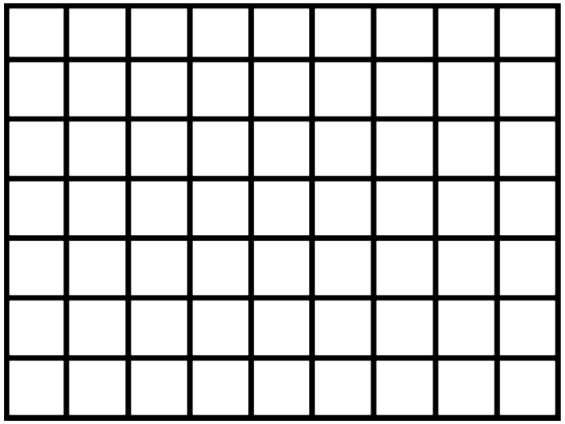
(Connected) Vertex Cover, Feedback Vertex Set, Cycle Packing. A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-instance.



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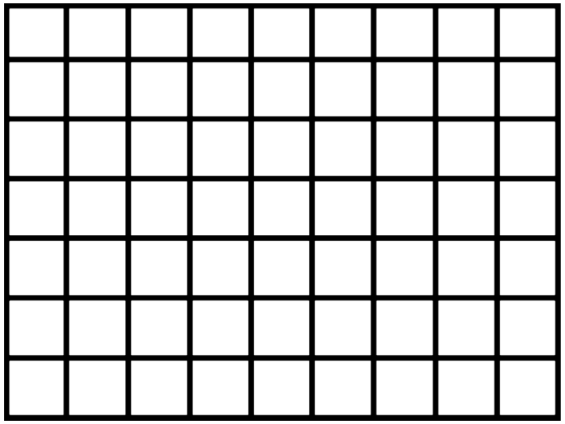

No!

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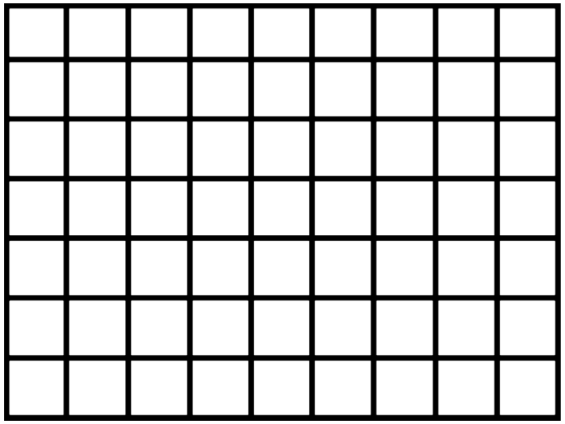

Yes!

Introduction: Branch on Cliques

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Case Distinction:

1. Clique of size k^ε? Branch, parameter decreases by ~k^ε. (No such clique? $\gamma=O(k^\varepsilon)$.)
2. Grid minor of size $c\sqrt{k}$x$c\sqrt{k}$? Yes/No. (No such grid? tw(G) $=O(\sqrt{k}\cdot\gamma)$.)
3. Treewidth $O(k^{0.5+\varepsilon})$. Solve in time $O^*(2^{O(k^{0.5+\varepsilon})})$ or $O^*(2^{O(k^{0.5+\varepsilon}\log k)})$.

Introduction: Branch on Cliques

Decomposition I(b). Given w and a UDG G of max. clique size γ, in time $O^*(2^{O(w\cdot\gamma)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\gamma)$.

Case Distinction: **Tradeoff**

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2. Grid minor of size $c\sqrt{k}$x$c\sqrt{k}$? Yes/No. (No such grid? tw(G) $=O(\sqrt{k}\cdot\gamma)$.)
3. Treewidth $O(k^{0.5+\varepsilon})$. Solve in time $\boldsymbol{O^*(2^{O(k^{0.5+\varepsilon})})}$ **or** $\boldsymbol{O^*(2^{O(k^{0.5+\varepsilon}\log k)})}$.

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Case Distinction:

1. Clique of size $k^{0.25}$? Branch, param. decreases by ~$k^{0.25}$. (No such clique? $\gamma=O(k^{0.25})$.)
2. Grid minor of size $c\sqrt{k}$x$c\sqrt{k}$? Yes/No. (No such grid? tw(G) $=O(\sqrt{k}\cdot\gamma)$.)
3. Treewidth $O(k^{0.75})$. Solve in time $O^*(2^{O(k^{0.75})})$ or $O^*(2^{O(k^{0.75}\log k)})$.

Introduction: Branch on Cliques

Decomposition I(b). Given w and a UDG G of max. clique size γ, in time $O^*(2^{O(w\cdot\gamma)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\gamma)$.

Case Distinction: **Time $O^*(2^{O(k^{0.75})})$ or $O^*(2^{O(k^{0.75}\log k)})$.**

1. Clique of size $k^{0.25}$? Branch, param. decreases by ~$k^{0.25}$. (No such clique? $\gamma=O(k^{0.25})$.)
2. Grid minor of size $c\sqrt{k}$x$c\sqrt{k}$? Yes/No. (No such grid? tw(G) $=O(\sqrt{k}\cdot\gamma)$.)
3. Treewidth $O(k^{0.75})$. Solve in time $O^*(2^{O(k^{0.75})})$ or $O^*(2^{O(k^{0.75}\log k)})$.

Introduction: Branch on Cliques

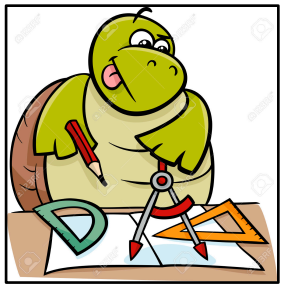
Decomposition I(b). Given w and a UDG G of max. clique size γ, in time $O^*(2^{O(w \cdot \gamma)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w \cdot \gamma)$.

What about…

- Better running time?
- Problems where the existence of a large grid does not seem useful?

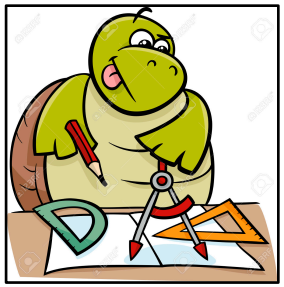
Outline

- Introduction
- **Decomposition II: General Decomposition**
- Decomposition III: Contraction Decomposition



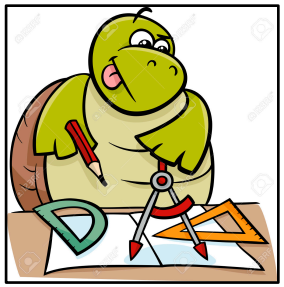
Decomposition II(a): Statement

Decomposition II(a). Given w and a UDG G, in time $O^*(2^{O(w)})$ either determine that G has a wxw grid minor or compute a tree decomposition where each bag consists of $O(w)$ ``cells'' (cliques).



Decomposition II(a): Clique-Grid Graph

Unit disk graph *G* → clique-grid graph *G'*.

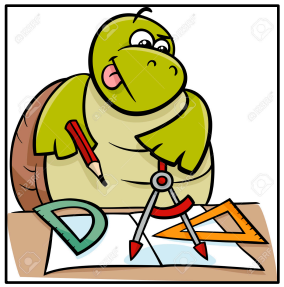


Decomposition II(a): Clique-Grid Graph

Unit disk graph G → clique-grid graph G'.

Clique-grid graph. There is f: $V(G) \rightarrow [t]\text{x}[t']$ such that

1. For all $(i,j) \in [t]\text{x}[t']$, $f^{-1}(i,j)$ is a clique.
2. For all $\{u,v\} \in E(G)$, $|i-i'| \leq 2$ and $|j-j'| \leq 2$, where $f(u)=(i,j)$ and $f(v)=(i',j')$.



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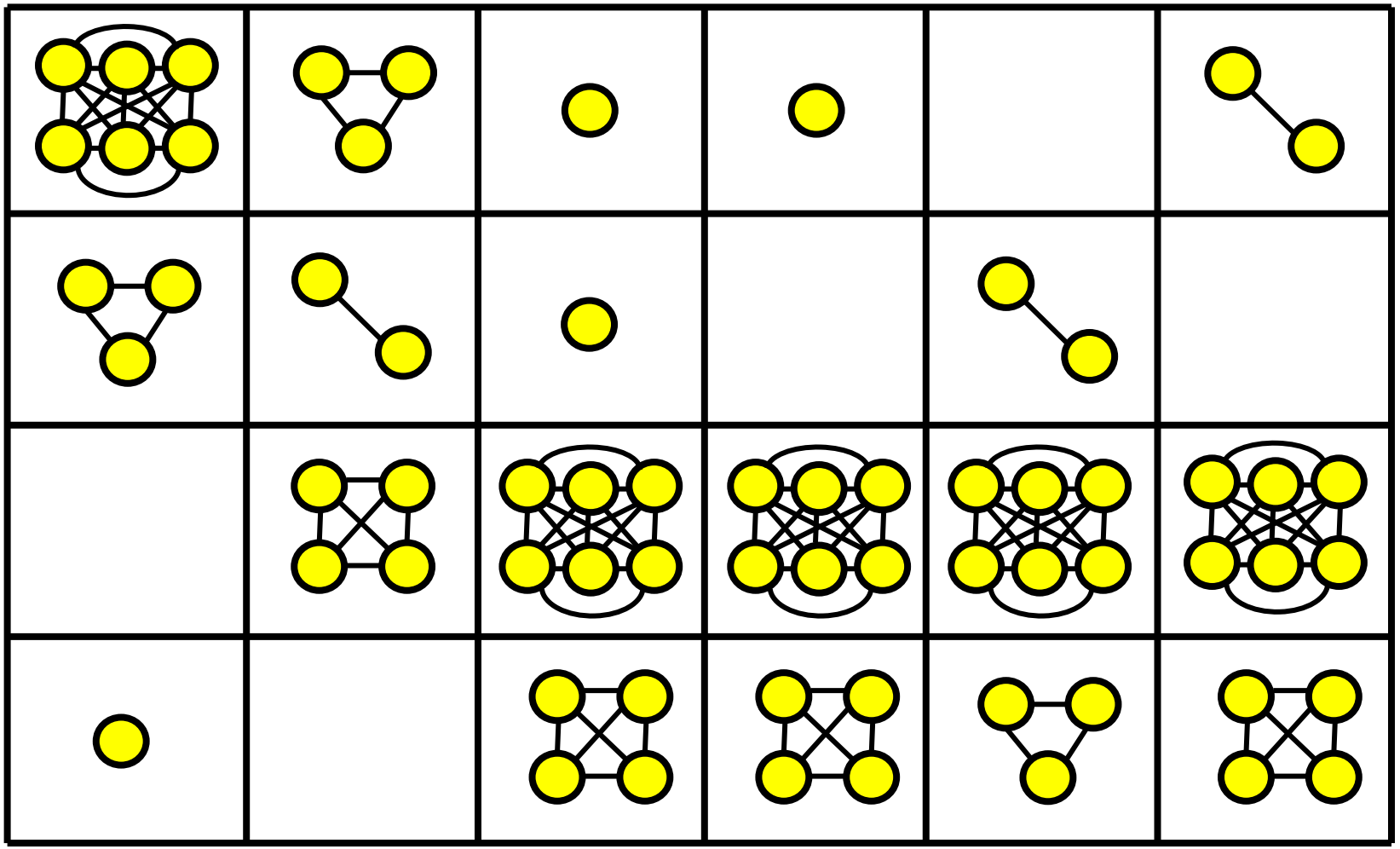

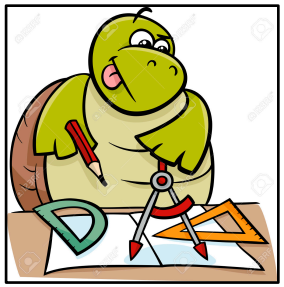

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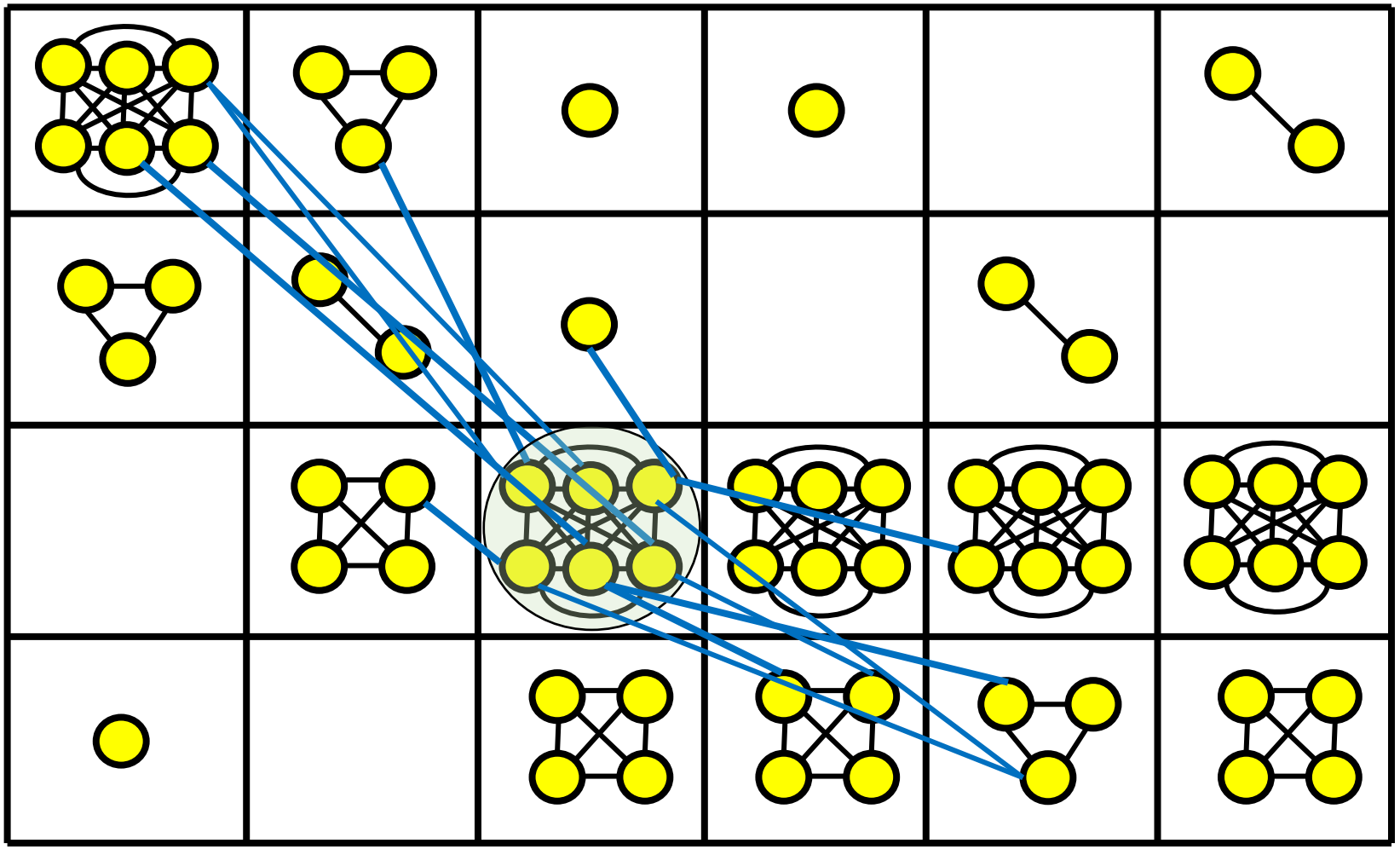
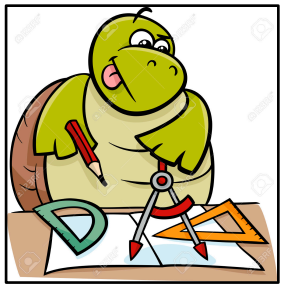
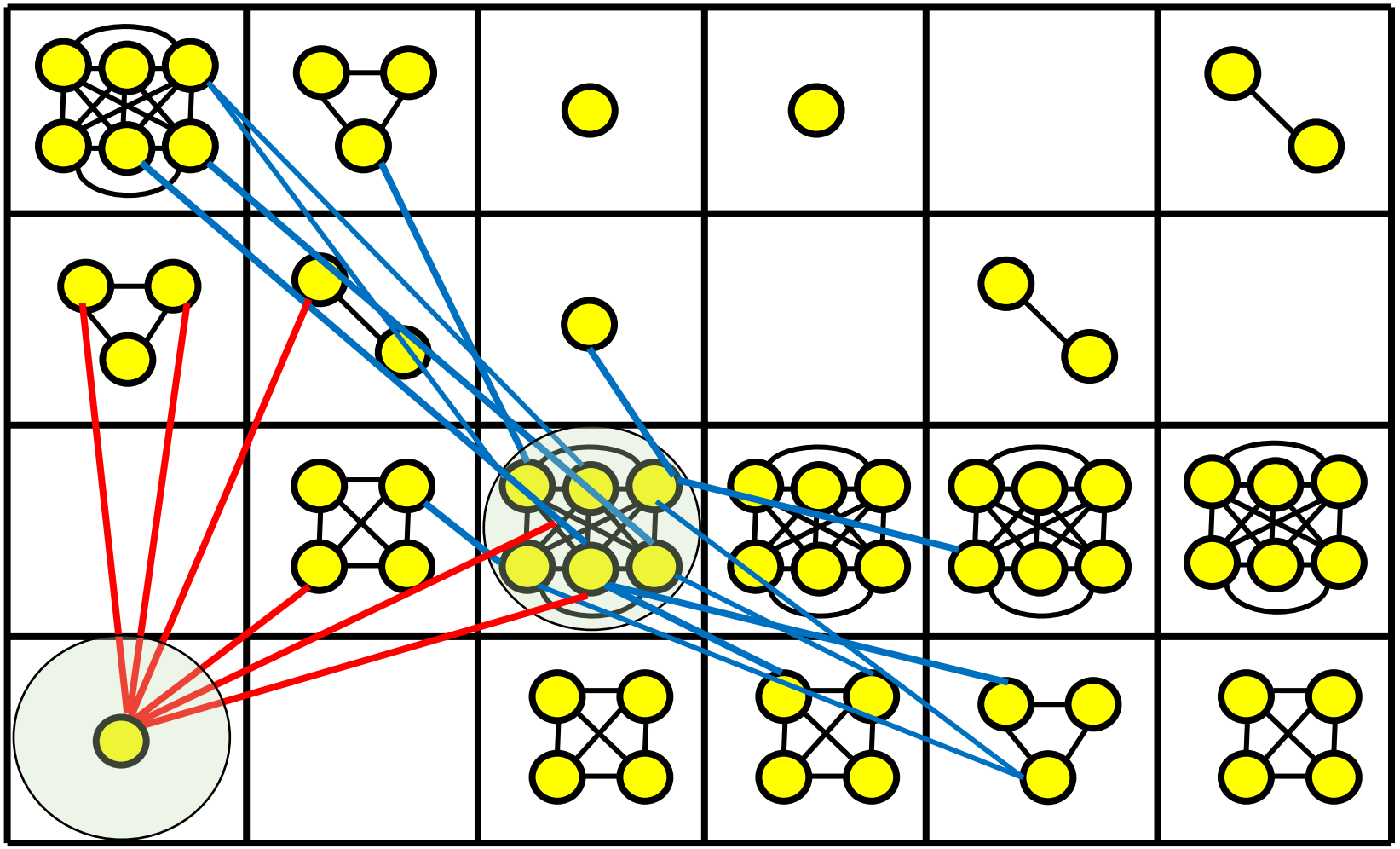
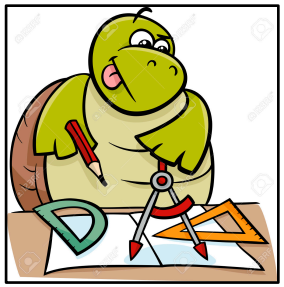




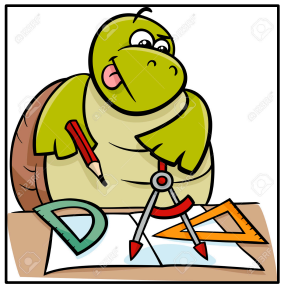
Decomposition II(a): Clique-Grid Graph





Decomposition II(a): Backbone Graph

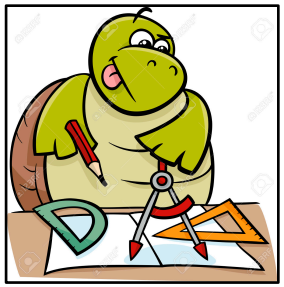
Unit disk graph *G* → clique-grid graph *G'* → backbone graph *B*.



Decomposition II(a): Backbone Graph

Unit disk graph *G* → clique-grid graph *G'* → backbone graph *B*.

Backbone graph (informal). Keep minimum number of vertices such that any two cells that have adjacent vertices in *G* also have adjacent vertices in *B*.



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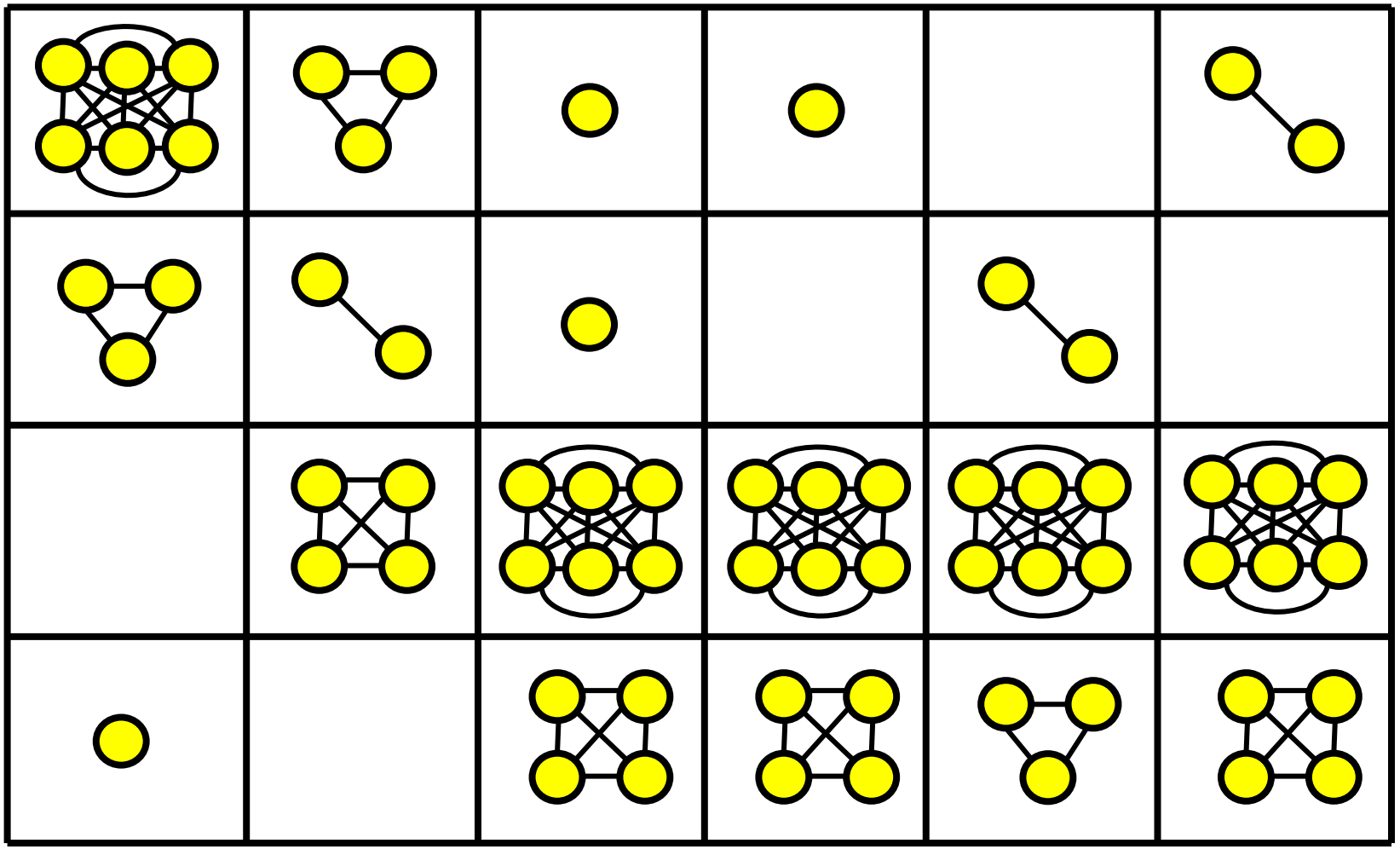

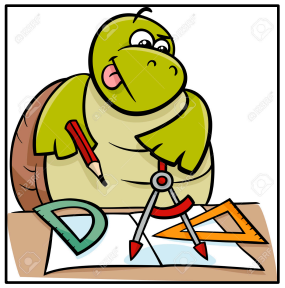

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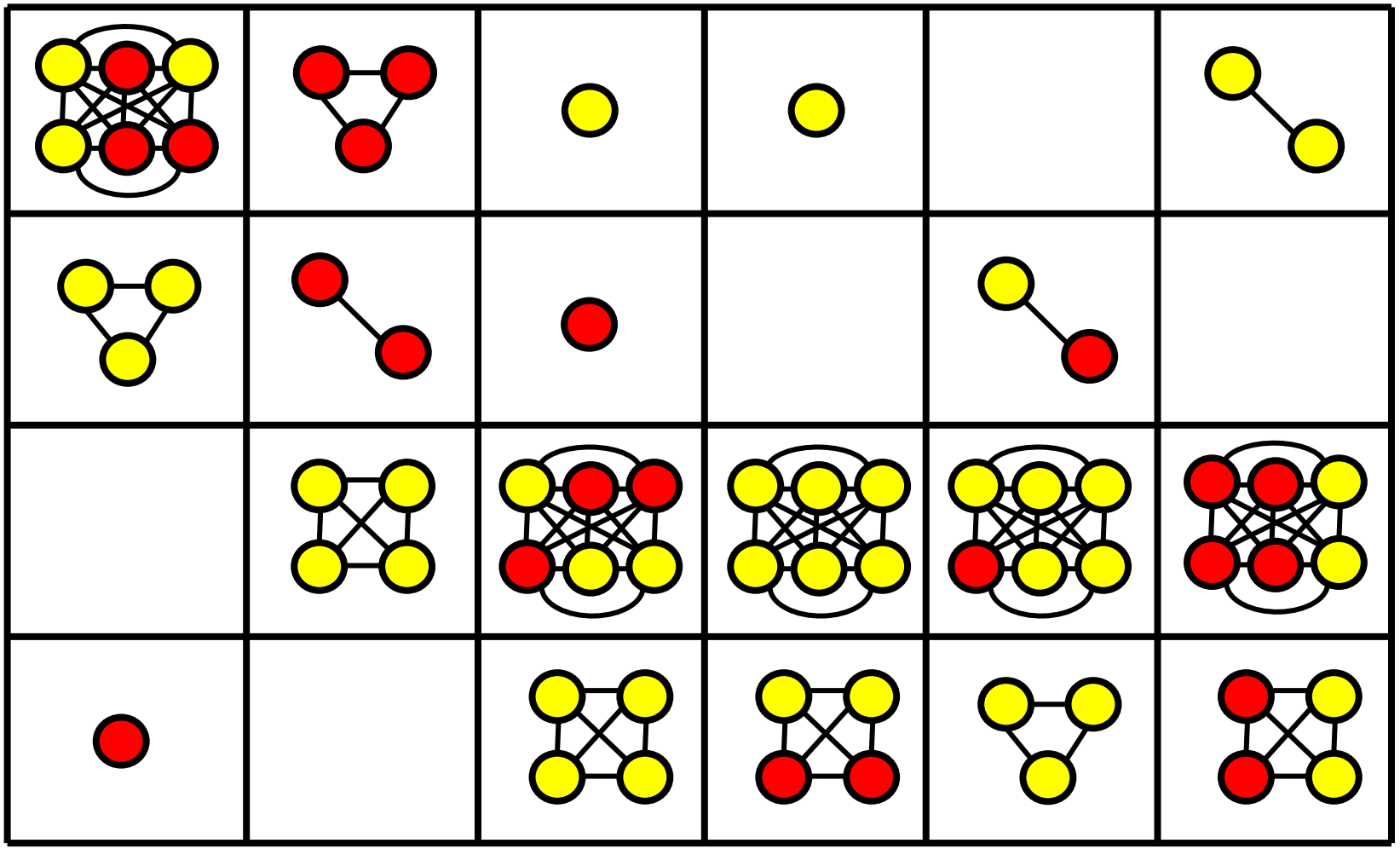

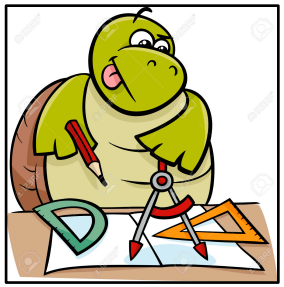

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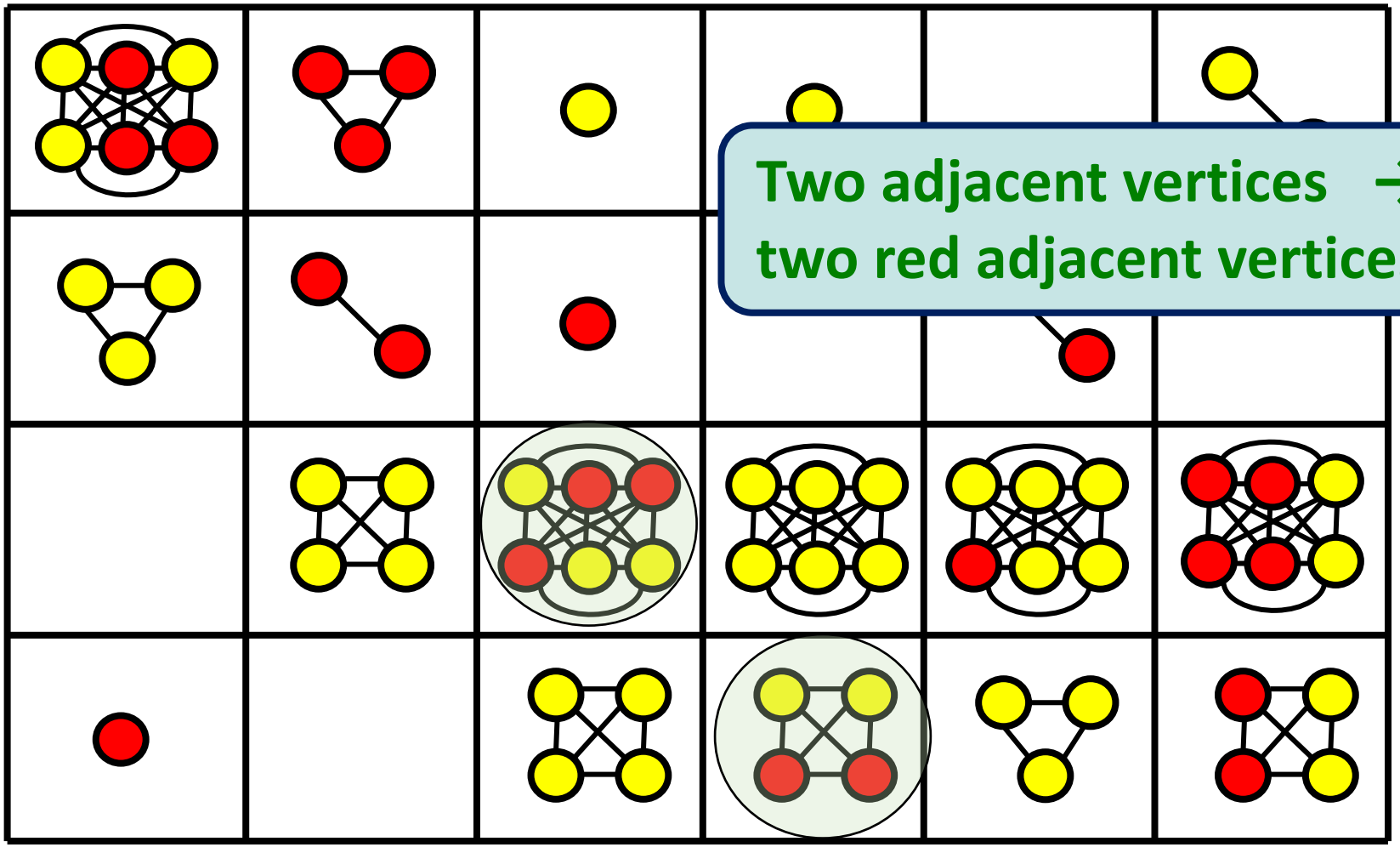
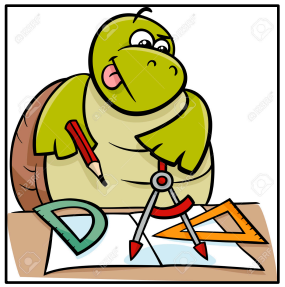
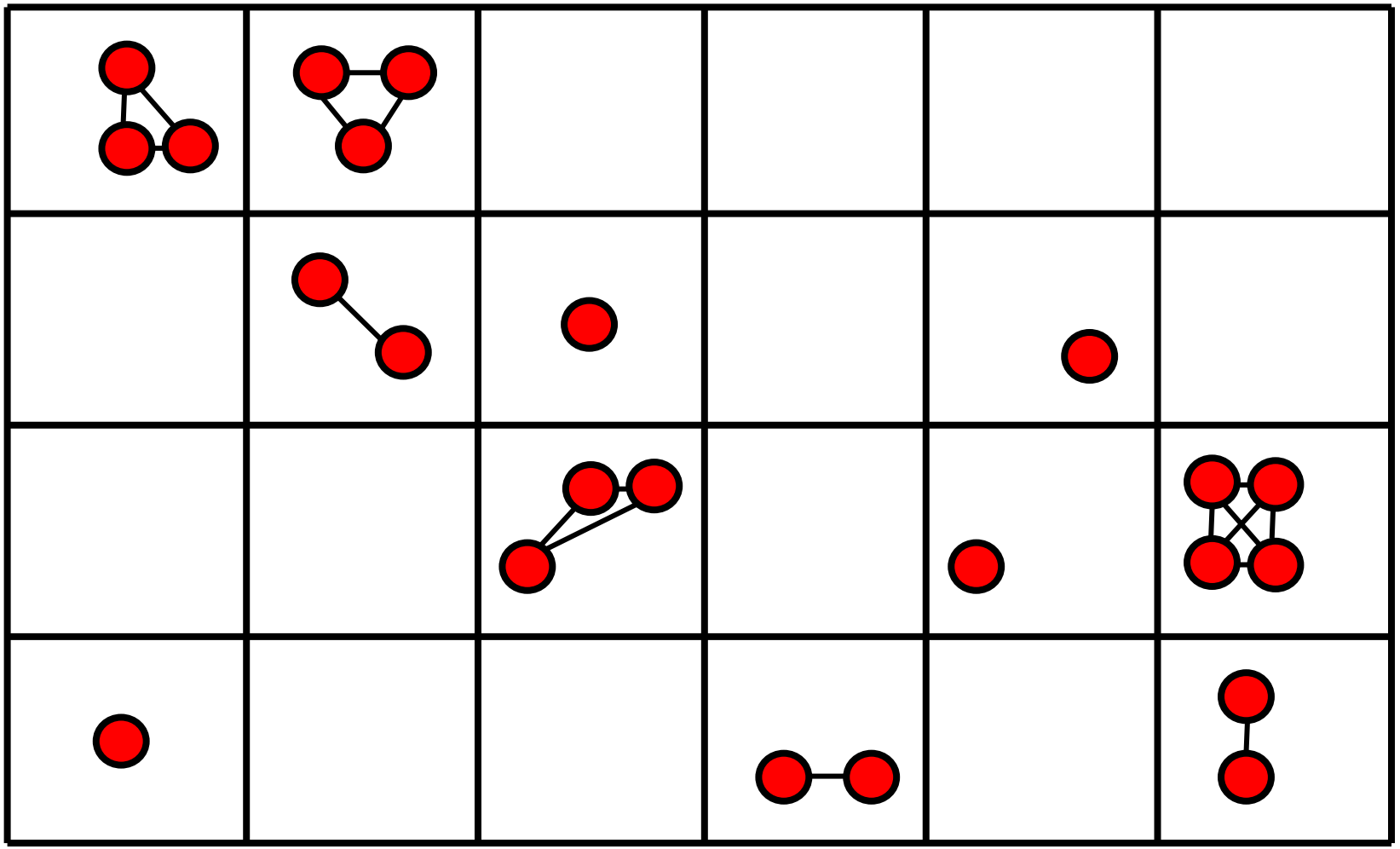
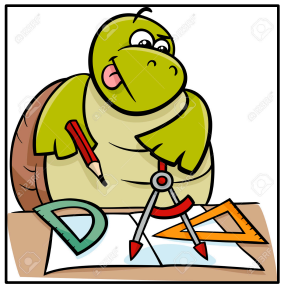




Decomposition II(a): Backbone Graph





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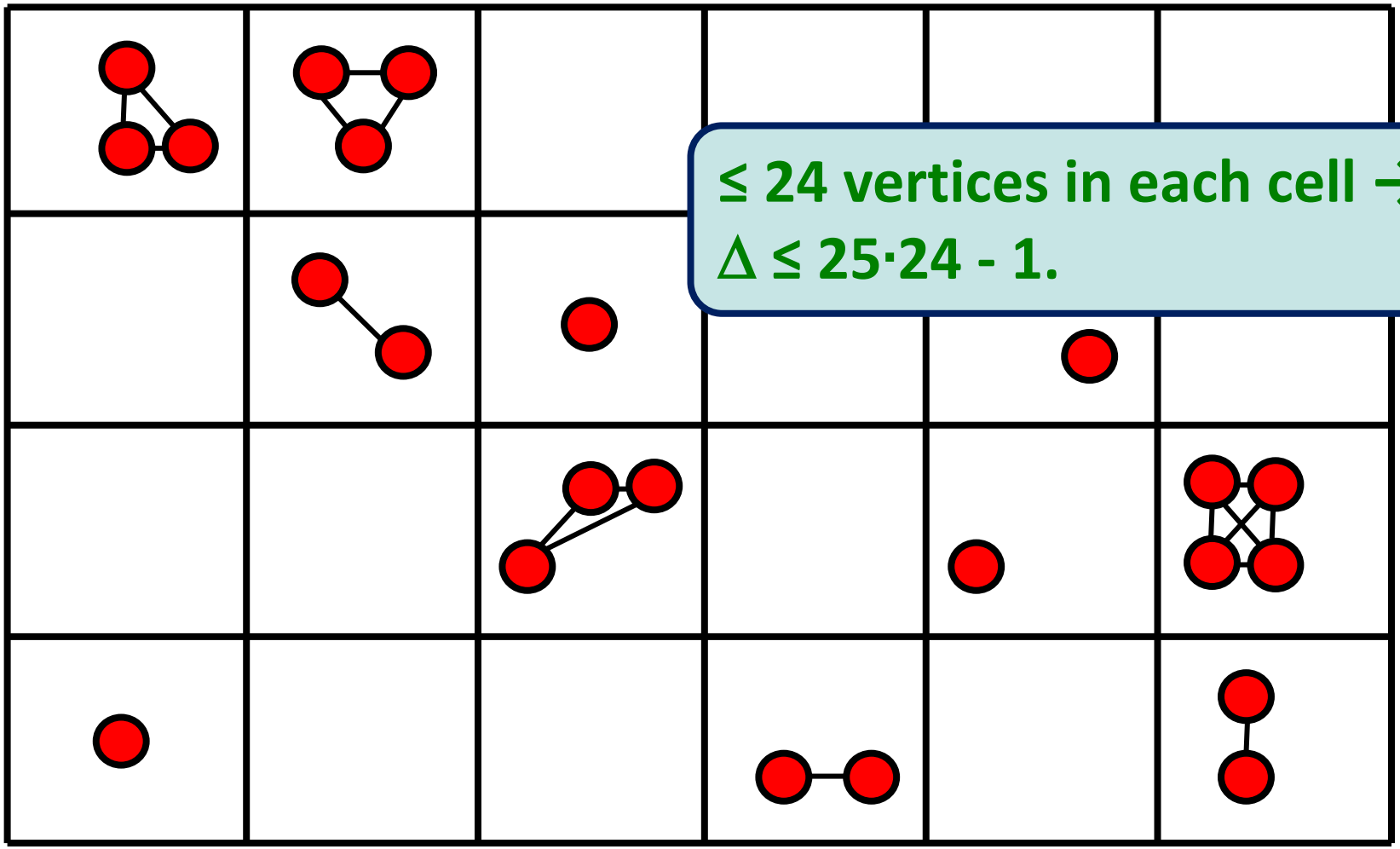
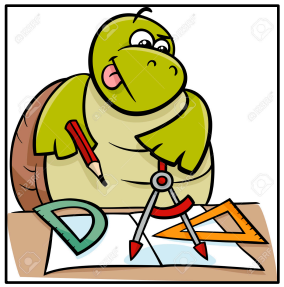


≤ 24 vertices in each cell →
Δ ≤ 25·24 - 1.

Introduction: Branch on Cliques

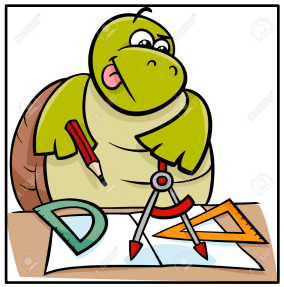
Decomposition I(b). Given w and a UDG G of max. degree Δ, in time $O^*(2^{O(w\cdot\Delta)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\Delta)$.



Decomposition II(a): Proof Sketch

Decomposition I(b). Given w and a UDG G of max. degree Δ, in time $O^*(2^{O(w\cdot\Delta)})$ either determine that G has a wxw grid minor or output a tree decomp. of width $O(w\cdot\Delta)$.

The backbone graph we compute for a UDG G is a UDG in itself, and its max. degree is bounded by a fixed constant.



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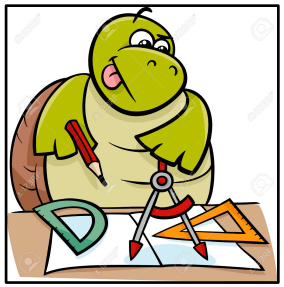
So in time $O^*(2^{O(w)})$... We either know that B has a wxw grid minor or output a tree decomp. of B of width $O(w)$.



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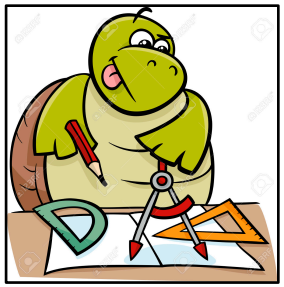
B is a subgraph of G.



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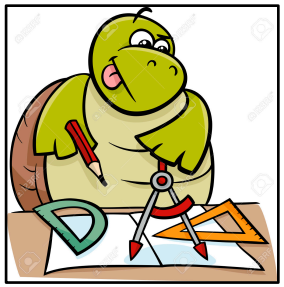
So in time $O^*(2^{O(w)})$... We either know that G has a wxw grid minor or output a tree decomp. of G of width $O(w)$.

Replace every vertex of B by all the vertices in its cell. (Not difficult to verify that this results in a tree decomp. of G.)



Decomposition II(a): Statement

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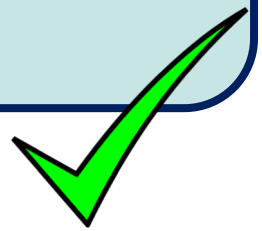

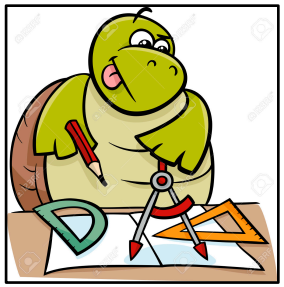

Decomposition II(a): Applications

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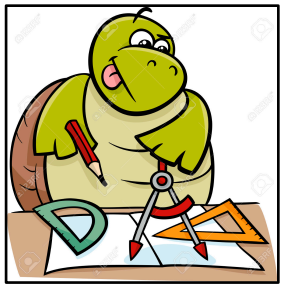
How to use this decomposition?



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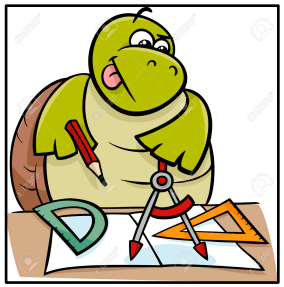
How to use this decomposition?

- Bound the max clique size by $O(k)$.



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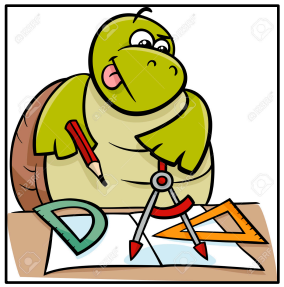
How to use this decomposition?

- Bound the max clique size by $O(k)$.
  (Easy for VC/FVS/Cycle Pack./Long Cycle/Exact Cycle).



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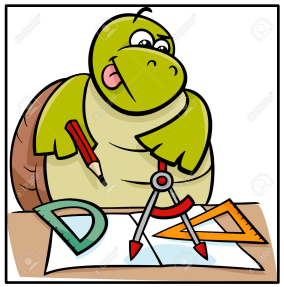
How to use this decomposition?

- Bound the max clique size by $O(k)$.
- A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-instance.



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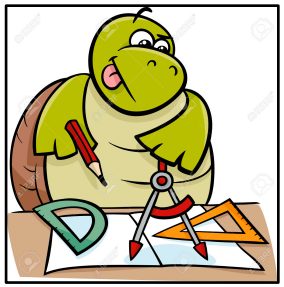
(Does not work for Exact Cycle!)



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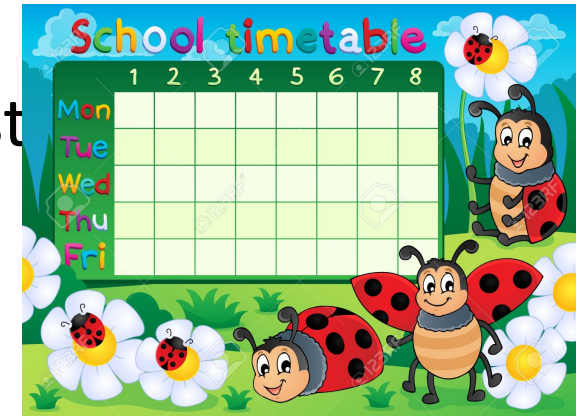
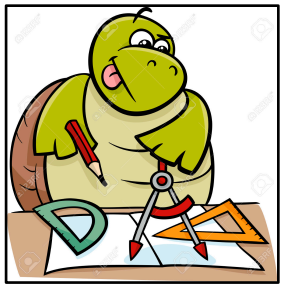
How to use this decomposition?

- Bound the max clique size by $O(k)$.
- A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-inst
- Special tree decomposition? DP.





Decomposition II(a): Applications

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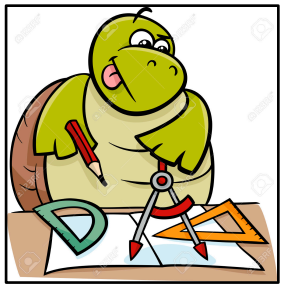
How to use this decomposition?

- Bound the max clique size by $O(k)$.
- A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-instance.
- Special tree decomposition? DP.

Easy for VC/FVS: In each bag (of size $O(k^{1.5})$), only $O(\sqrt{k})$ may not be chosen.



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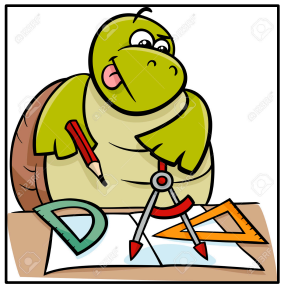
- A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-instance.
- Special tree decomposition? DP.

Easy for VC/FVS: Time $\boldsymbol{O^*(2^{O(k^{0.5}\log k)})}$.



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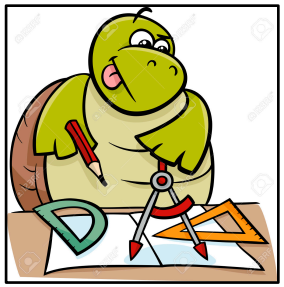
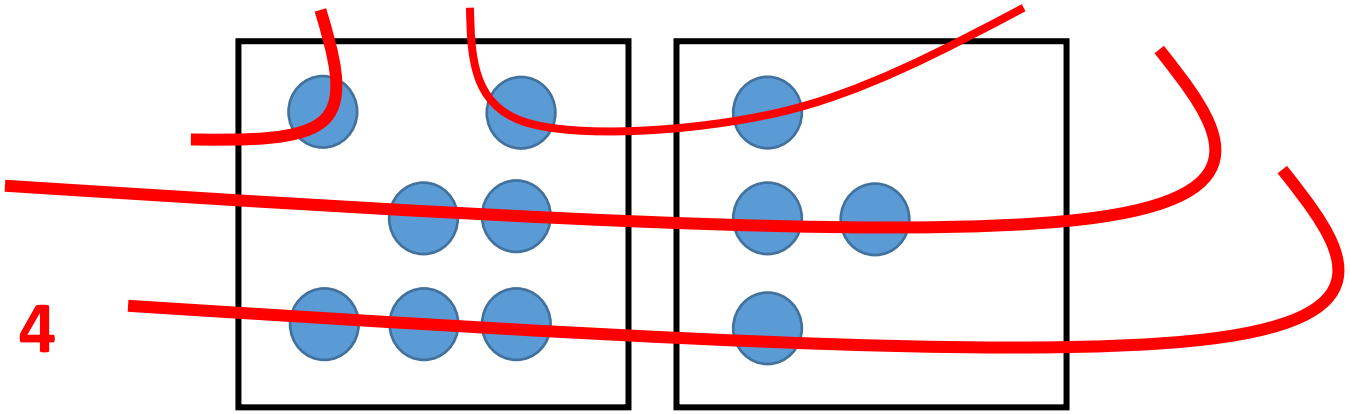
What about Cycle Packing and Long Cycle/Path?

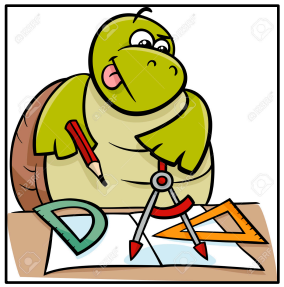


Decomposition II(a): Applications

Existence of a solution where the interaction between any two cells is bounded by $O(1)$.

Cycle Packing. How many cycles ``pass'' through two cliques?





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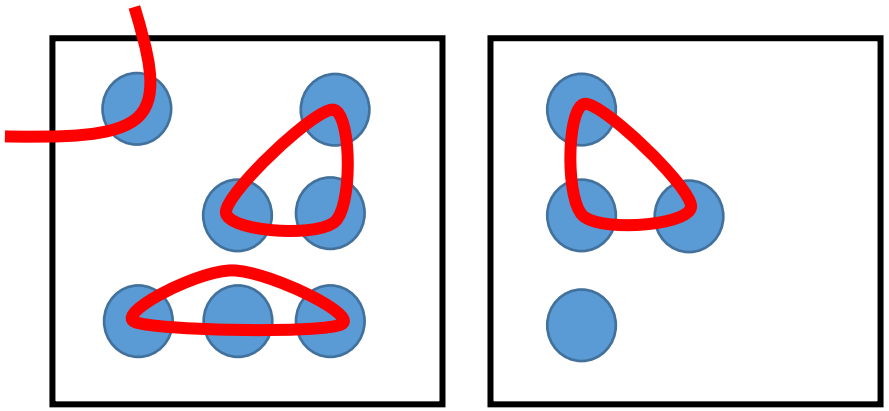
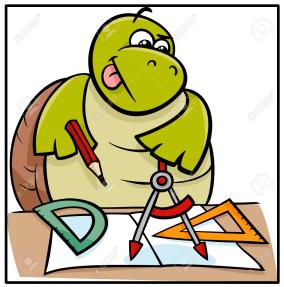
4





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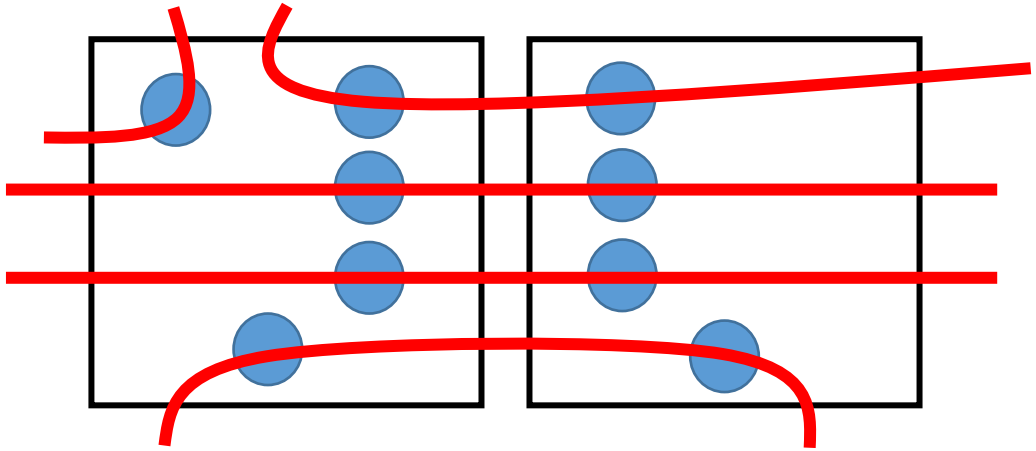
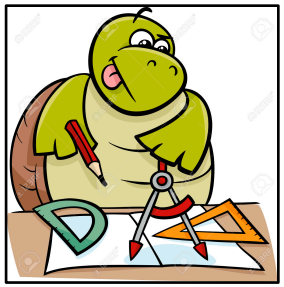


5



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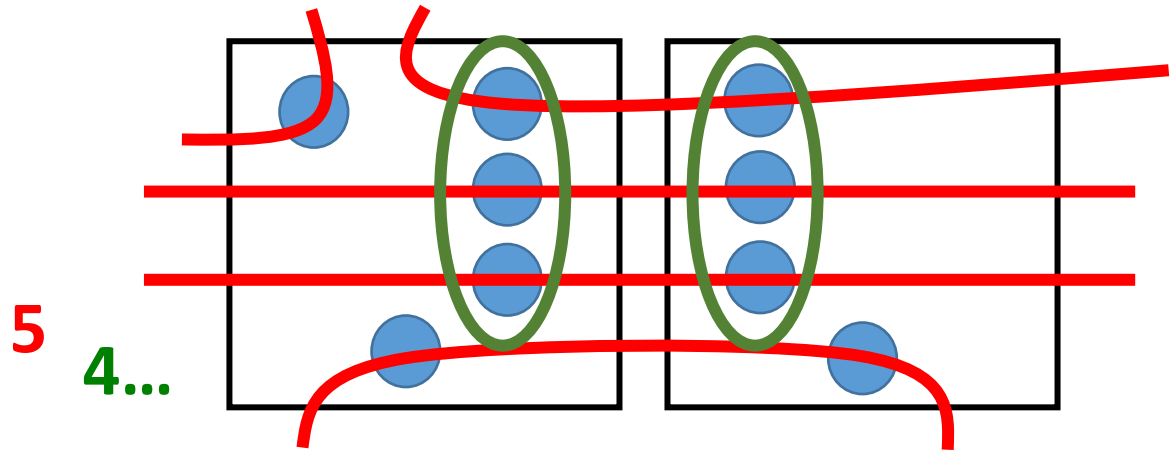

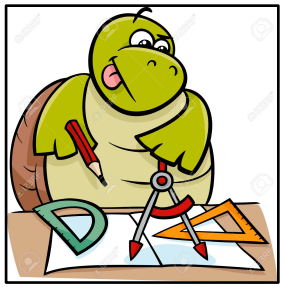

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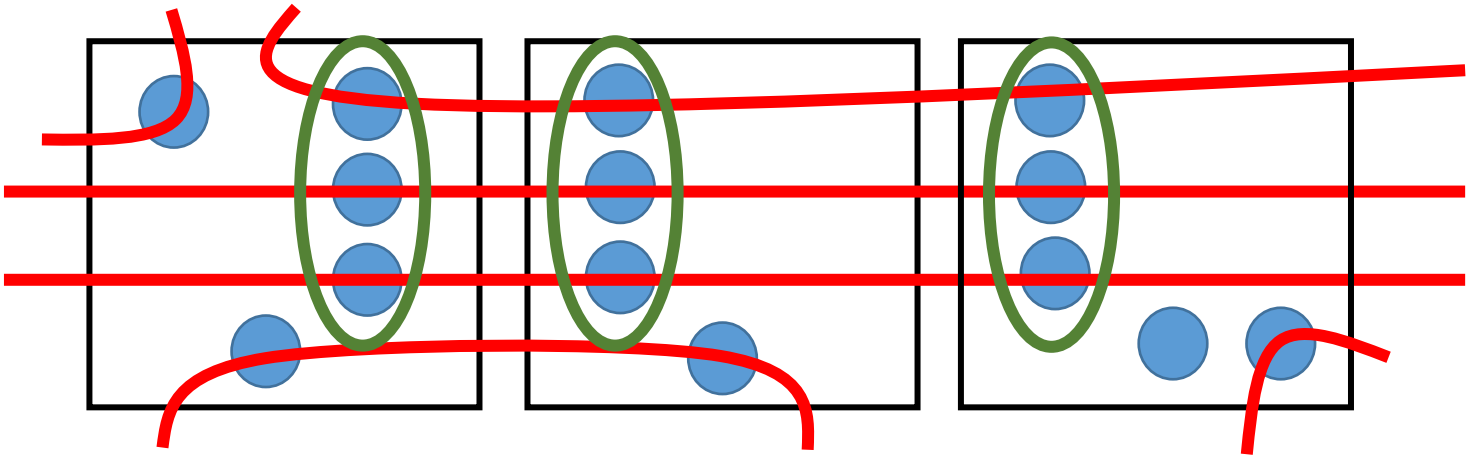
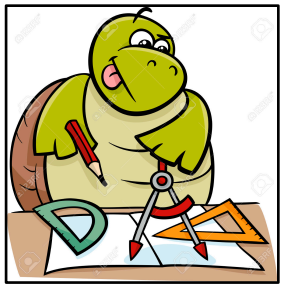
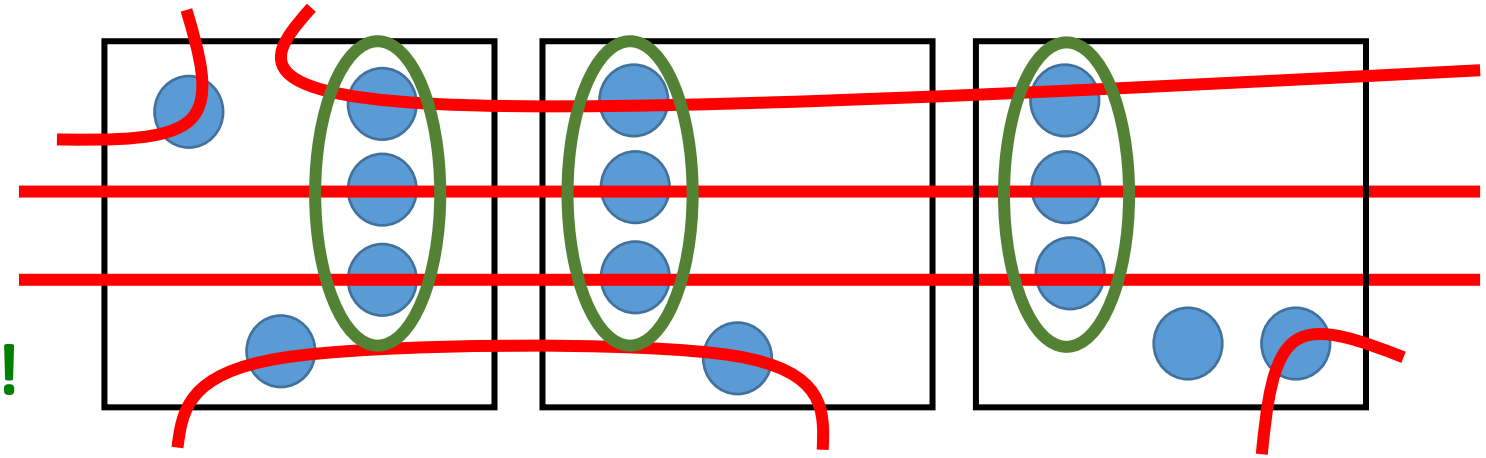


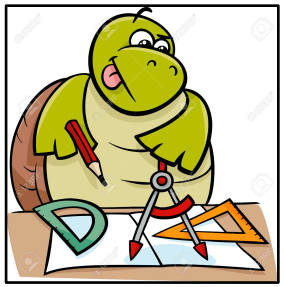


Decomposition II(a): Applications

Existence of a solution where the interaction between any two cells is bounded by *O*(1).

Cycle Packing. Every cycle that passes through a cell uses at most two of its vertices, so the interaction is *O*(1).





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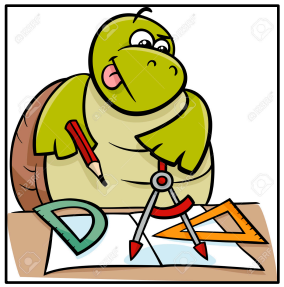
Existence of a solution where the interaction between any two cells is bounded by $O(1)$.

Long Cycle/Path and Exact Cycle. We also have an exchange argument that bounds the interaction between any two cells.



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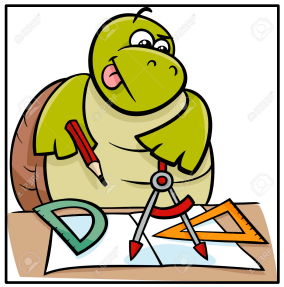
Note. The exchange is done in a way that ensures bounded interaction for all pairs of cells simultaneously.



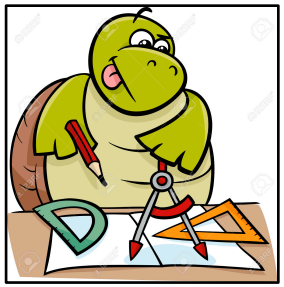
Decomposition II(a): Applications

Decomposition II(a). Given w and a UDG G, in time $O^*(2^{O(w)})$ either determine that G has a wxw grid minor or compute a tree decomposition where each bag consists of $O(w)$ ``cells'' (cliques).

How to use this decomposition?

- Bound the max clique size by $O(k)$.
- A $c\sqrt{k}$x$c\sqrt{k}$ grid minor? Yes/No-instance.
- Special tree decomposition? DP.

What about Cycle Packing and Long Cycle/Path?



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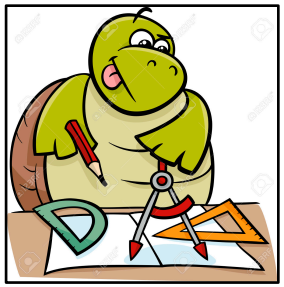
Cycle Packing/Long Cycle/Path: In each bag (of size $O(k^{1.5})$), only $O(\sqrt{k})$ are ``reserved'' for the future.



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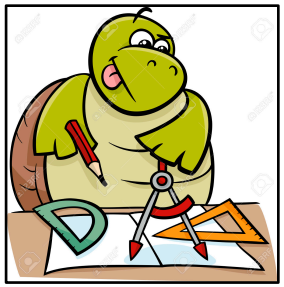
Cycle Packing/Long Cycle/Path: Time $\boldsymbol{O^*(2^{O(k^{0.5}\log k)})}$.

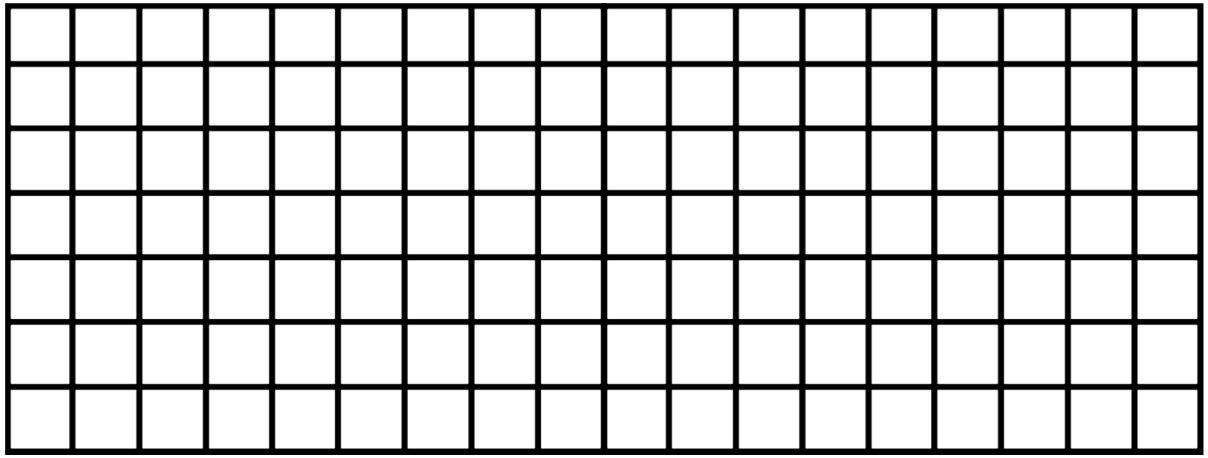


Decomposition II(b): Outline

Decomposition II(b): Based on Baker's Technique.

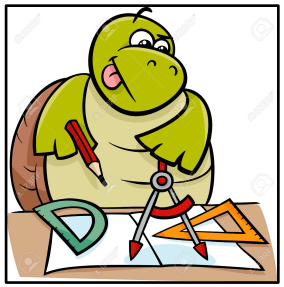
1. Label **pairs of consecutive** columns/rows of the clique-grid graph by 1,…,k+1. ``Guess'' column and row labels whose removal results in a 2$k$x2$k$ ``piece'' that has a solution (if one exists).

1 1 2 2 3 3 4 4 1 1 2 2 3 3 4 4 1 1



Exact Cycle

k=3



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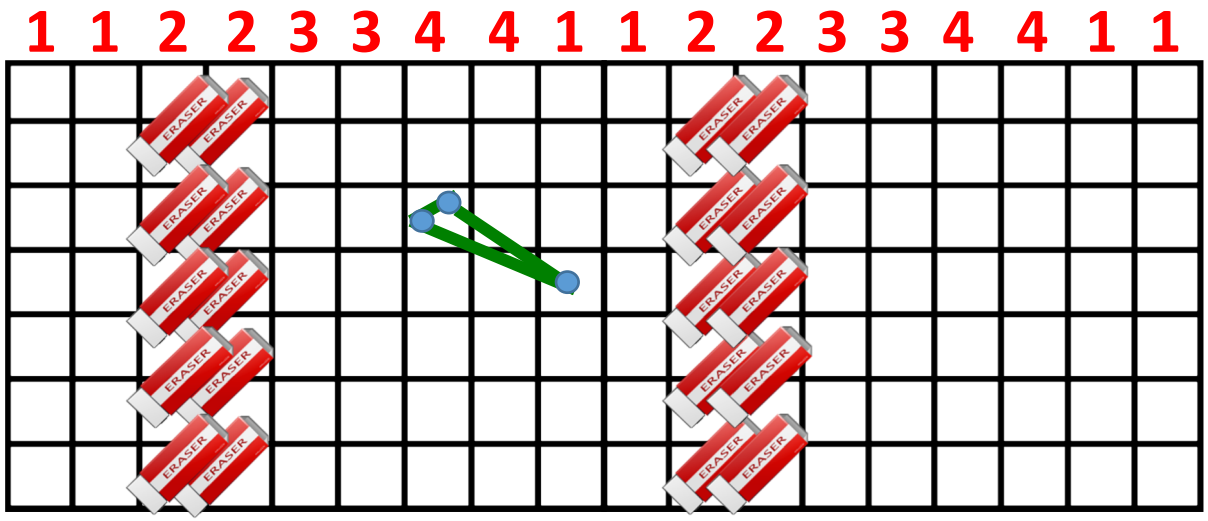

Exact Cycle

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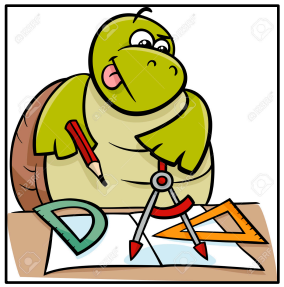
Deleting 2 (or 3) is fine.



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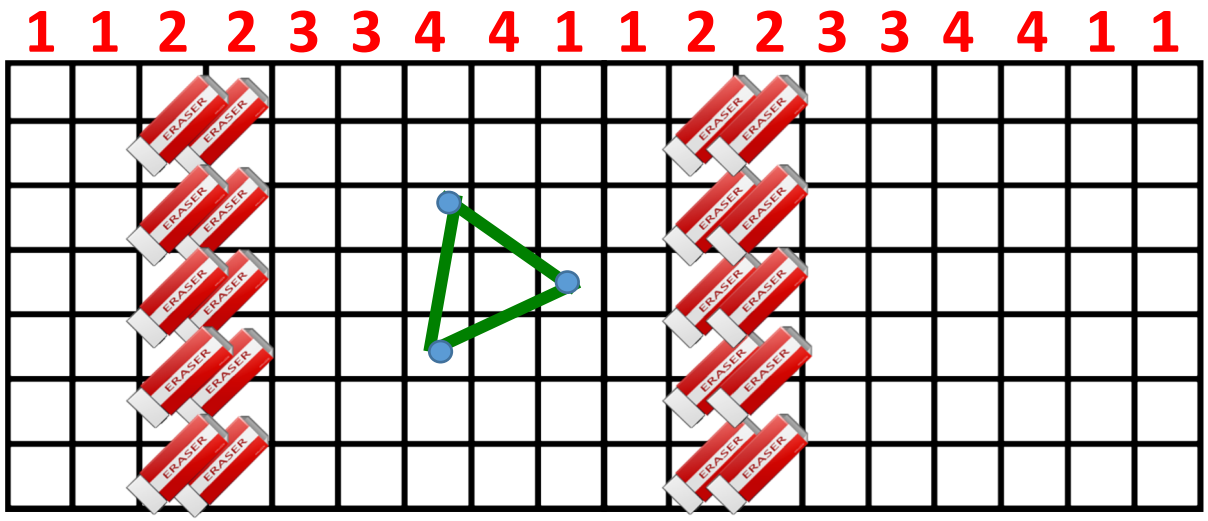

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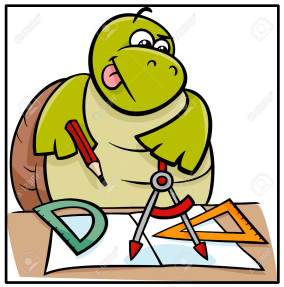

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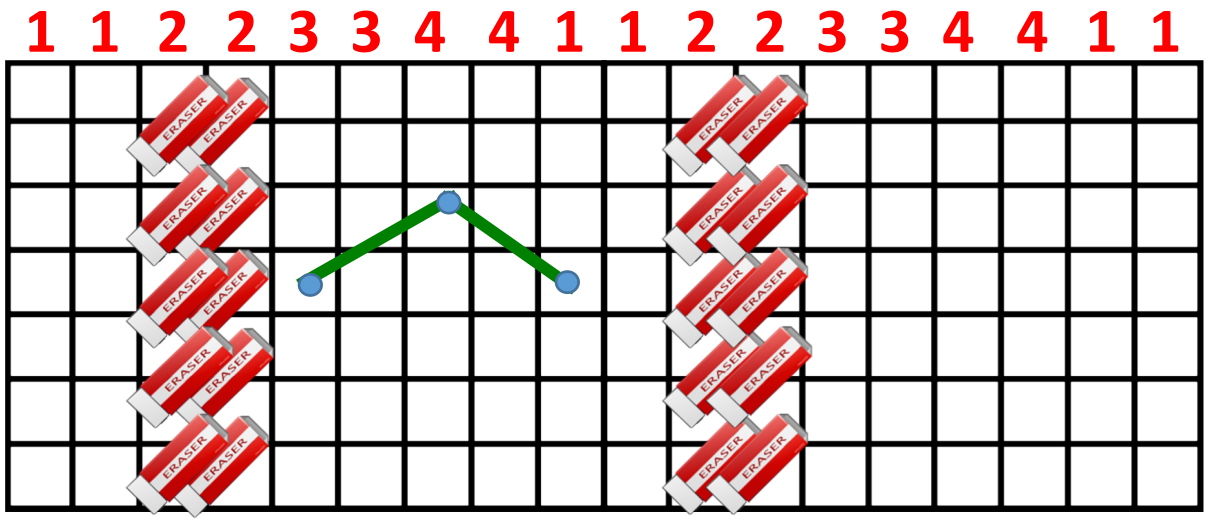

Exact Path

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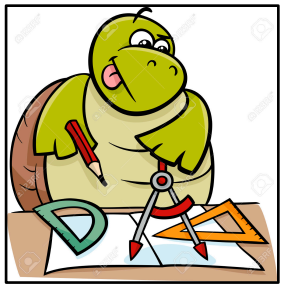

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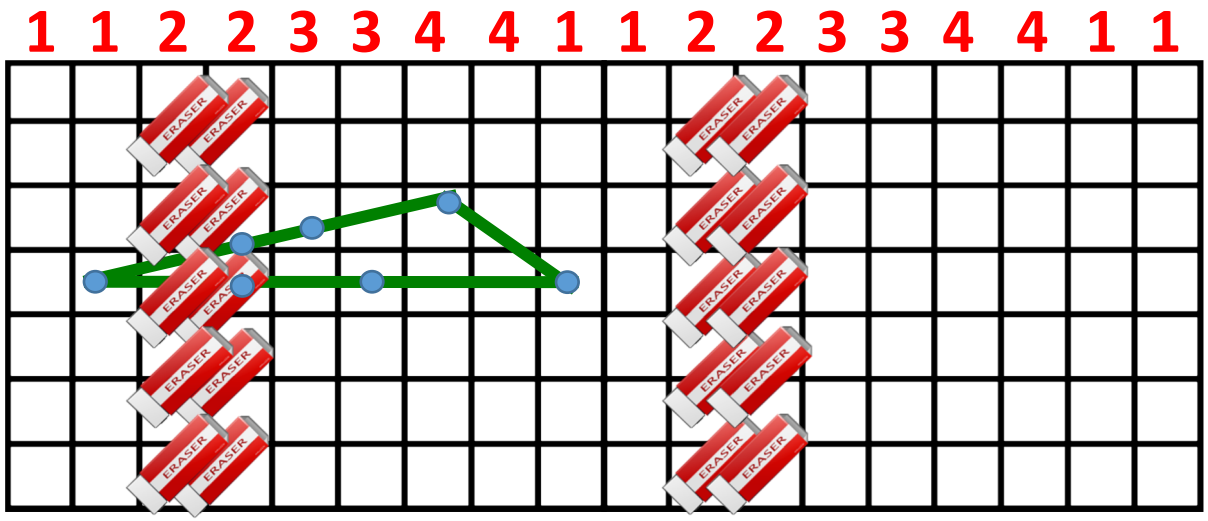

Long Cycle?

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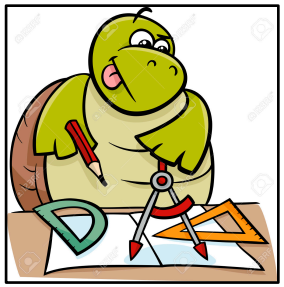
???



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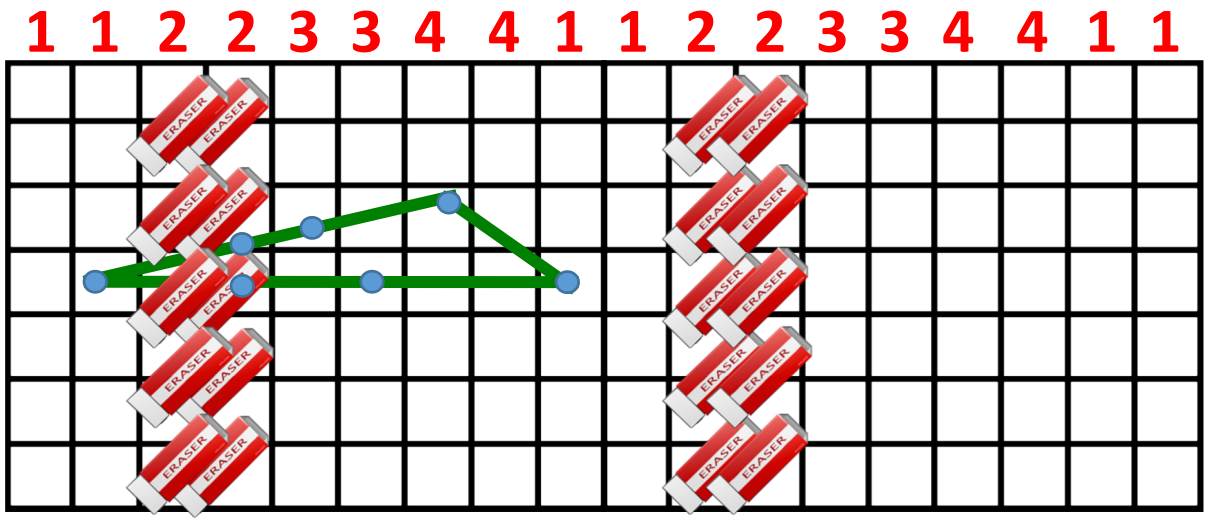

Long Cycle

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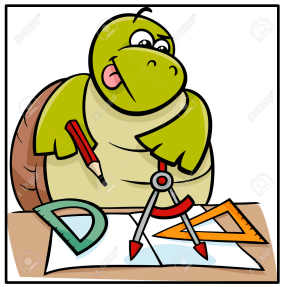
With a minor (known) trick, can handle this case.



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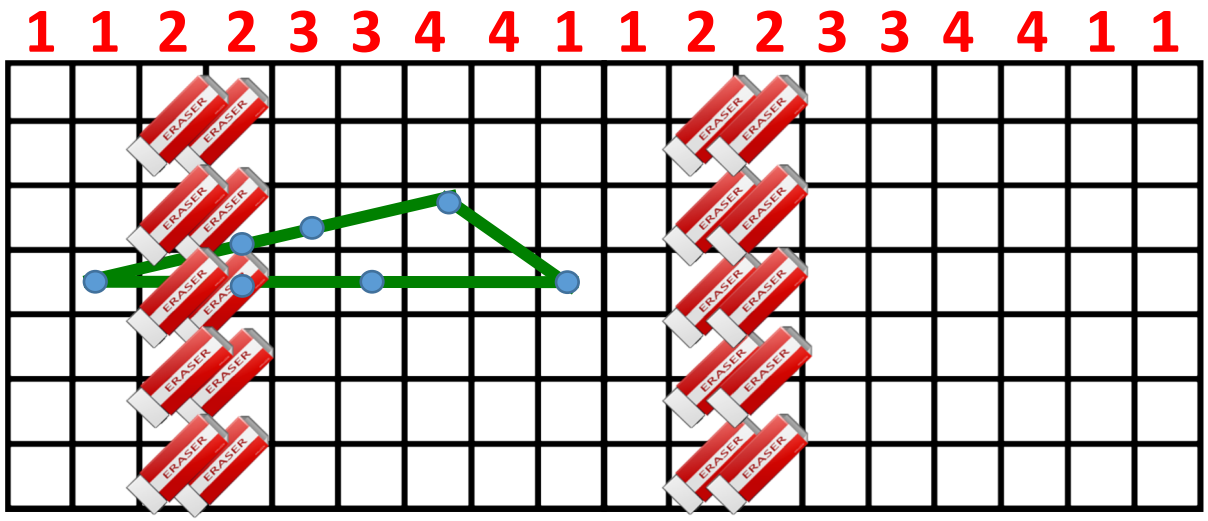
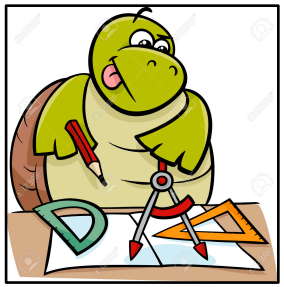


FVS?

***k*=3**

???



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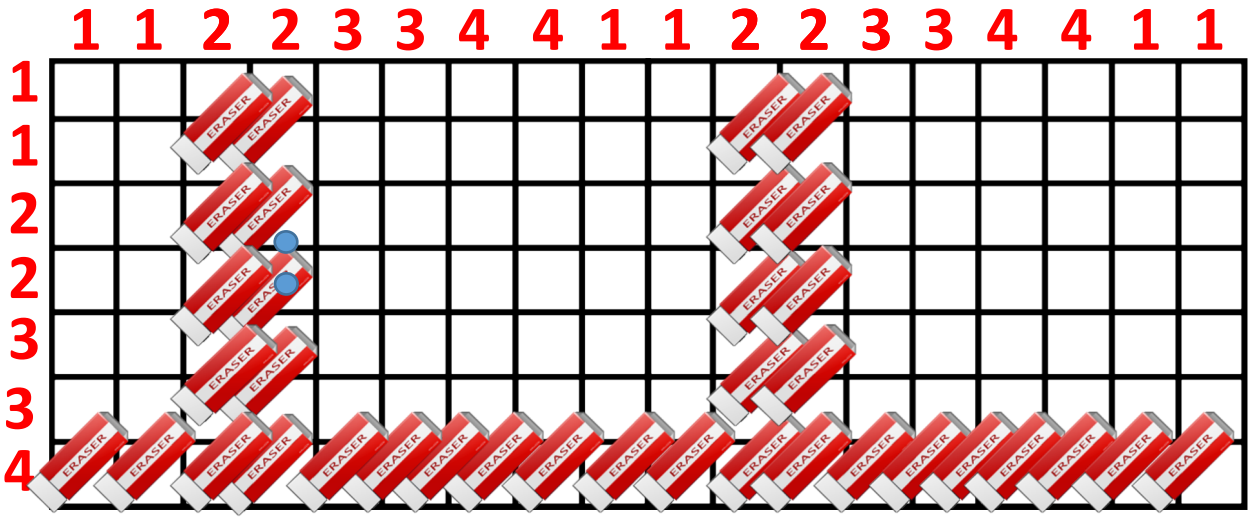
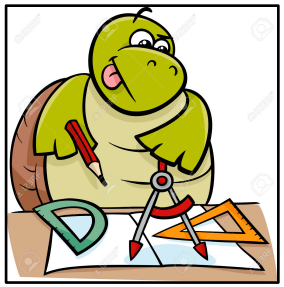


Repeat for rows.



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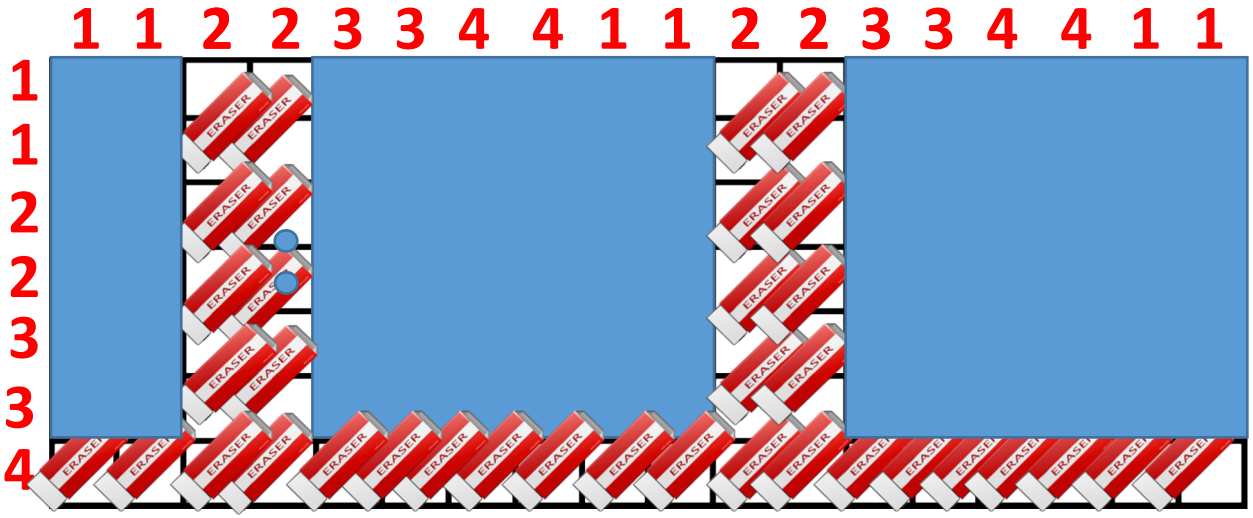

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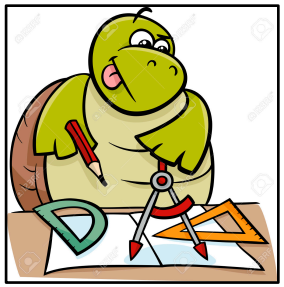

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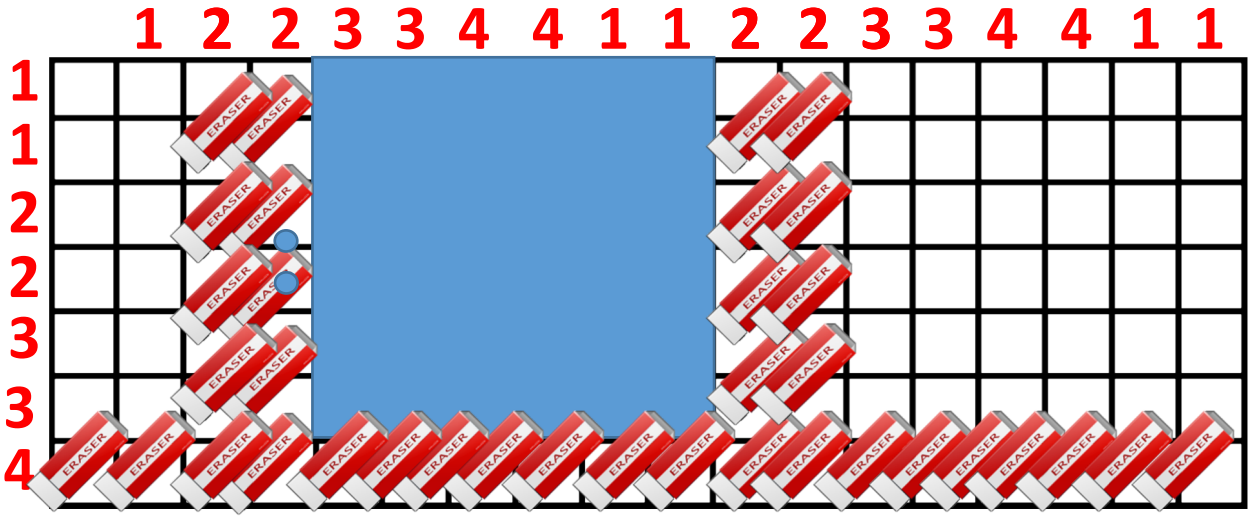

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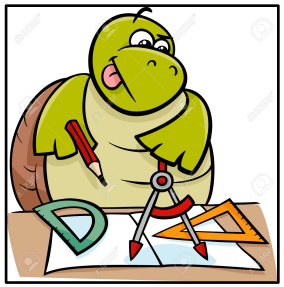
Now, zoom into a ``piece''.



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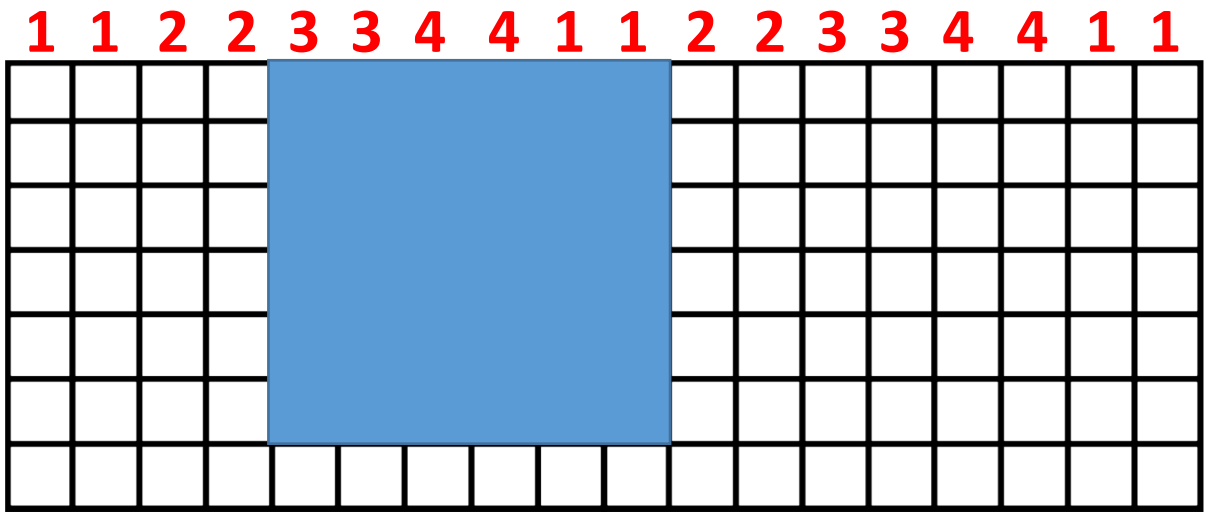

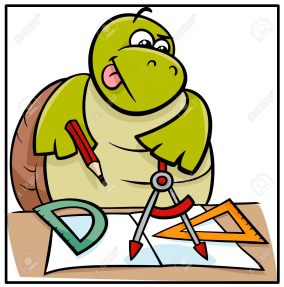

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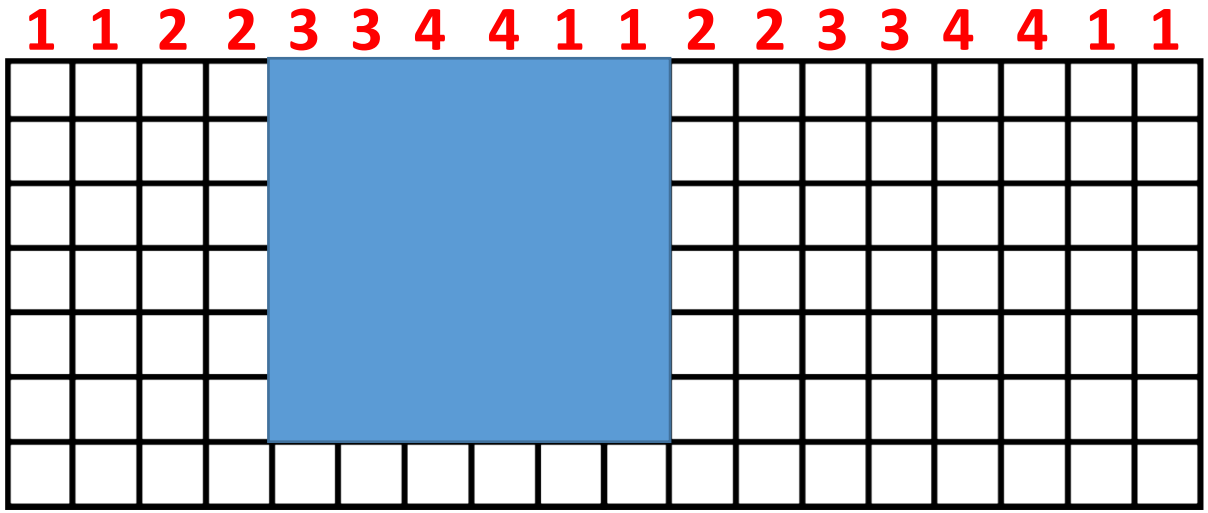
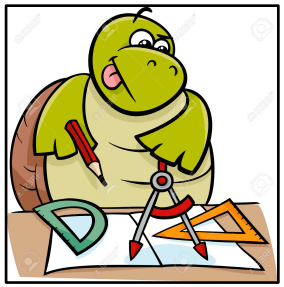


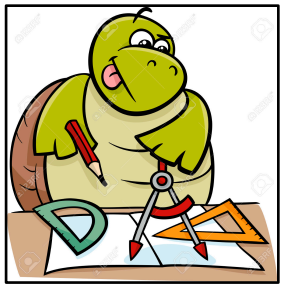
Such a label must exist for Long/Exact Cycle/Path



Decomposition II(b): Outline

Decomposition II(b): Based on Baker's Technique.

3. ``Guess'' which vertices in the columns and rows of the chosen labels are used, and delete the rest of the vertices in these columns and rows.



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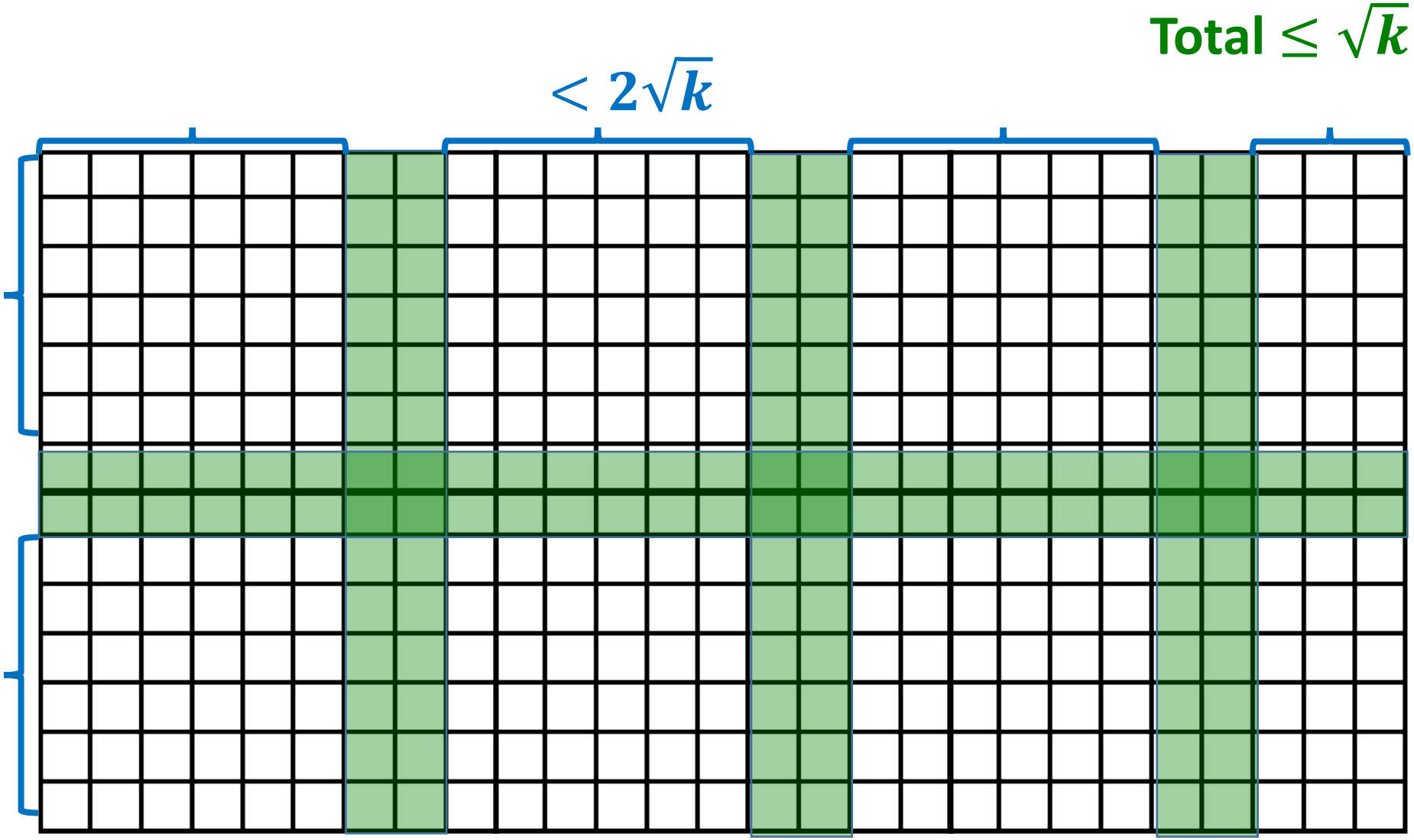
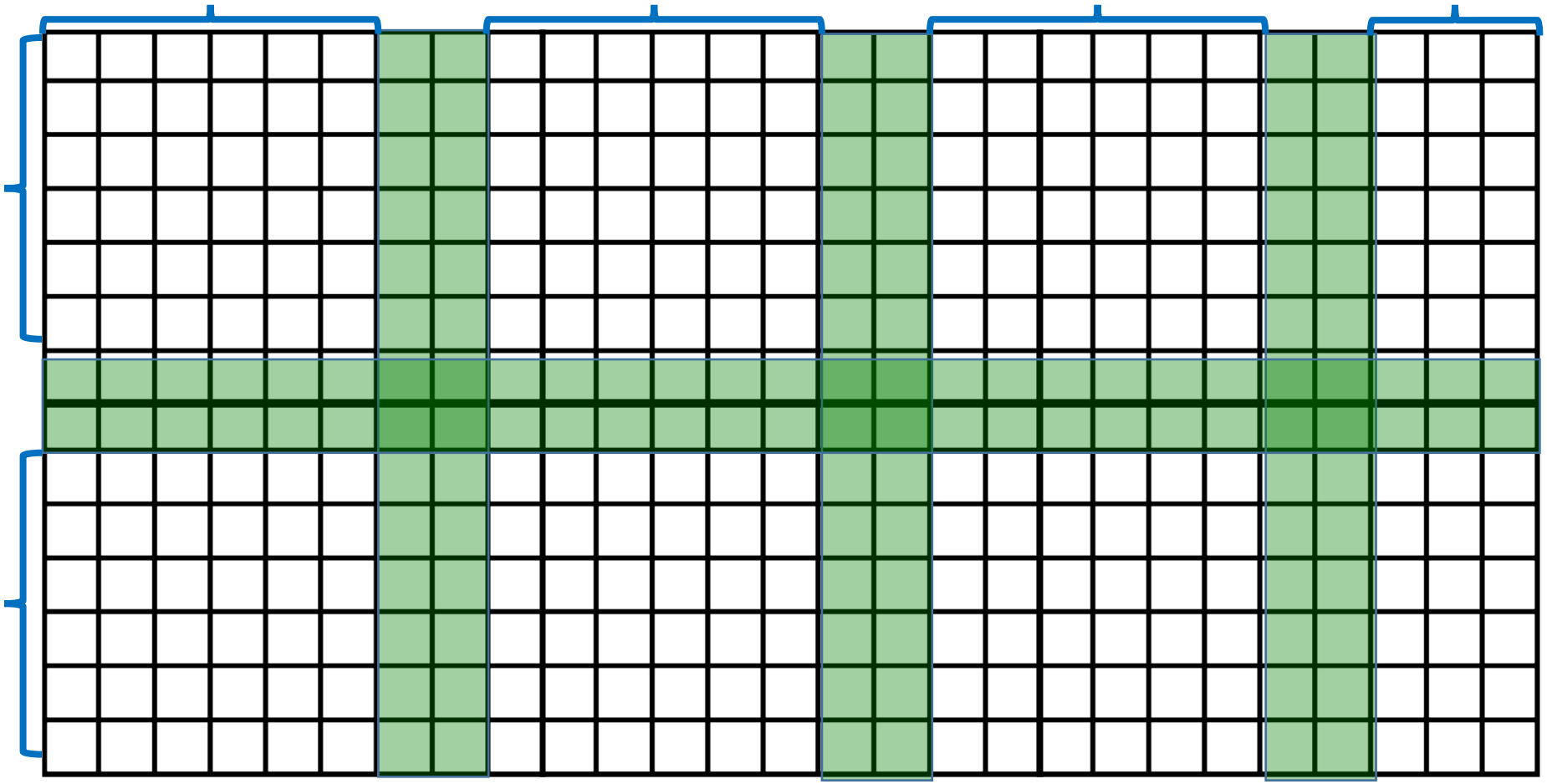
4. At this point, we can construct a tree decomposition such that each of its bags consists of $O(\sqrt{k})$ cells.

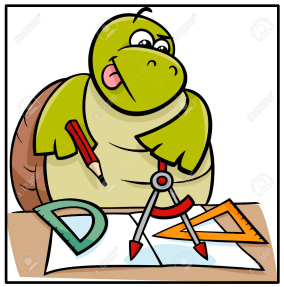

$< 2\sqrt{k}$
Total $\leq \sqrt{k}$

- Insert the $< \sqrt{k}$ green cells that are occupied to all bags.
- In each ``subpiece'', insert **three** columns at a time.

Total $\leq \sqrt{k}$

$< 2\sqrt{k}$





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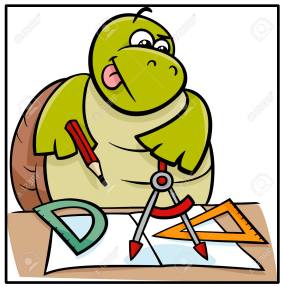

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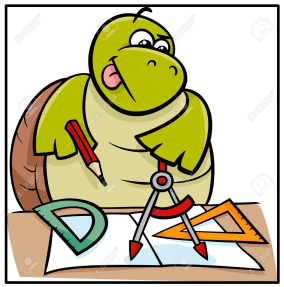
(Solve Long/Exact Cycle/Path in time $O^*(2^{O(k^{0.5}\log k)})$.)



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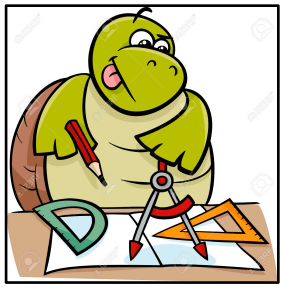
Note. We constructed $O^*(2^{O(k^{0.5}\log k)})$ decompositions.

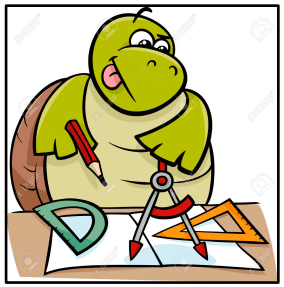
Outline

- Introduction
- Decomposition II: General Decomposition
- **Decomposition III: Contraction Decomposition**



Vertex/Edge Decomposition

Vertex/Edge Decomposition Theorem. Let G be a graph, and k be a positive integer. Then, the vertex/edge set of G can be partitioned into k+1 sets (possibly empty) such that any k of these sets (alternatively, deleting any set) induce a graph of treewidth $f(k)$.



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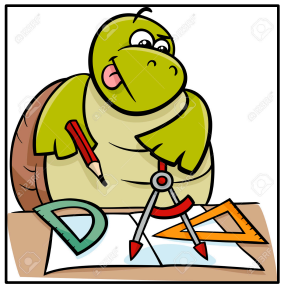

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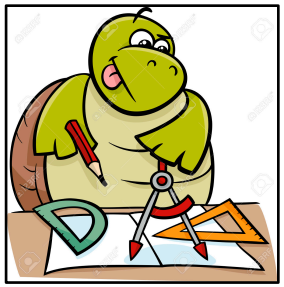
Planar graphs. [Baker, J. ACM'94]

Bounded genus, apex-minor-free. [Eppstein, Algorithmica'00]

***H*-minor-free.** [Demaine, Hajiaghayi, Mohar, FOCS'05], [DeVos, Ding, Oporowski, Sanders, Reed, Seymour, Vertigan, JCTB'04]



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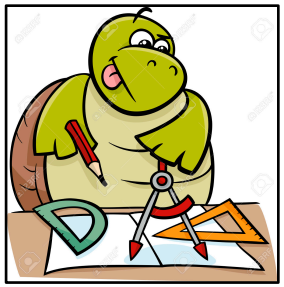

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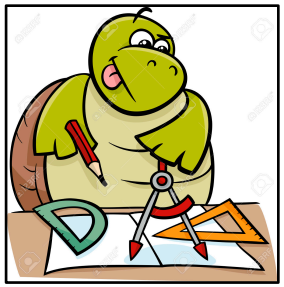
Note. Does not exist for UDGs: consider a large clique!



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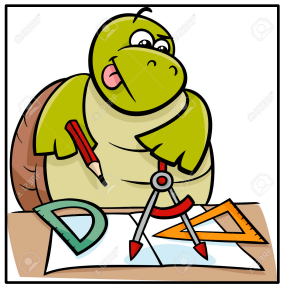
However... We sidestepped this issue previously by thinking of cells as units.



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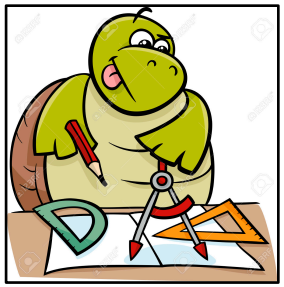
Useful to design approx. and parameterized algorithms for ``local'' problems.

For ``non-local'' problems …



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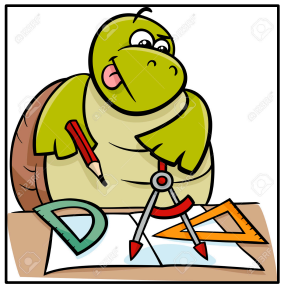

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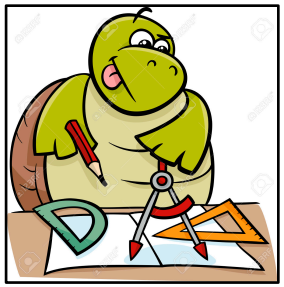

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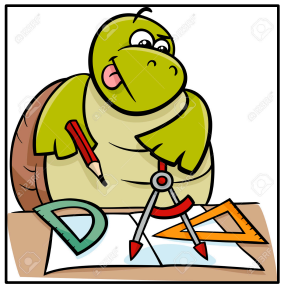

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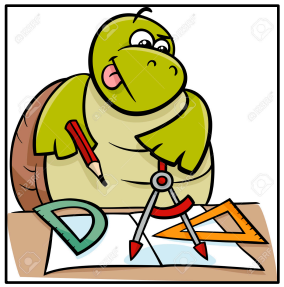

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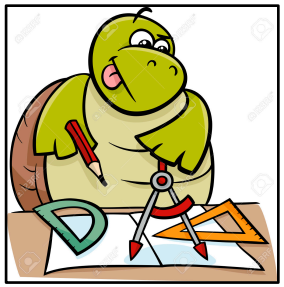

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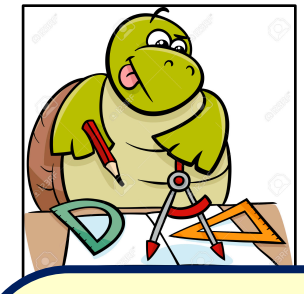
How to use? Suppose that we seek a sol. S of k edges. Then, there is a ``piece'' that does not intersect $S$. After we contract the piece, $S$ is preserved. Now, we solve a ``similar'' problem on a graph of bounded treewidth, and lift the sol. to the original problem.



Weak Contraction Decomposition

Weak CDT. Let G be a graph, and $k \in \mathbb{N}$. There exist fixed constants $\alpha, \beta \in \mathbb{N}$ (independent of G and k), and subsets of edges of G, $E_1, E_2, \ldots, E_{\alpha k}$, such that

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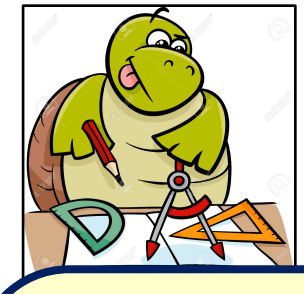

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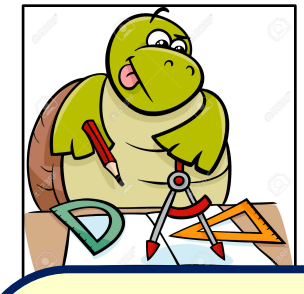

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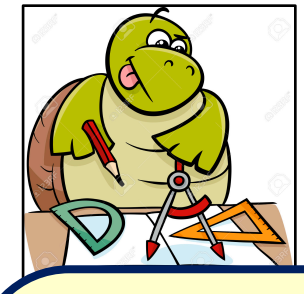

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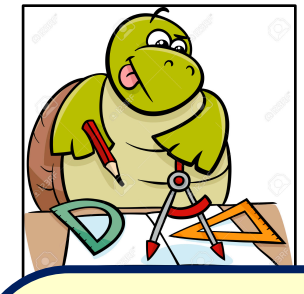
$E_1, E_2, \ldots, E_{\alpha k}$: **CDT.** A partition.

Weak CDT. ``Almost'' a partition.



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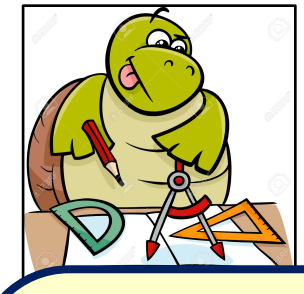
Weak CDT suffices for algorithmic applications.



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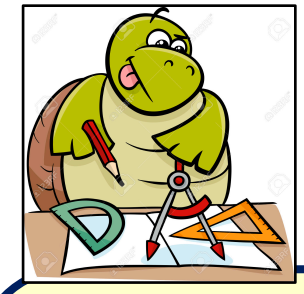
Theorem (CDT in UDGs). Given a **UDG** G and $k \in \mathbb{N}$, such subsets $E_1, E_2, \ldots, E_{\alpha k}$ exist and are computable in polynomial time.



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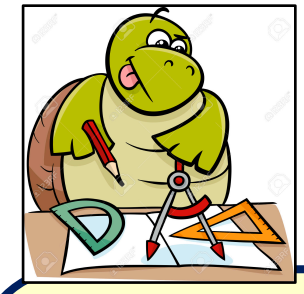
Recall. UDGs do not admit a Vertex/Edge Decomposition.



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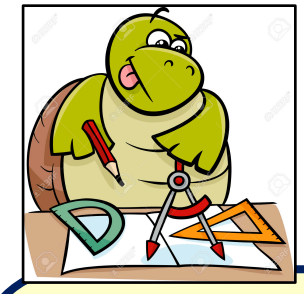
Theorem (CDT in UDGs). Given a **UDG** G and $k \in \mathbb{N}$, such subsets $E_1, E_2, \ldots, E_{\alpha k}$ exist and are computable in polynomial time.

The construction that gives rise to this result is the one we will use in our applications; sidestep the quadratic dependency on k.



Weak Contraction Decomposition

Theorem (Informal). There is a polynomial-time algo. that, given a UDG G and $k \in \mathbb{N}$, outputs a collection of $O(k)$ tree decompositions of G with the following properties.

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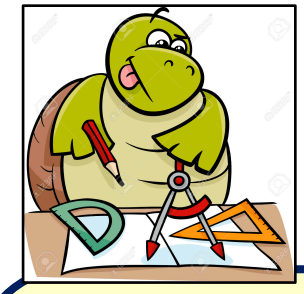

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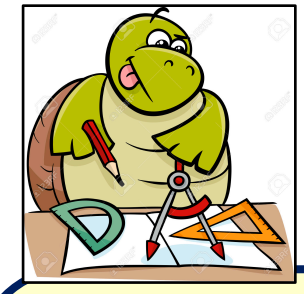
For any bag B and any edge $\{u,v\} \in S$ where $u,v \in B$, u,v belong to either diff. pieces or a single piece **that induces a clique**.



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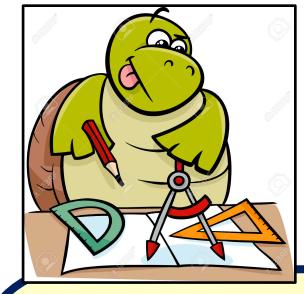

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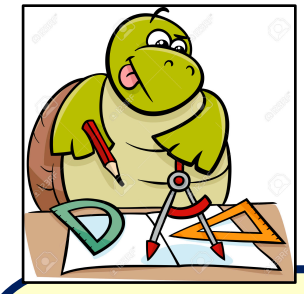
What to do with the clique pieces? (We need to contract all pieces to have linear treewidth.)



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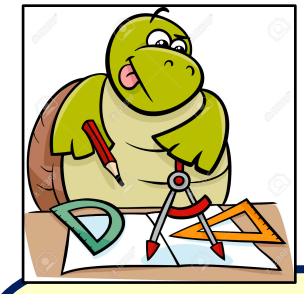

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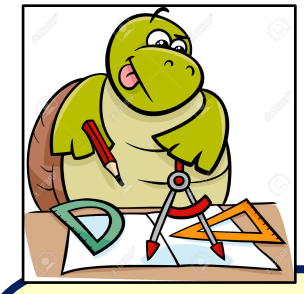
For some cut problems, because cliques are hard to break, we can prove that the total num. of vertices in the cliques that S intersects is small. **(Will see example soon.)**



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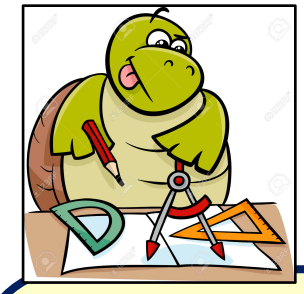
Utilize the previous observation to solve the problem using DP over the tree decomposition.



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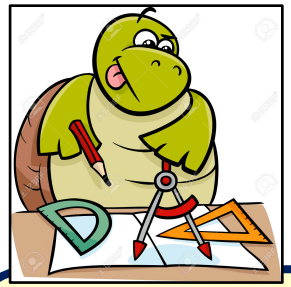
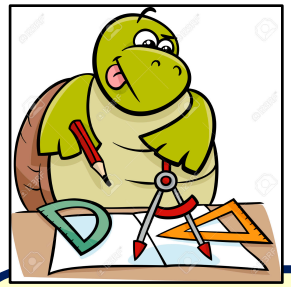
The proof of this theorem is quite technical, so we will not see it here.



Weak Contraction Decomposition

Applications (on UDGs).

- Minimum Bisection is solvable in time $2^{O(k)}n^{O(1)}$.
- s-Way Cut (and Steiner Cut) is solvable in time $k^{O(k)}n^{O(1)}$.
- Edge Multiway Cut Uncut is solvable in time $k^{O(k)}n^{O(1)}$.



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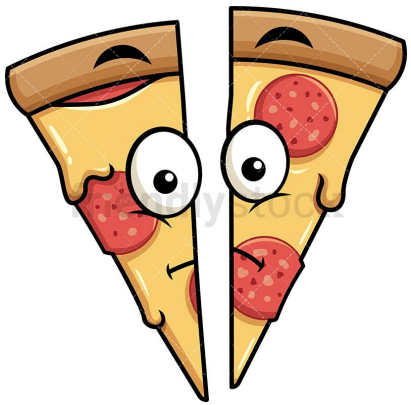
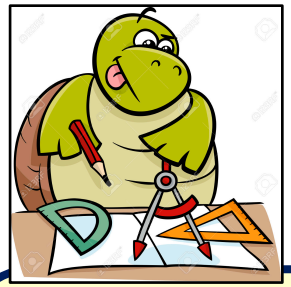
Given a graph G and a non-negative integer k, decide whether there is a partition (A,B) of $V(G)$ such that $|A|=|B|$ and $|E(A,B)| \leq k$.





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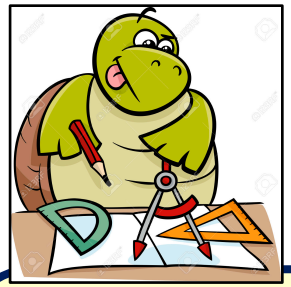
Given a graph G and a non-negative integer k, decide whether there is a subset X of $E(G)$ such that $|X| \leq k$ and G-X has at least s connected components.



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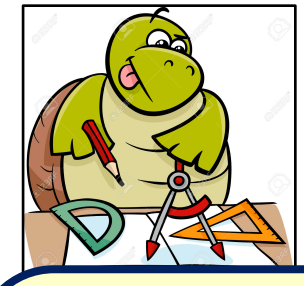
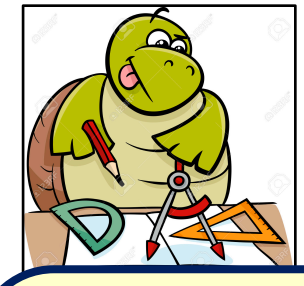
Given a graph G, a terminal set $T \subseteq V(G)$, an equivalence relation R on T, and a non-negative integer k, decide if there is $X \subseteq E(G)$ such that $|X| \leq k$ and in $G-X$ two terminals u,v belong to the same connected component iff $u \sim_R v$.



Example: Bisection

Claim (informal). Consider a bag B of a weak CDT that is ``good'' for a solution (X,Y). Then,

1. Every non-clique piece is contained in either X or Y.
2. The total number of vertices in clique pieces that cross (X,Y) is at most $2k$.



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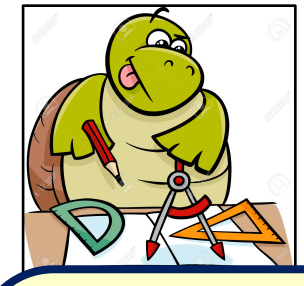
The number of edges contributed by a clique piece that crosses (X,Y) is at least the size of its not small side, which is at least half its size.



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In the DP: (i) Which cliques cross and how they cross ($2^{O(k)}$); (ii) For all non-crossing pieces—what is their side ($2^{O(k)}$); (iii) additional info (number of edges, size of side).

Summary

- Introduction: Bounded Degree/Clique Size; Branch on Cliques
- Decomposition II(a+b): General Decomposition
- Decomposition III(a+b): Contraction Decomposition