Bistellar and Edge Flip Graphs

of

Triangulations in the Plane

—

Geometry and Connectivity

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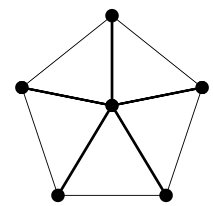
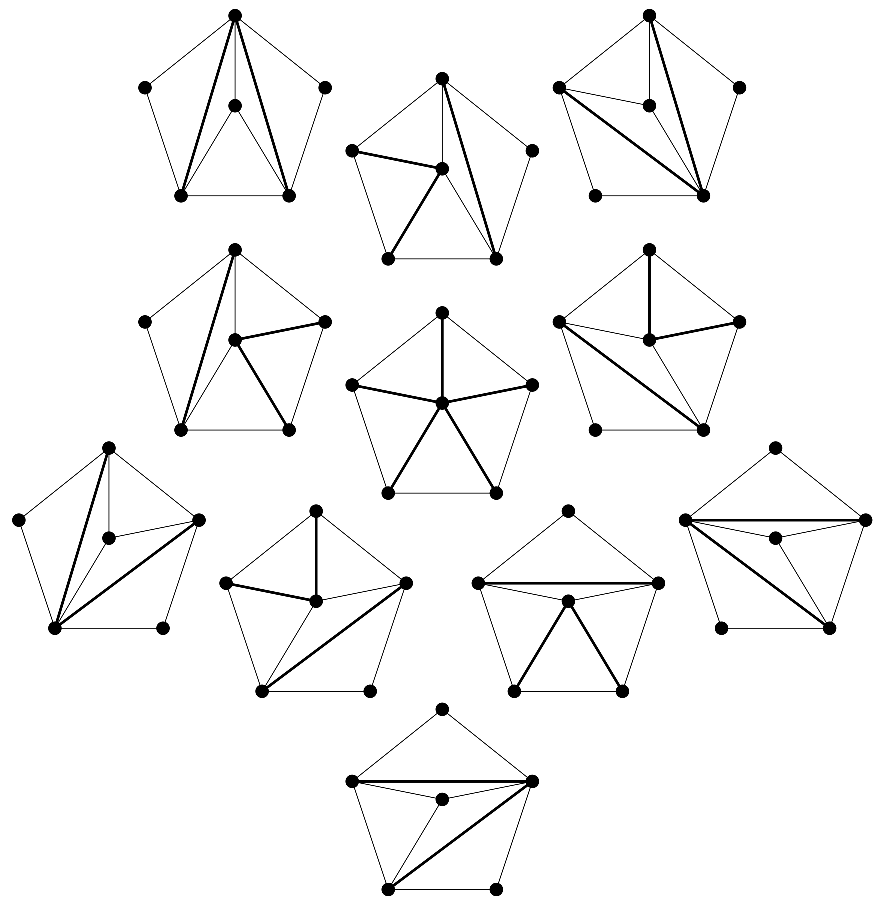
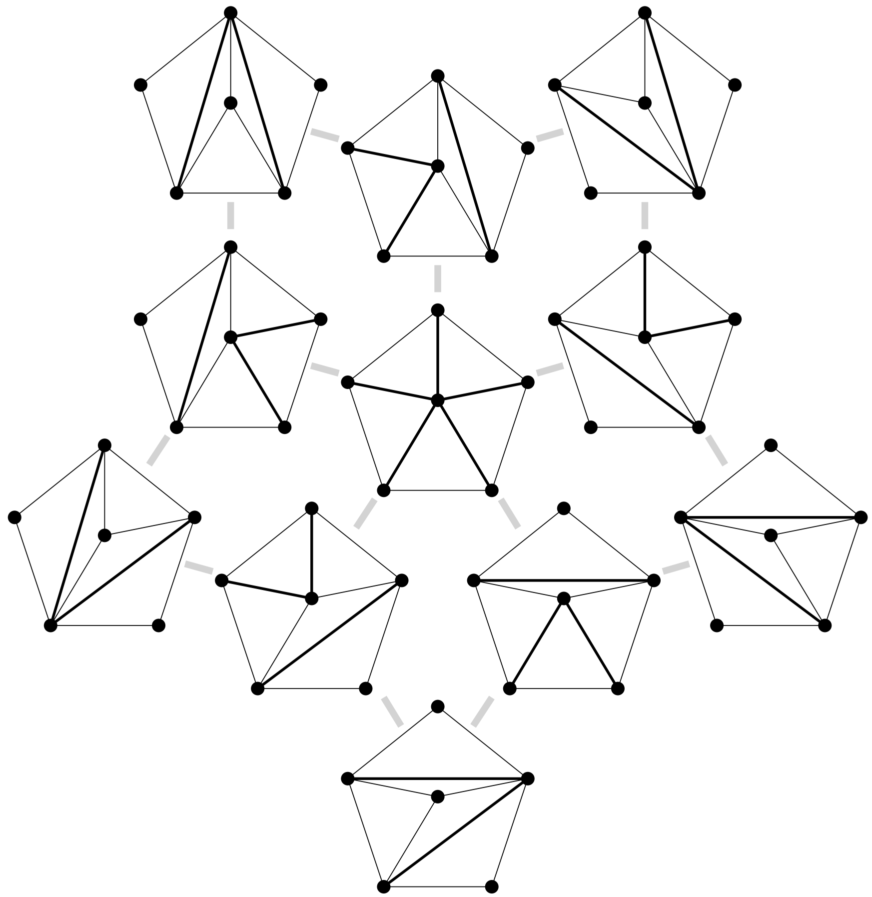
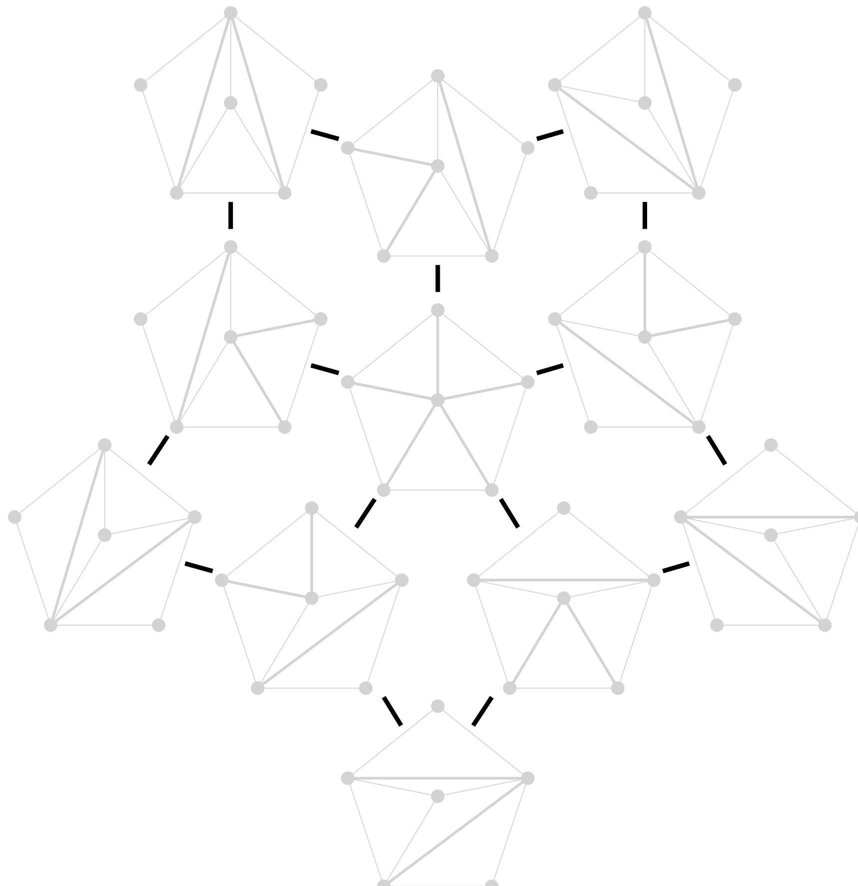
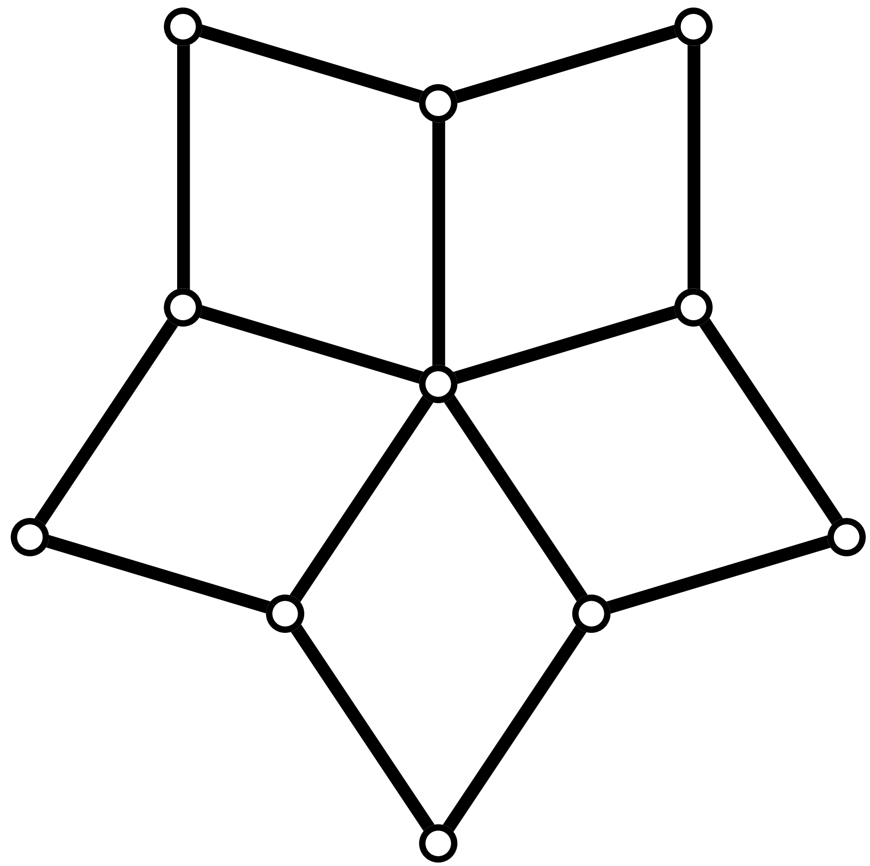
FUB-TAU Workshop, Tel Aviv, Israel Sep 23, 2019

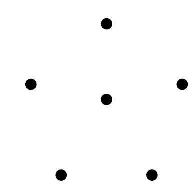


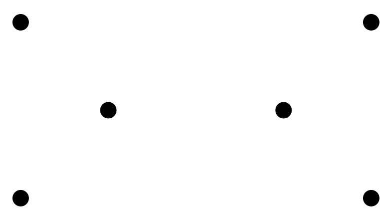


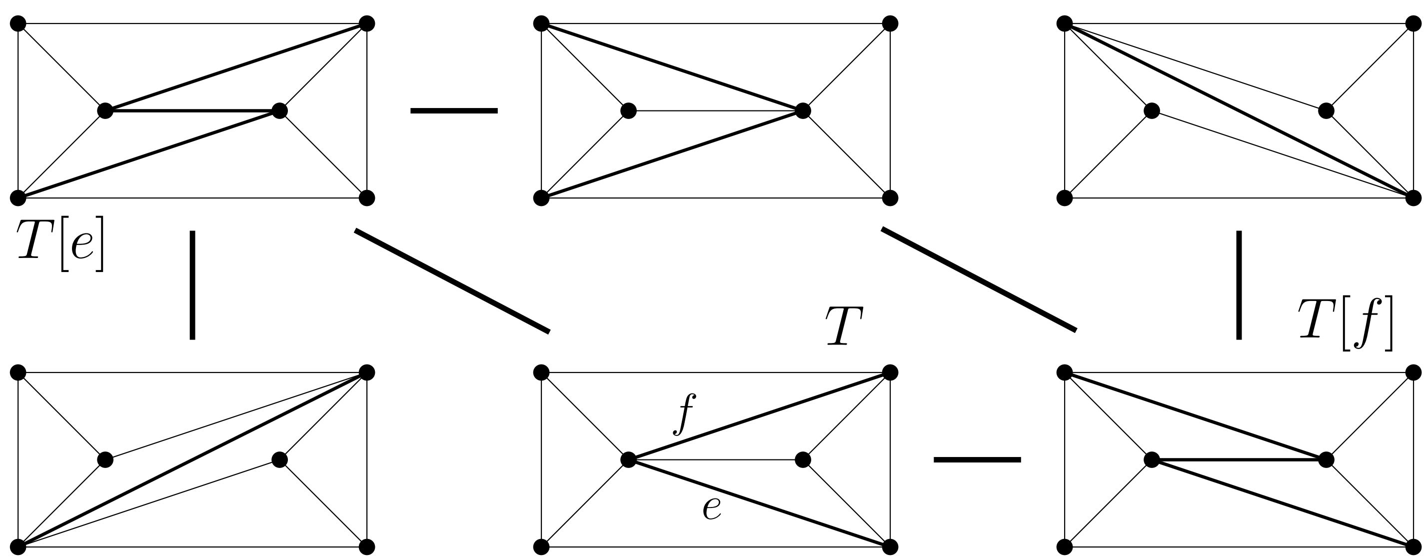


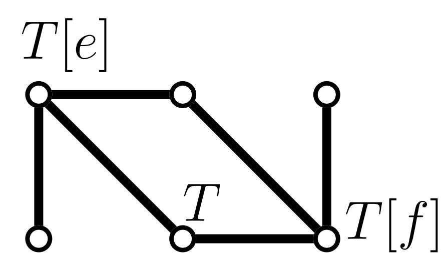




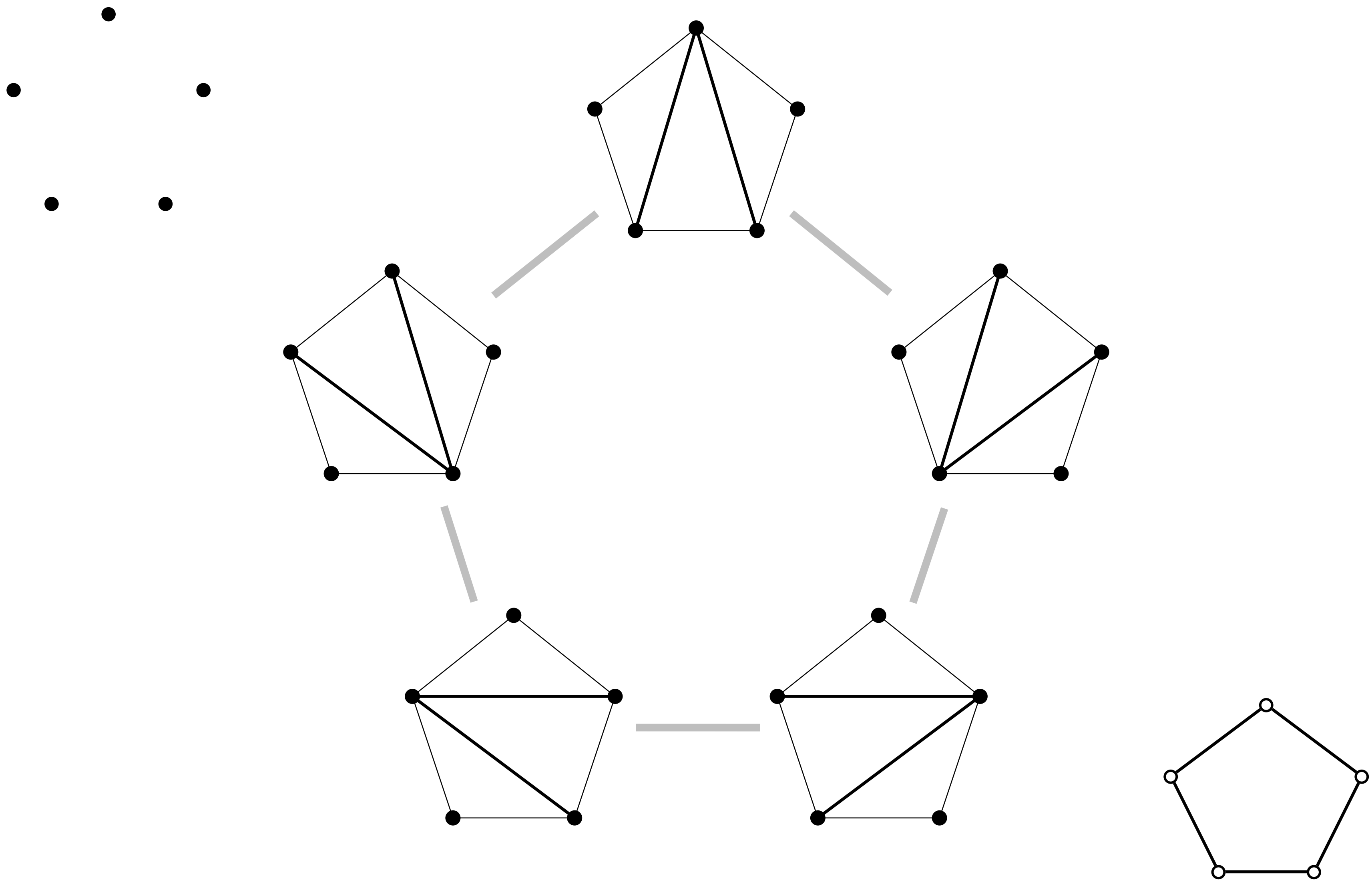
Flip graph of triangulations of 







(bold edges are flippable)



We are interested in the connectivity of the flip graph.

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. . . and its structure and "geometry", in general.

Connectedness – [Lawson, 1972]

In 1972 Charles Lawson proved connectedness of the flip graph (by exhibiting edge-flips towards a reference triangulation).

This allows local improvement algorithms (often heuristics) towards a "desired" triangulation (min-weight triangulation, avoiding small angles, Delaunay triangulation).

The diameter is known to be $O(n^2)$.

Here:

What is the edge- or vertex-connectivity flip graphs?

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Here:

What is the edge- or vertex-connectivity flip graphs?

What can we say about (sub-)structures in flip graphs?

Connectedness – A Bit of a Struggle

[Lawson, 1972]:

"The theorem in §*5* [i.e. of the connectivity of the flip graph] *was considered by Lawson and Weingarten in 1965, but the proof we had in mind at that time was somewhat obscure. Recent discussions with A.R. Curtis, R. Fletcher, M.J.B. Powell and J.K. Reid of the Atomic Energy Research Establishment Harwell, England, led to the notion of a* reference triangulation *and to the proof of the theorem given in this note."*

Obvious Upper Bound: Min-Degree δ of Flip Graph

Clearly, the minimum degree δ of a vertex in a graph is an upper bound on the connectivity of a graph. Note that

the degree of a vertex (i.e. a triangulation T) in the flip graph
=
the number of flippable edges in T.

Proposition [Hurtado, Noy, Urrutia, 1999] In any triangulation T of P the number of flippable edges is at least

$$n/2 - 2 \ .$$

For every n there is a set that shows this bound to be tight.

The min-degree of the flip graph is at least $\lceil n/2 - 2 \rceil$.

Connectivity Results

Theorem: *The flip graph of triangulations of a planar point set P, $|P|$ large enough, in general position is*

δ-vertex (and thus δ-edge) connected,

where δ is the minimum number of flippable edges in a triangulation of P.

Theorem: *The flip graph of triangulations of any planar point set P, $n := |P|$, in general position is*

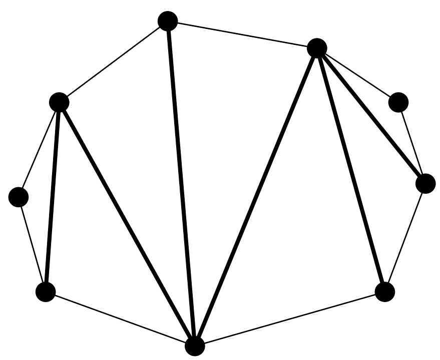
$\lceil n/2 - 2 \rceil$-vertex connected.

... and $(n-3)$-vertex connecitivity for the bistellar flip graph of partial triangulations.

. . . geometry of the flip graph?

Convex Position – Triangulations of Convex n-Gons

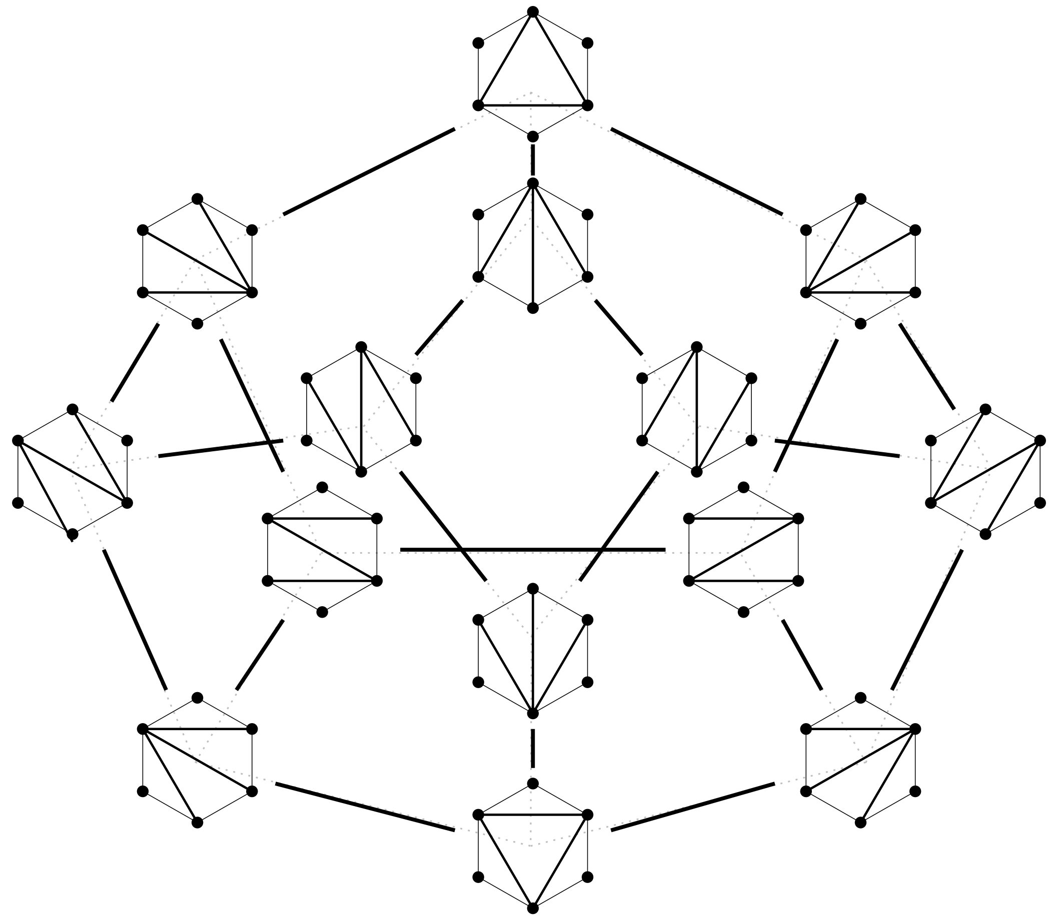
P convex position := vertices of a convex polygon.



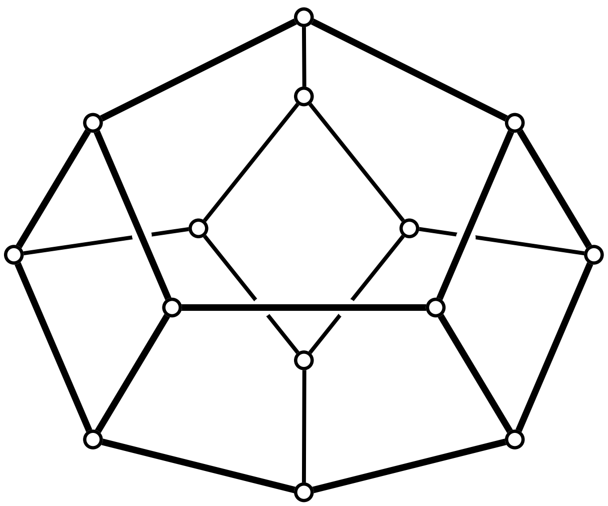
In a triangulation of a convex n-gon, there are $n-3$ inner edges, all of which are flippable. That is, the flip graph of n points in convex position is $(n-3)$-regular.

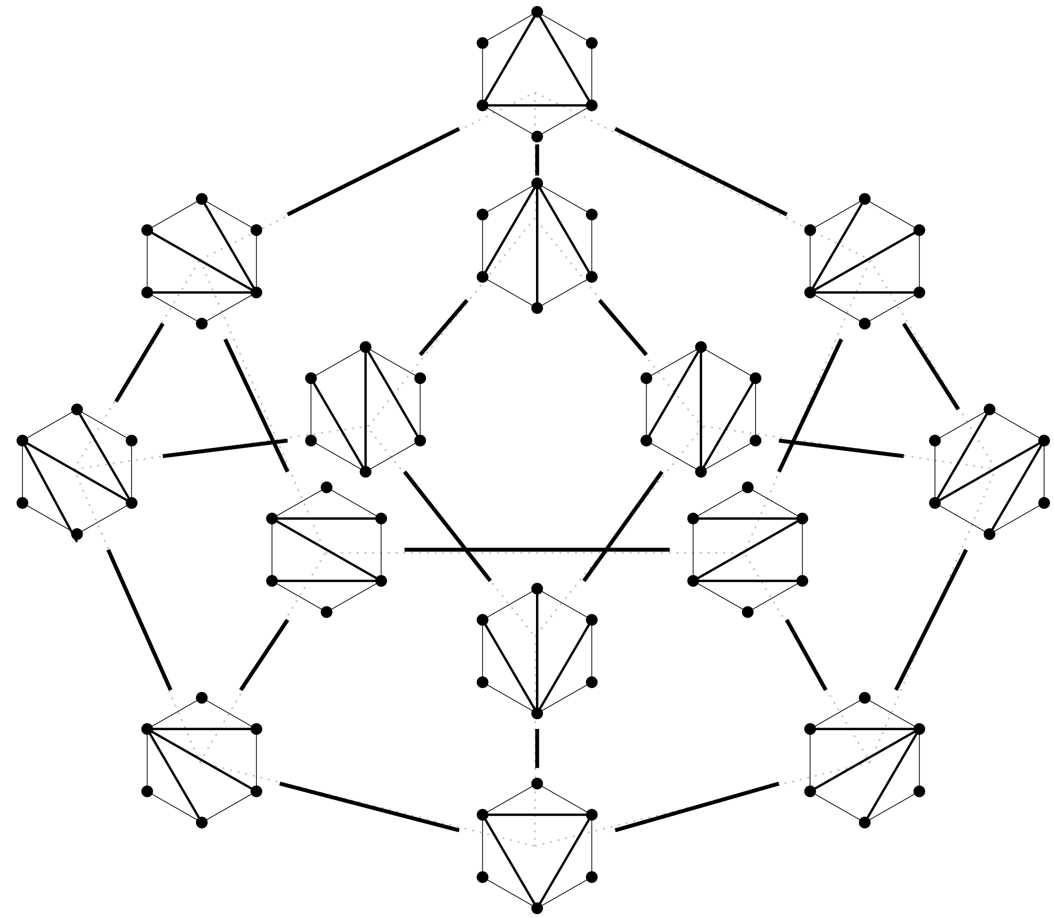
The number of triangulations of a convex n-gon is $C_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2} = \Theta(n^{-3/2}4^n)$ (Catalan Number), which is neither the minimal, nor the maximal possible number of triangulations for a general n-point set.

Example – Flip Graph of Hexagon



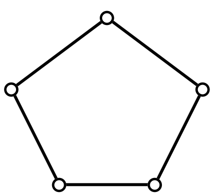
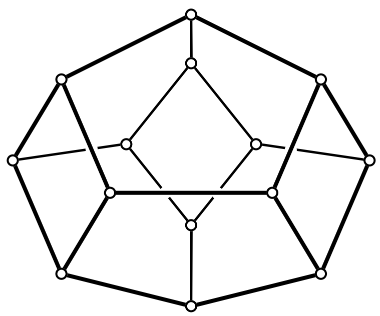
Example – Flip Graph of Hexagon





Flip Graph of Convex Polygons — Associahedra

The flip graph of a convex n-gon is the 1-skeleton of a convex polytope — the associahedron (of order $n-1$) of dimension $n-3$ [Dov Tamari, 1951; Jim Stasheff, 1961].

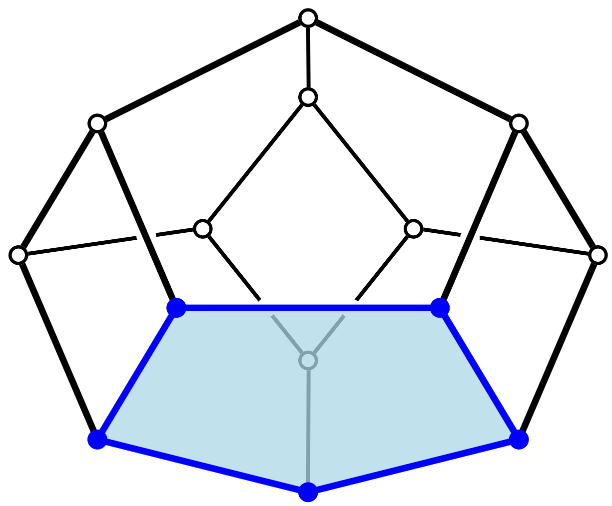
 ,  , . . .

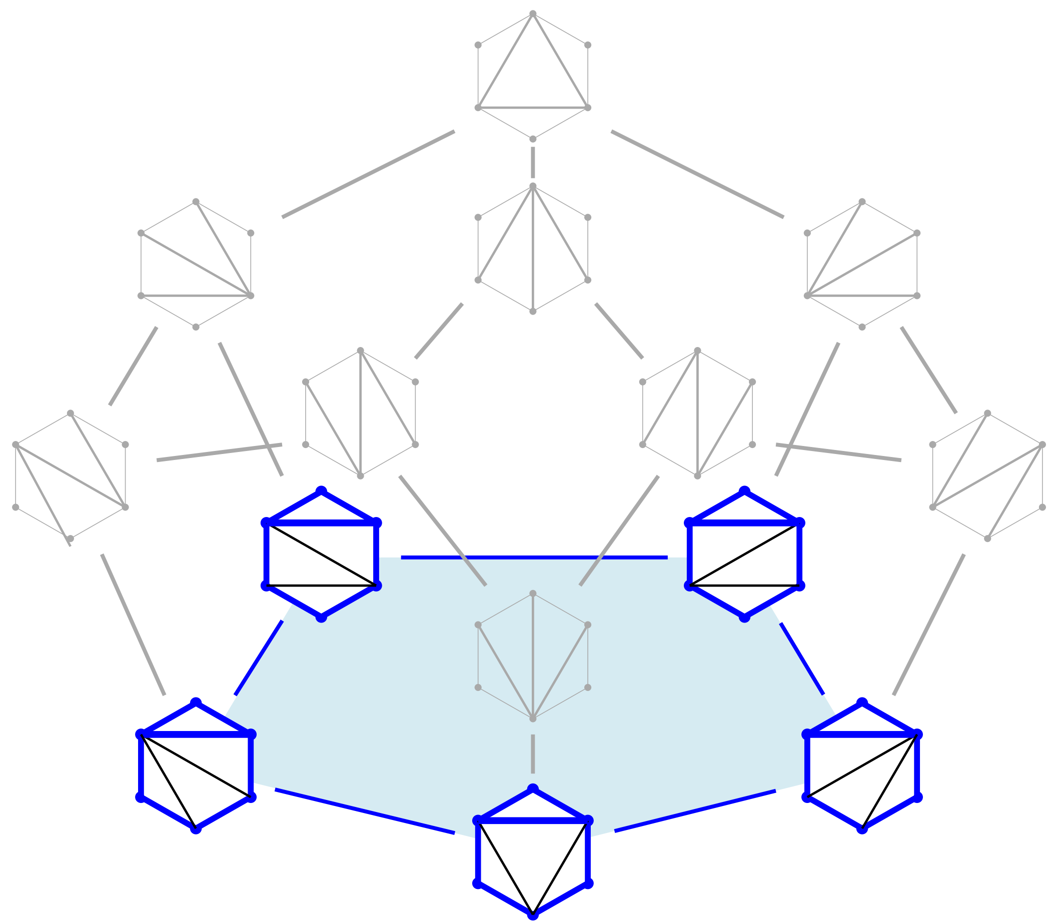
Balinski's Theorem [1961]
The 1-skeleton of a convex d-polytope is at least d-vertex connected.

$\Rightarrow$ The flip graph of n points in convex position is $(n-3)$-vertex connected.

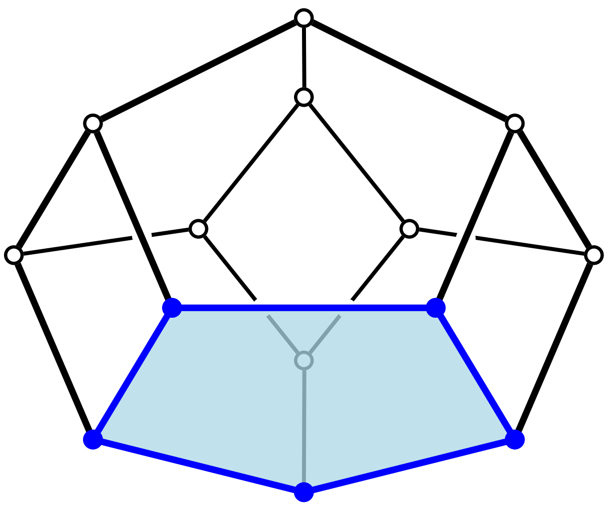
Can we expect similar structure in general (nonconvex position)? – See later.

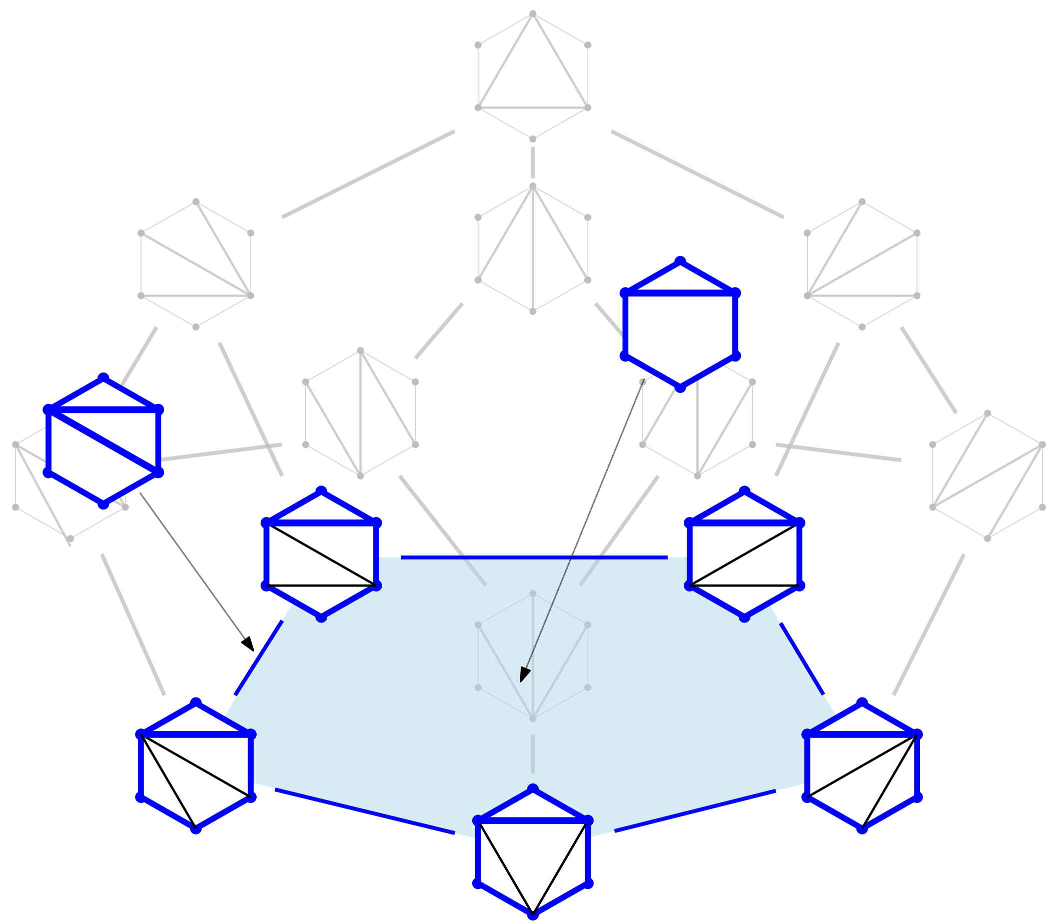
Example – Flip Graph of Hexagon





Example – Flip Graph of Hexagon





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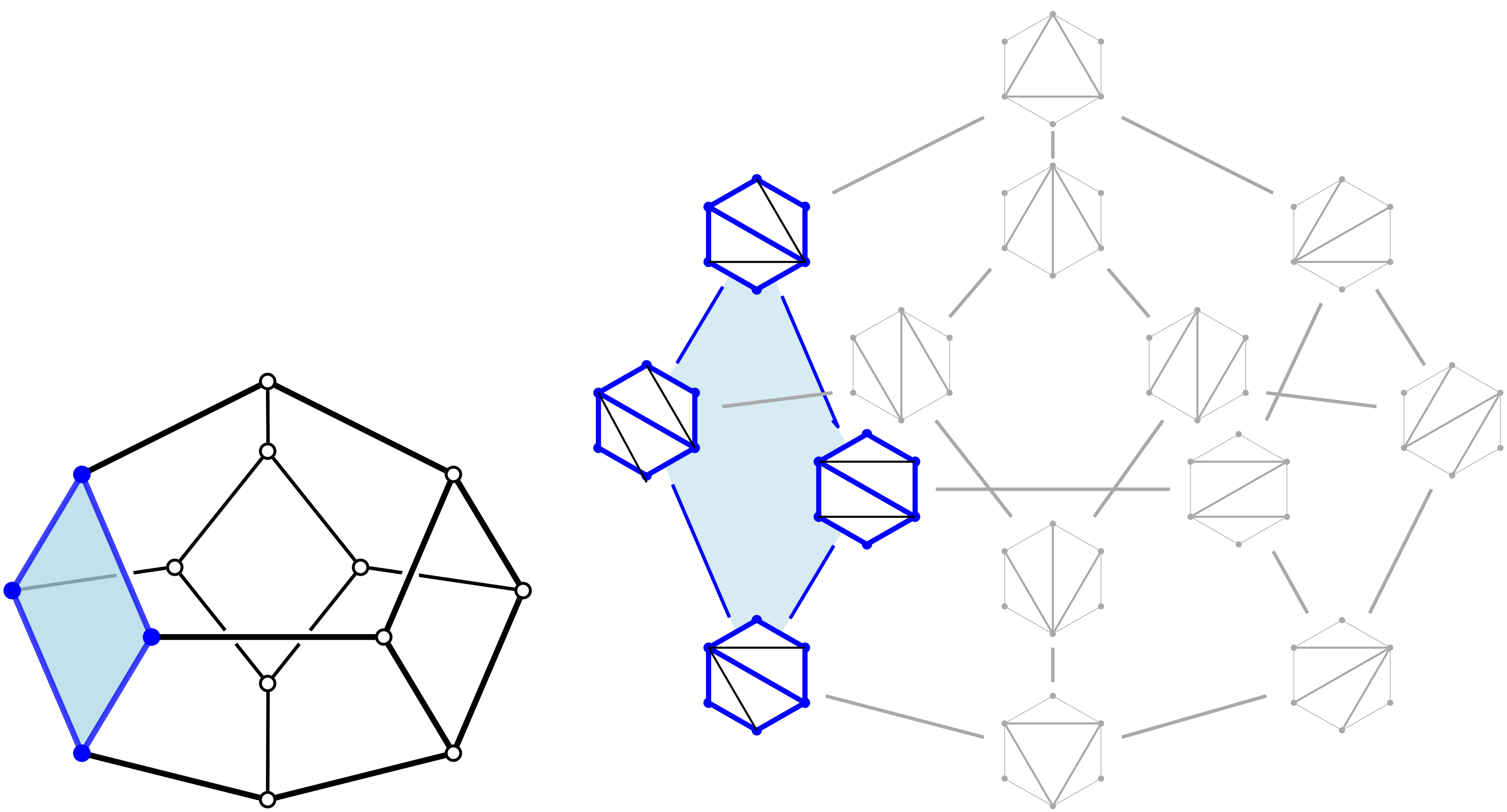

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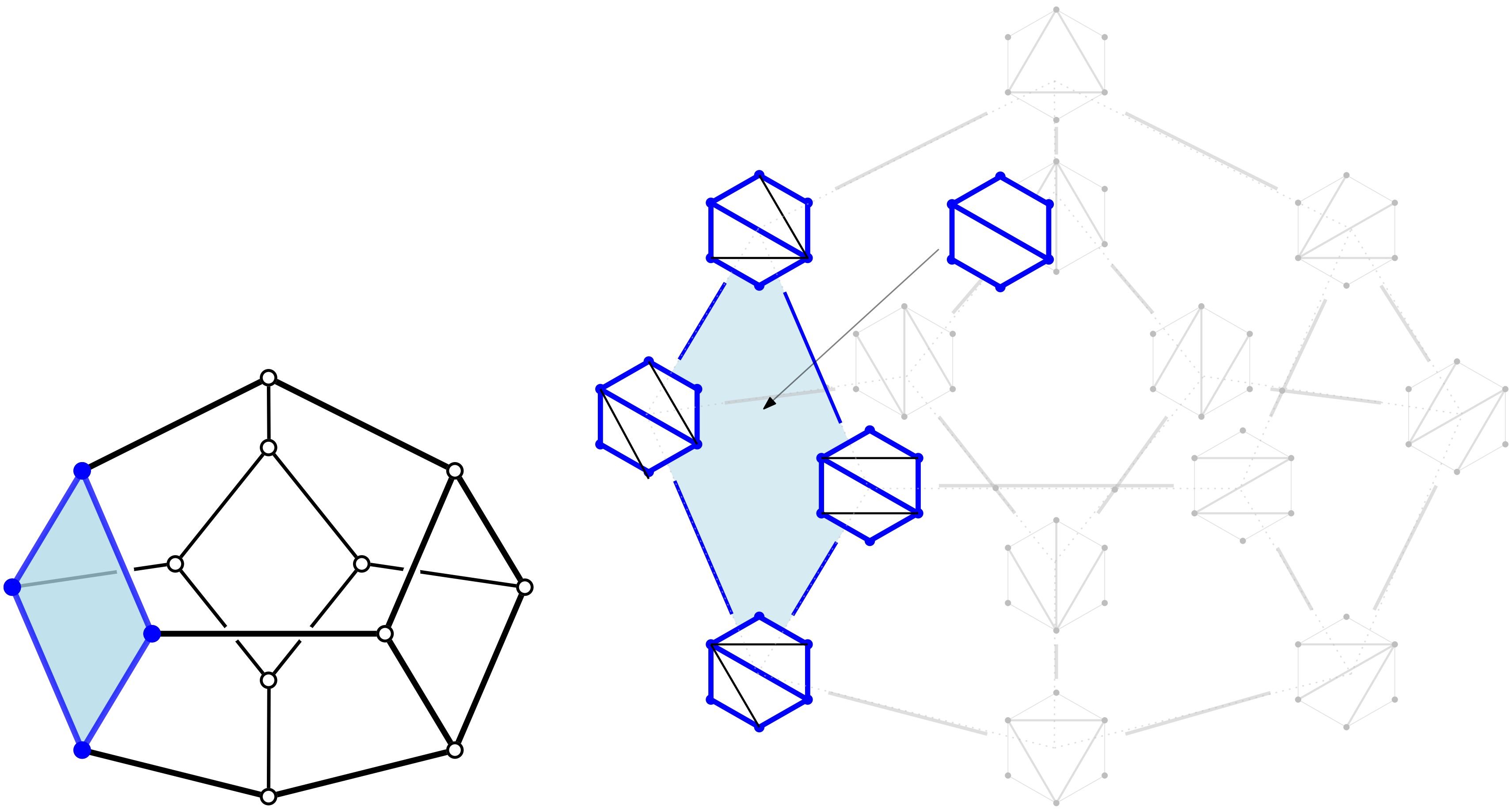
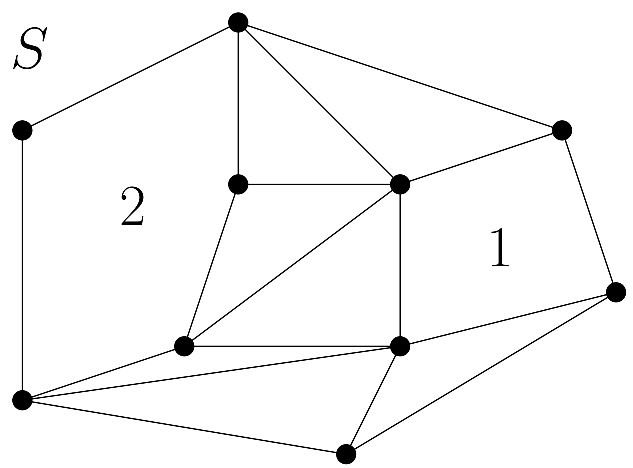


Subdivision (= Convex Decomposition)

A graph $S = (P, E)$ is a *subdivision of P* if it is plane, all convex hull edges of P are in E, there are no isolated vertices, and all bounded faces are convex. The *slack*, $\mathrm{sl}\, S$, of a subdivision is the number of edges it takes to triangulate S. $\mathcal{T}\langle S \rangle$ is the set triangulations refining S.



$\mathrm{sl}\, S = 2 + 1 = 3$

$\mathrm{sl} = 0$: triangulation

$\mathrm{sl} = 1$: edge flip

$\mathrm{sl} = i$: "i-face" . . . think

(of a polytope that doesn't exist)

"Geometry" Results

Theorem: *The flip graph of triangulations of <u>any</u> planar point set P, $n := |P|$, in general position can be covered by 1-skeletons of $\lceil n/2 - 2 \rceil$-polytopes (contained in the flip graph).*

. . . and the bistellar flip graph of partial triangulations can be covered by $(n-3)$-polytopes.

I. Set-Up & Context $\leftarrow$ DONE!
triangulations, edge-flips in triangulations, flip graph, connectivity, associahedron, subdivisions

II. Connectivity in Graphs, Local Condition
Menger's Theorem and "Local Menger"

III./IIIa. Proof (for Edge Connectedness)
min-degree bound, $\lceil n/2 - 2 \rceil$-bound

IV. Regular Triangulations − Partial Triangulations
bistellar flips, secondary polytope *. . . probably not*

V. Open Problem
computing large slack subdivisions,
stronger connectivity (rapid mixing).

Triangulations (and their flip graphs), also in higher dimensions, have a rich literature, cf. e.g.

"Triangulations"
by De Loera, Rambau, and Santos, 2010,

or

"Geometry and Topology for Mesh Generation"
by Edelsbrunner, 2001.

II.

Connectivity in Graphs
Local Condition

Menger's Theorem, 1927

Let $k \geq 2$ be an integer. A simple undirected graph G with at least two vertices

(1) is k-edge connected iff for any pair of distinct vertices u and v there are k pairwise edge-disjoint u-v-paths, and

(2) it is k-vertex connected iff for any pair of distinct vertices u and v there are k pairwise internally vertex-disjoint u-v-paths.

“Local” Condition

Let $k \geq 2$ be an integer. A simple undirected *connected* graph G with at least two vertices

(1) is k-edge connected iff for any pair of adjacent vertices u and v in G there are k pairwise edge-disjoint u-v-paths, and

(2) it is k-vertex connected iff G has at least three vertices and for any pair of vertices u and v at distance 2 there are k pairwise internally vertex-disjoint u-v-paths.

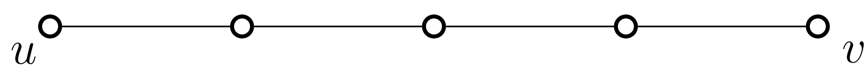
Proof – Edge Connectivity

k-edge connected $\Leftrightarrow$ for any pair of adjacent vertices u and v in G there are k pairwise edge-disjoint u-v-paths

$\Rightarrow$ Menger's Theorem

$\Leftarrow$

$$k = 4$$



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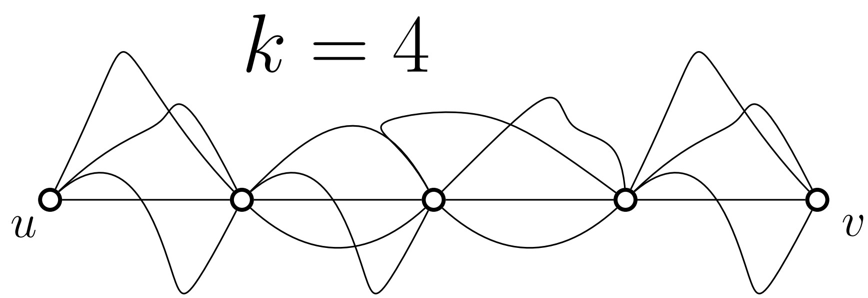

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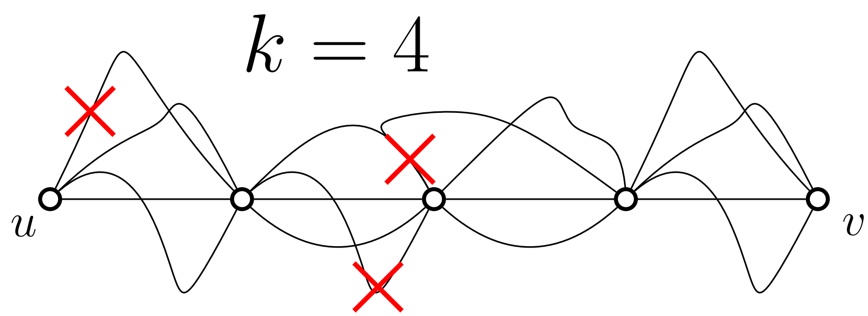


If we remove $k-1$ edges, for every edge $\{x, y\}$ on the u-v-path, at least one of the k supplied x-y-paths survive.

III.

Proof
Edge Connectivity

For the Proof (of Edge-Connectivity) . . .

(Recall that $T[e]$ is the triangulation obtained by flipping e in T.)

We want to show that for $T, T[e]$, $\deg T \leq \deg T[e]$, there are $\deg T$ edge-disjoint T-$T[e]$-paths.

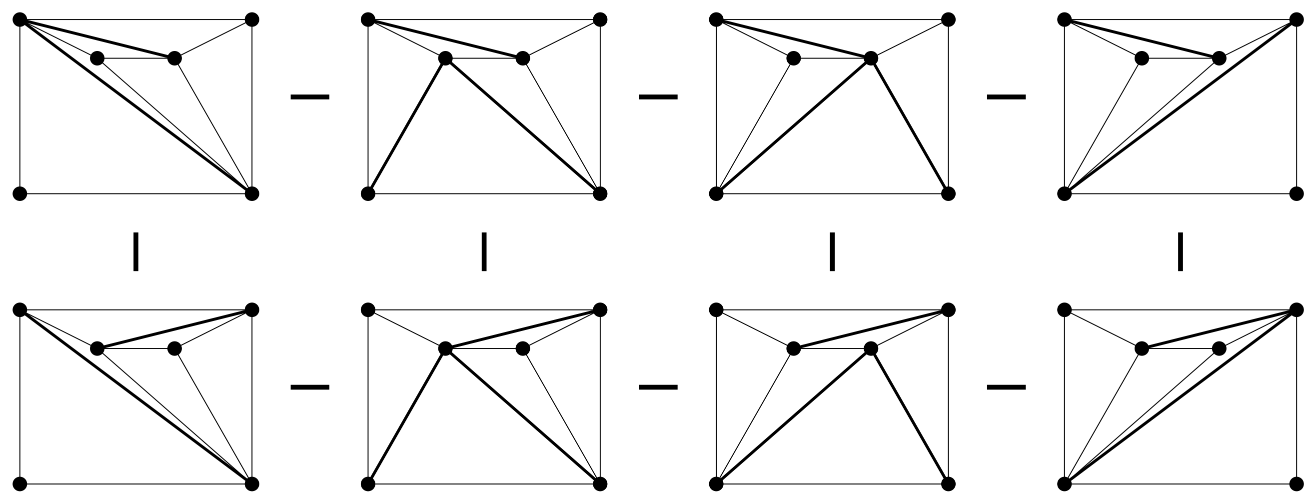
In other words, for each f flippable in T we need a path

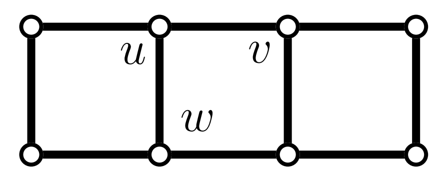
$$\mathsf{path}(f) = (T, T[f], \ldots, T[e])$$

with these paths pairwise edge disjoint. (E.g. $\mathsf{path}(e) := (T, T[e])$)

at least for P large enough

No 3 Edge-Disjoint u-v-Paths





We need $|P|$ large enough!

Two Flippable Edges e and f

independently flippable: e flippable in $T[f]$ and $T[e,f] = T[f,e]$.

weakly independently flippable:
e flippable in $T[f]$ and $T[e,f] \neq T[f,e]$.

dependently flippable: e not flippable in $T[f]$.

Two Flippable Edges e and f

Lemma. For e and f flippable in T,

e is flippable in $T[f]$ iff f is flippable $T[e]$.

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Two Flippable Edges e and f

Lemma. For e and f flippable in T,

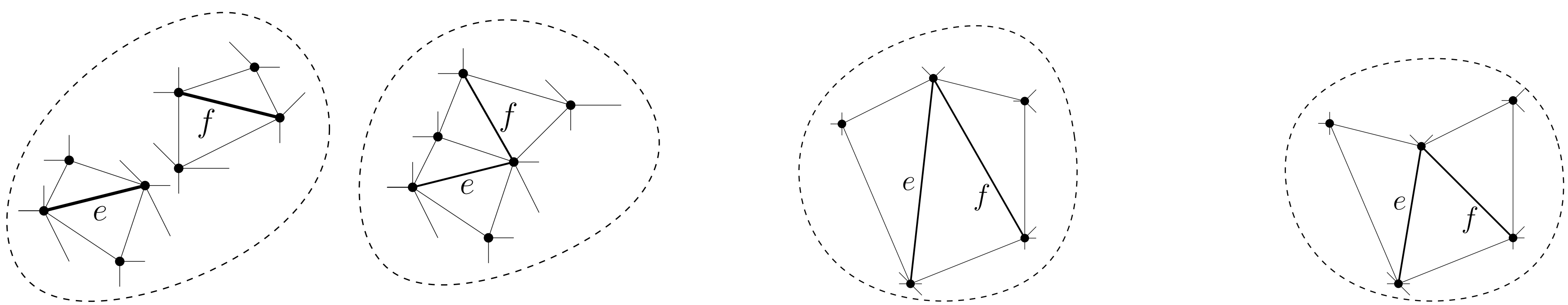
e is flippable in $T[f]$ iff f is flippable $T[e]$.

independently flippable: e flippable in $T[f]$ and $T[e, f] = T[f, e]$.
$\Leftrightarrow$ Removal of e and f in T creates two convex quadrilaterals.

weakly independently flippable:
e flippable in $T[f]$ and $T[e, f] \neq T[f, e]$.
$\Leftrightarrow$ Removal of e and f in T creates a convex pentagon.

dependently flippable: e not flippable in $T[f]$.
$\Leftrightarrow$ Removal of e and f in T creates a nonconvex pentagon.

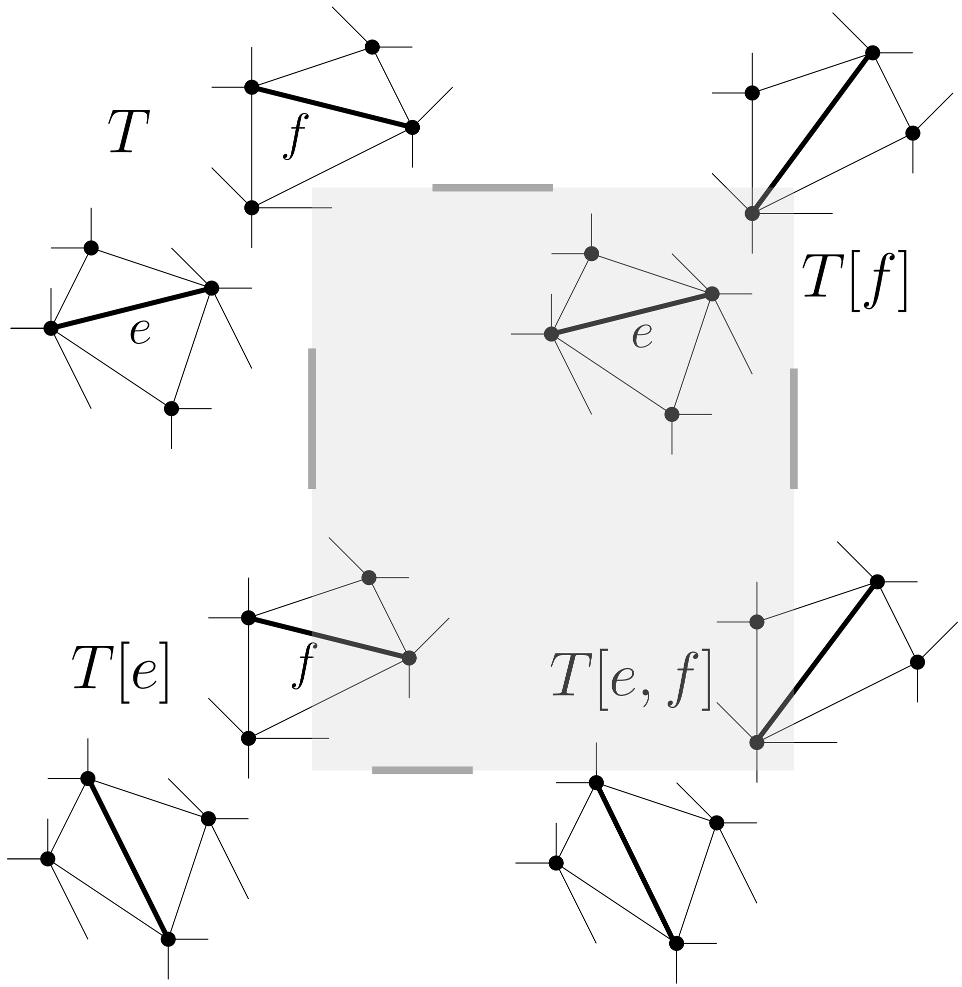
Independently/Weakly Independently/Dependently Flippable



independently weakly independently dependently

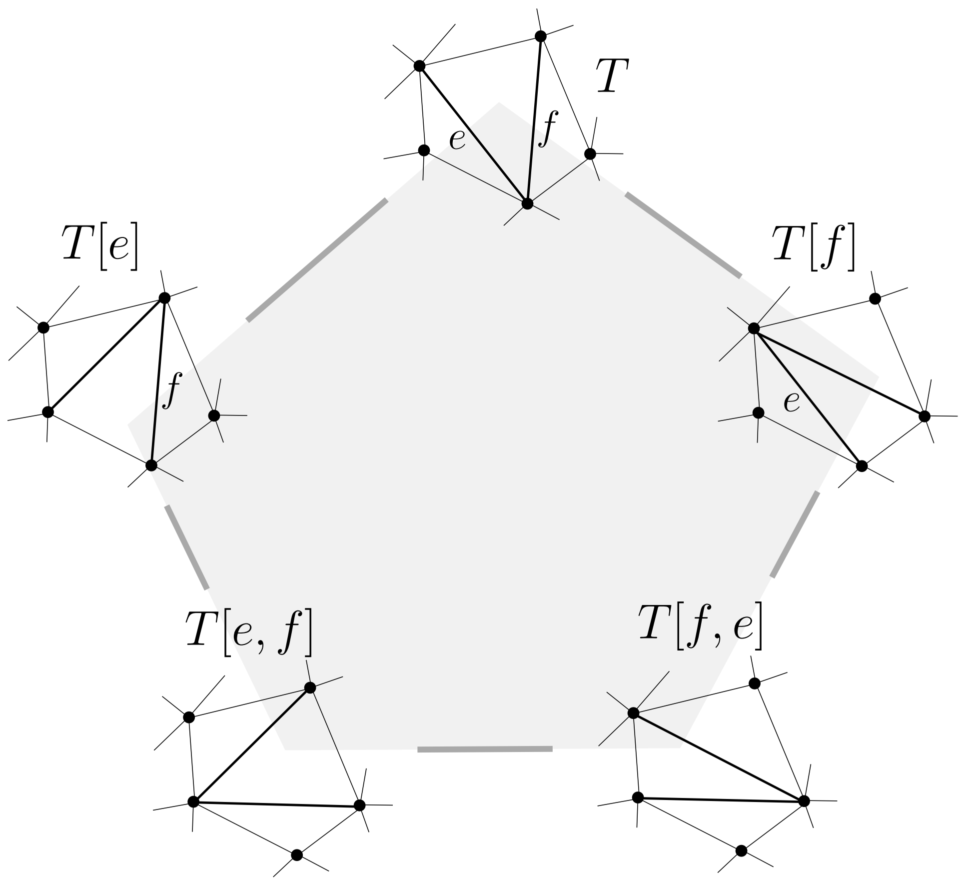
$\underbrace{\qquad\qquad\qquad\qquad}$ $T_{-e,f}$ is a subdivision

e and f Independently Flippable



path(f) :=
$(T, T[f], T[f,e], T[e])$

e and f Weakly Independently Flippable



path(f) :=

$(T, T[f], T[f,e], T[e,f], T[e])$

e and f Dependently Flippable

I.e. f is not flippable in $T[e]$. Since deg $T \leq$ deg $T[e]$, there must be $f^* \in \mathsf{E}(T[e])$ flippable in $T[e]$ but not in T.

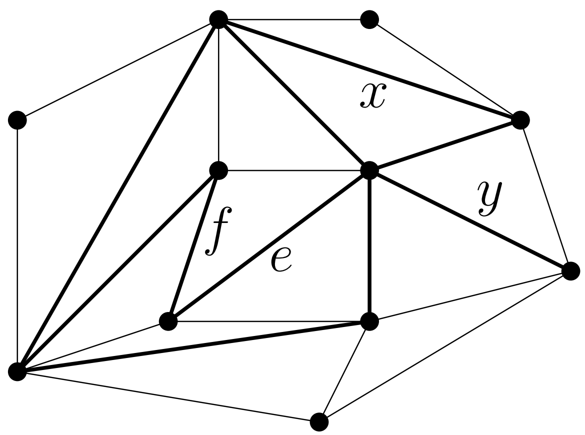
For "far away" independently flippable edges x and y in T let

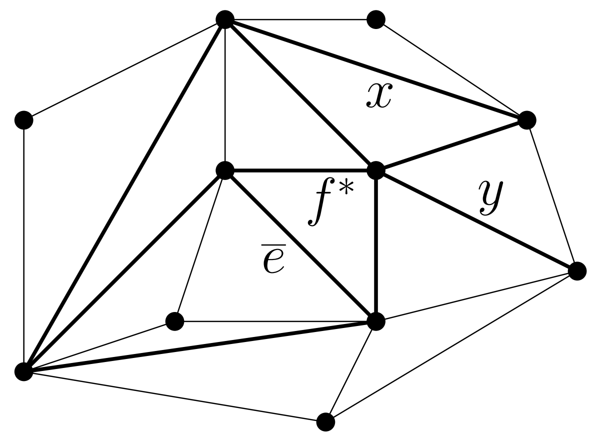
$$\begin{aligned} \mathsf{path}(f) \;:=\; & (T, T[f], T[f,x], \overbrace{T[f,x,y]}^{=T[x,y,f]}, T[x,y], \\ & \underbrace{T[e,x,y]}_{=T[x,y,e]}, \underbrace{T[e,f^*,x,y]}_{=T[e,x,y,f^*]}, T[e,f^*,x], T[e,f^*], T[e]) \end{aligned}$$

with trace $(\{f\},\{f,x\},\{f,x,y\},\{x,y\},\{x,y\},\{f^*,x,y\},\{f^*,x\},\{f^*\})$

(i) x and y exist, since if n is large enough, there are many flippable edges. (ii) Such a case can happen at most twice, since both f and f^* must be edges of the quadrilateral created by removal of e.

e and f Dependently Flippable

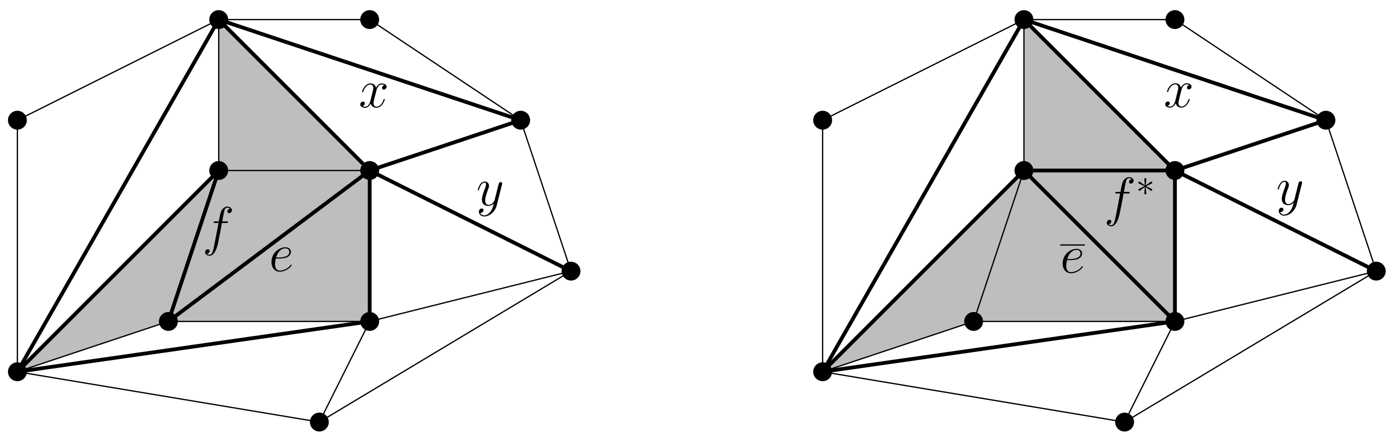




$\mathsf{path}(f) \;:=\; (T, T[f],$

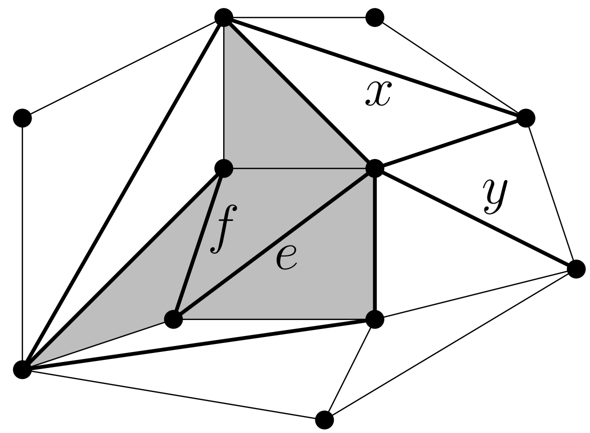
$, T[e, f^*], T[e])$

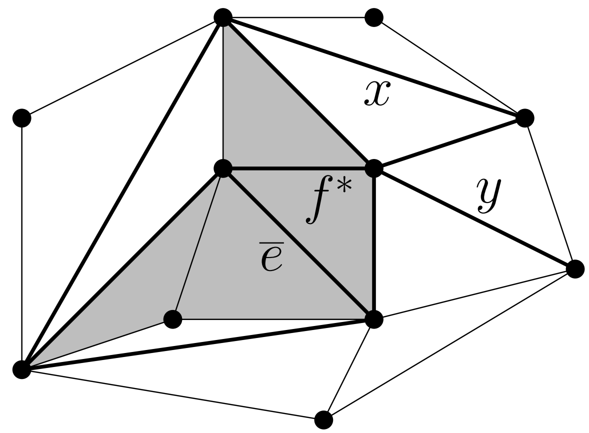
e and f Dependently Flippable



$$\text{path}(f) \;:=\; (T, T[f], T[f,x], T[f,x,y], T[x,y],$$
$$, T[e, f^*], T[e])$$

e and f Dependently Flippable





$$\text{path}(f) \;:=\; (T, T[f], T[f,x], T[f,x,y], T[x,y],$$
$$T[e,x,y], T[e,f^*,x,y], T[e,f^*,x], T[e,f^*], T[e])$$

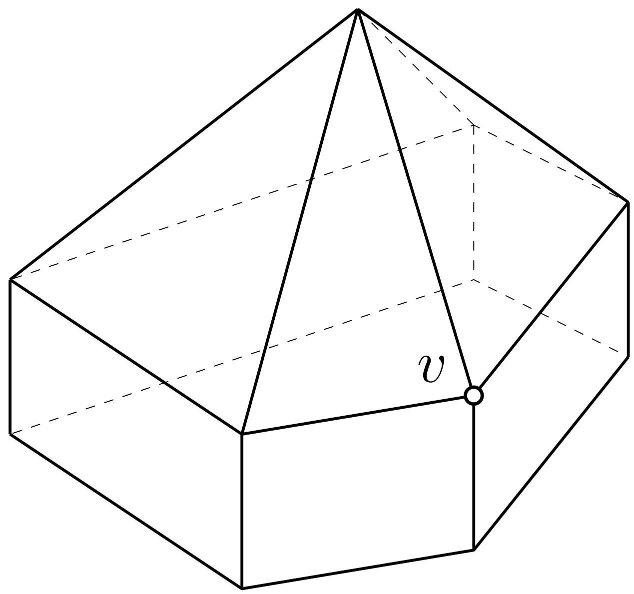
Need to make sure that paths are disjoint.

IIIa.

$\lceil n/2 - 2 \rceil$-Bound For All P

Link of a Vertex (Vertex Figure) in a Polytope

"Cut off vertex by hyperplane."



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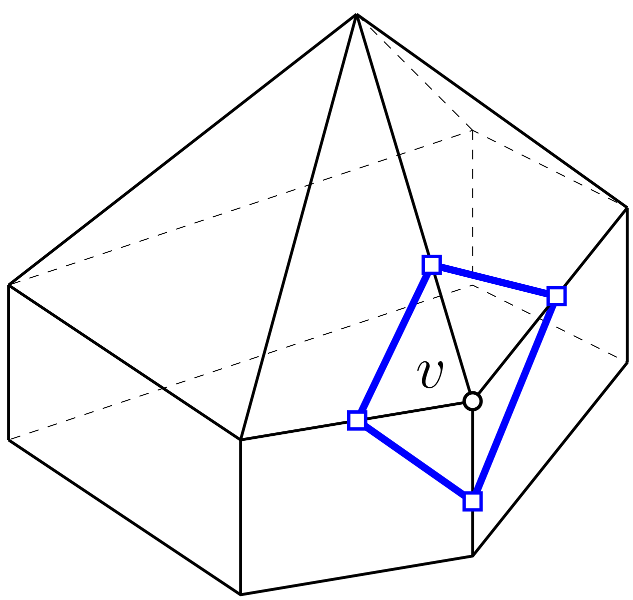

Link of a vertex v in a polytope $\mathcal{P}$:
link-vertices = $\mathcal{P}$-edges incident to v.
link-edges = pairs of $\mathcal{P}$-edges incident to common 2-face of $\mathcal{P}$.

Link of a Vertex (Vertex Figure) in a Polytope

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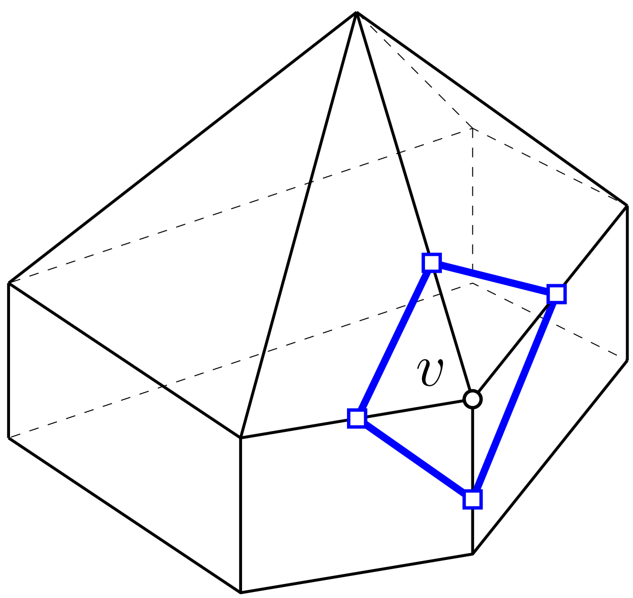

Link of a vertex v in a polytope $\mathcal{P}$:
link-vertices = $\mathcal{P}$-edges incident to v.
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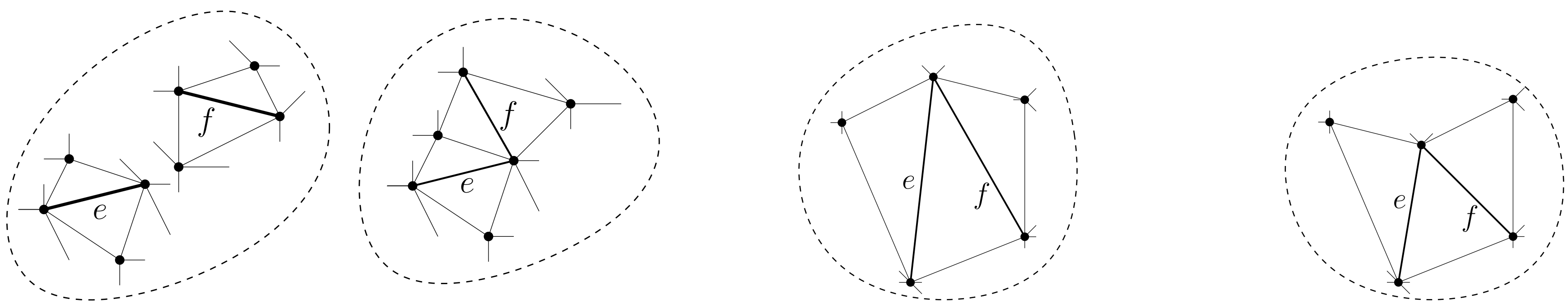
Link of a vertex T in flip graph:
link-vertices = $\{T_{-e} | e$ flippable in $T\}$.
link-edges = pairs $\{T_{-e}, T_{-f}\}$ incident to common "2-face" $T_{-e,f}$.

The Link of a Triangulation

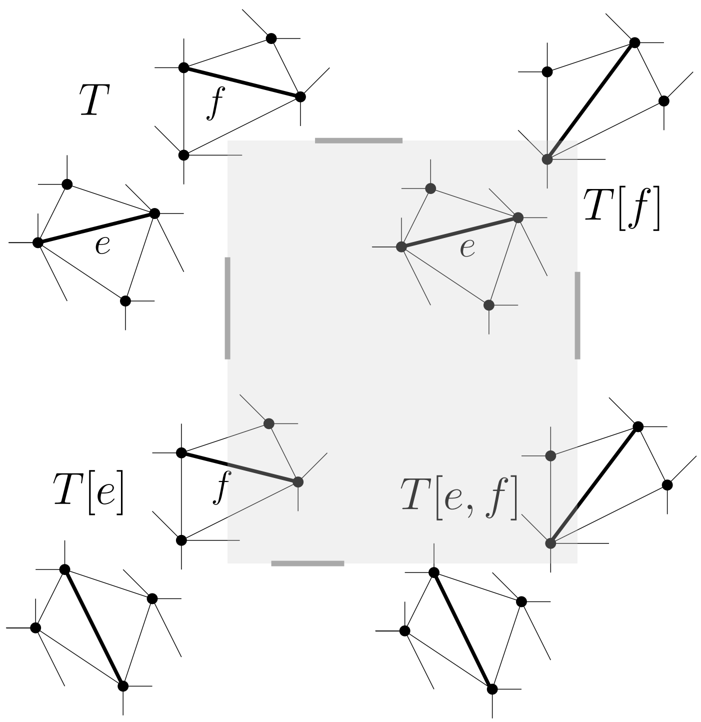
We call edges e and f in a triangulation T *compatible* if they are either independently or weakly independently flippable $\Leftrightarrow$ $T_{-e,f}$ is a subdivision $\Leftrightarrow$ edges T_{-e} and T_{-f} in the flip graph are incident to a common 2-face $T_{-e,f}$.



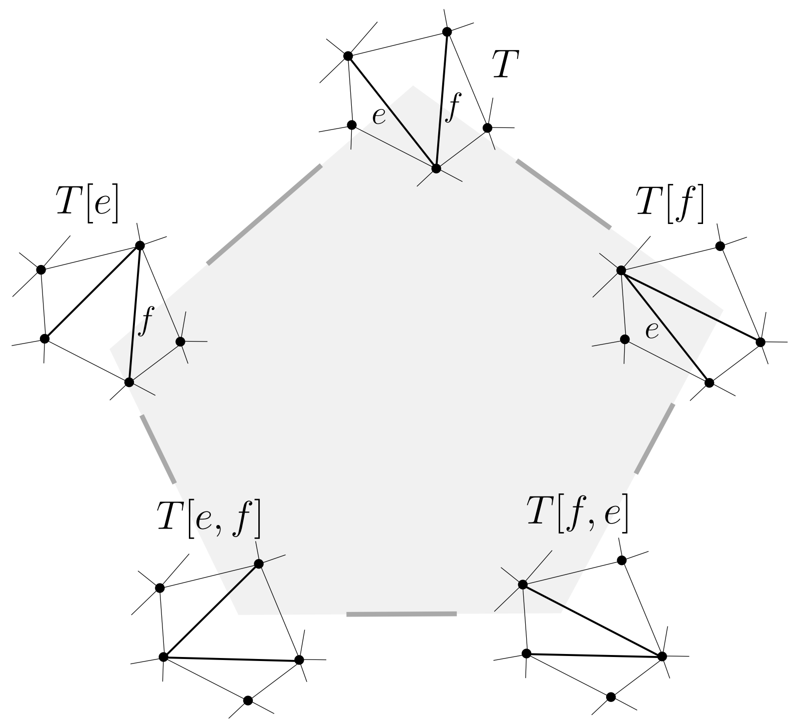
independently | weakly independently | dependently

$\underbrace{\text{independently} \qquad \text{weakly independently}}_{T_{-e,f} \text{ is a subdivision}}$

allows T-avoiding $T[e]$-$T[f]$-paths



$(T[e], T[e, f], T[f])$



$(T[e], T[e, f], T[f, e], T[f])$

The Link of a Triangulation

We call edges e and f in a triangulation T *compatible* if they are either independently or weakly independently flippable $\Leftrightarrow$ $T_{-e,f}$ is a subdivision $\Leftrightarrow$ edges T_{-e} and T_{-f} in the flip graph are incident to a common 2-face $T_{-e,f}$.

The *link of a triangulation T* is the graph with

$$\text{vertex set } \{e \mid e \text{ flippable in } T\}$$

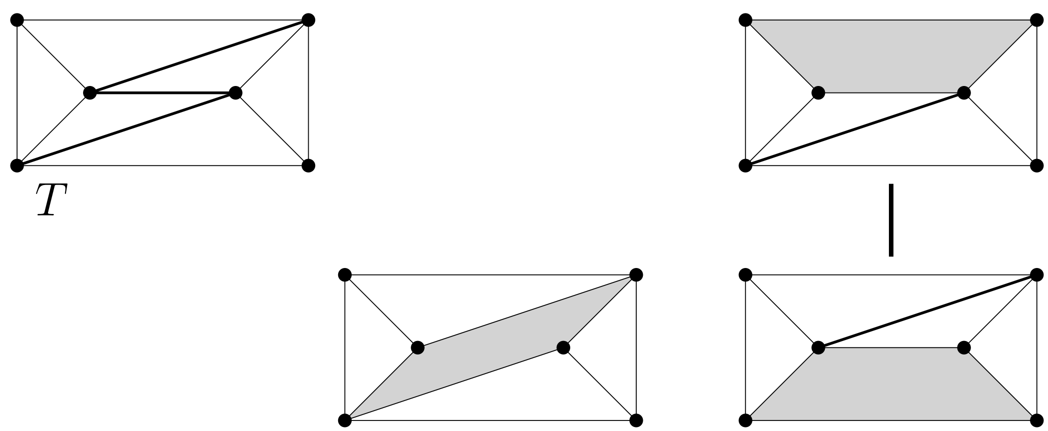
should be $\{T_{-e} \mid \ldots$

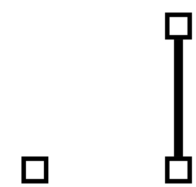
and

$$\text{edge set } \{\{e,f\} \mid e \text{ and } f \text{ compatible in } T\} \; .$$

If $\{e,f\}$ is an edge in the link of T then there is a $T[e]$-$T[f]$-path (of length 2 or 3) different from $(T[e],T,T[f])$ in the flip graph of P.

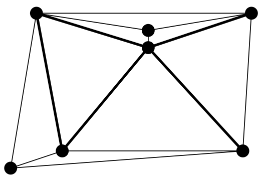
The Link of a Triangulation



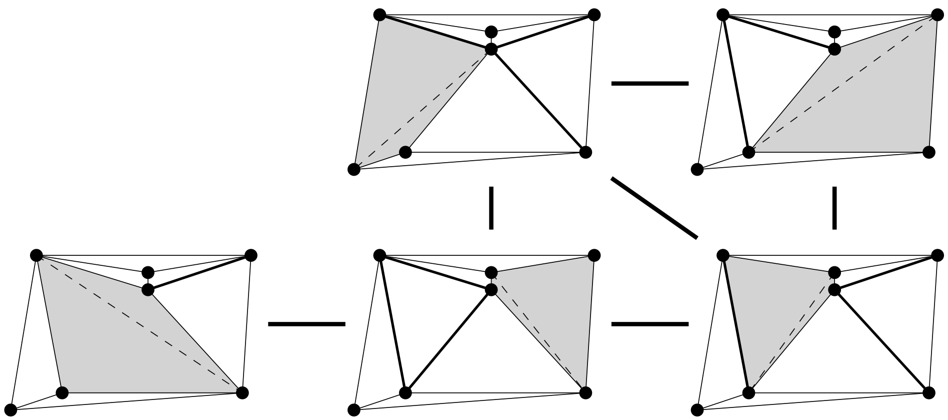


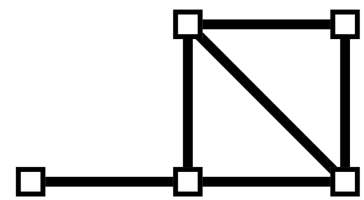
link of T

The Link of a Triangulation



T

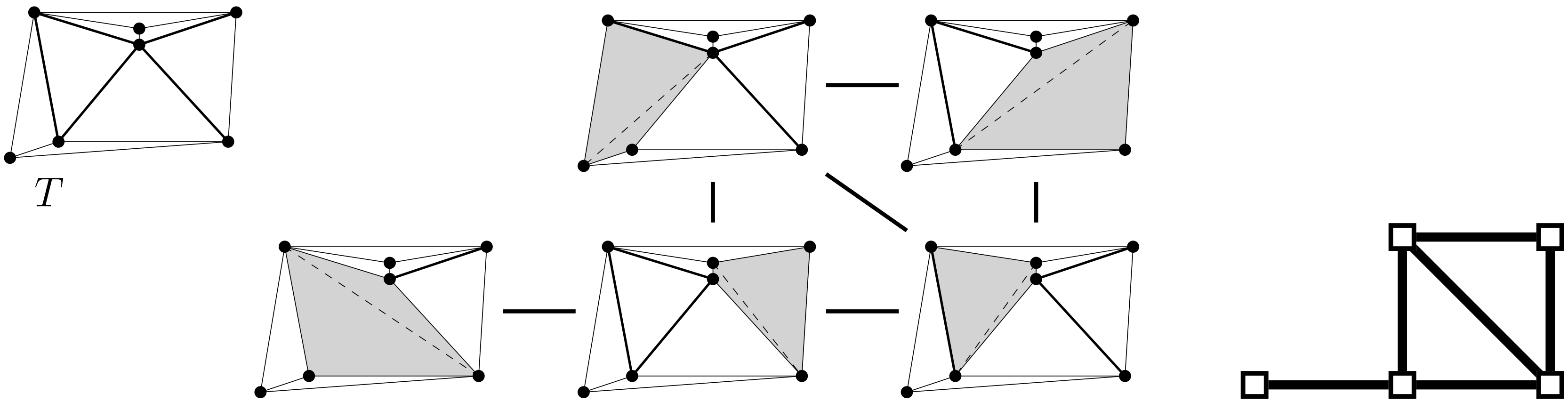




link of T

Doesn't really look like a link of a vertex in a polytope!

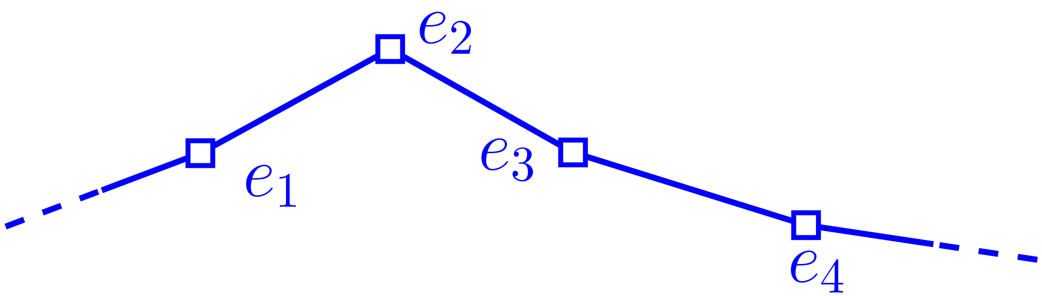
The Link of a Triangulation



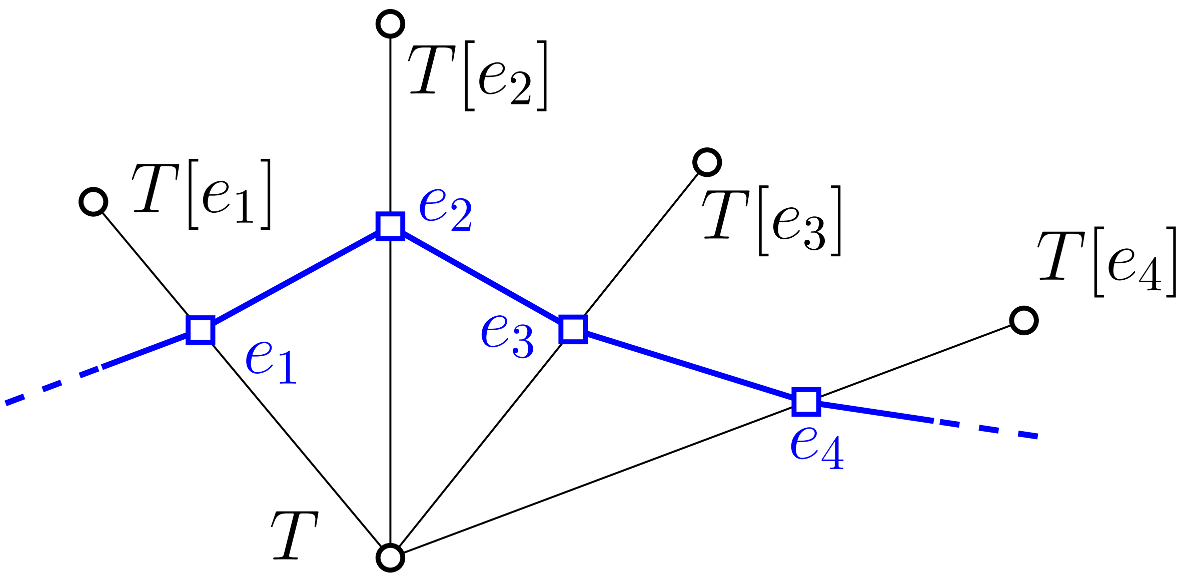
link of T

Doesn't really look like a link of a vertex in a polytope!
But disjoint link-paths translate to disjoint flip-graph-paths.

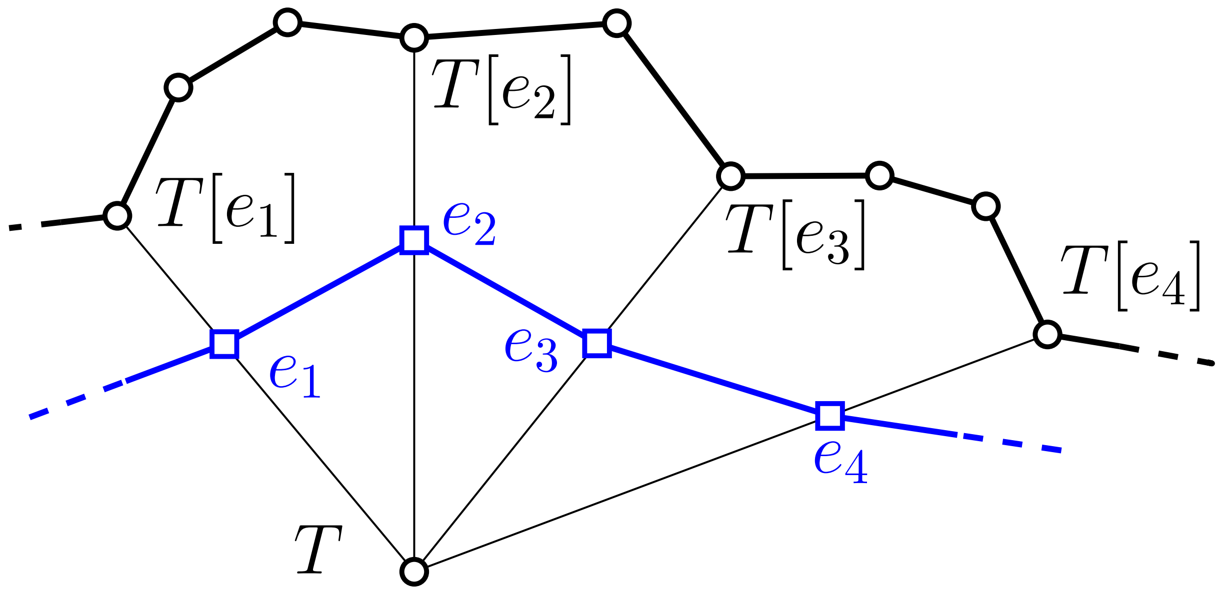
Link Path to Flip-Graph-Path



Link Path to Flip-Graph-Path



Link Path to Flip-Graph-Path



Every Link is Dense and Highly Connected

Lemma. In every link, the min-degree is at least $n/2 - 3$.

$\Leftarrow$ Coarsening Lemma

Lemma. The complement of a link of a triangulation is C_4-free.

Lemma. If the complement of a graph G is C_4-free, then G is δ-vertex connected (δ min-degree in G).

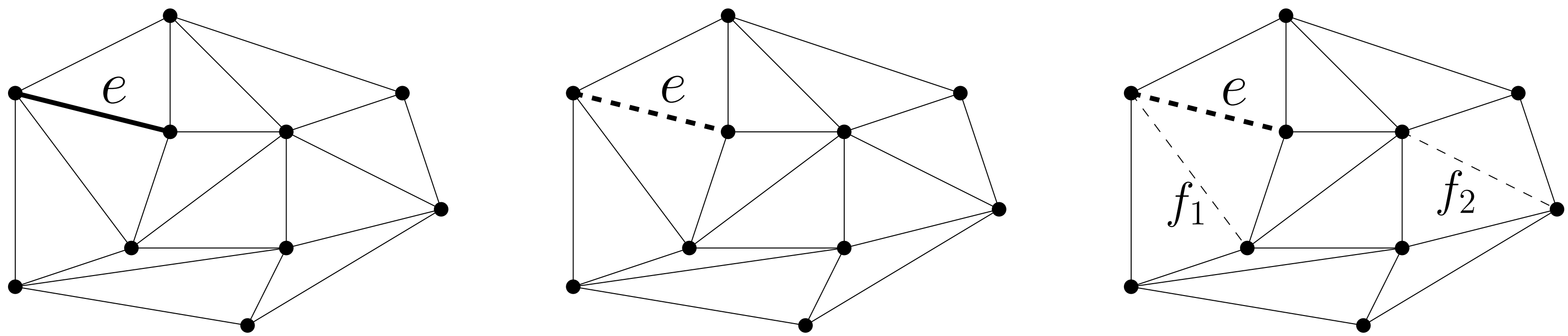
Theorem. (i) Every link is $\lceil n/2 - 3 \rceil$-vertex connected.
(ii) The flip graph is $\lceil n/2 - 2 \rceil$-vertex connected.

Recall Local Menger. Given $T[e]$, $T[f]$ (vertices at distance 2), the link of T delivers $\lceil n/2 - 3 \rceil$ $T[e]$-$T[f]$-paths, plus the path $(T[e], T, T[f])$.

Coarsening Subdivisions

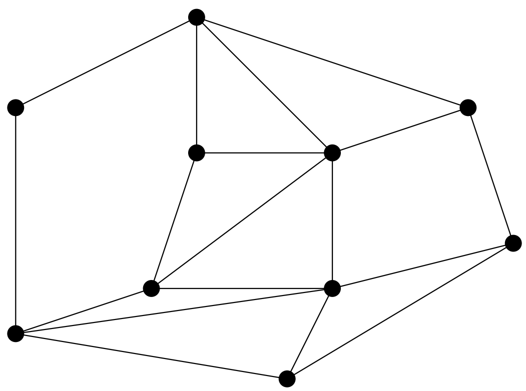
Coarsening Lemma. Every subdivision can be coarsened (by removal of edges) to a subdivision of slack at least $\lceil n/2 - 2 \rceil$.
(extending work by [Hoffmann, Schulz, Sharir, Sheffer, Toth, W. 2013]).

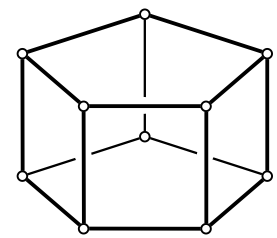
Edges removed in a coarsening of a triangulation are pairwise compatible.



Delivers $\lceil n/2 - 3 \rceil$ edges compatible with any given flippable edge e in $T \Rightarrow$ degree of e in the link of T is at least $\lceil n/2 - 3 \rceil$.

Coarsening Subdivisions





The set $\mathcal{T}\langle S\rangle$ of refinements of a subdivision S induces a subgraph of the flip graph isomorphic to a d-polytope, $d = \mathsf{sl}\, S$ (a product of associahedra) $\Rightarrow$ the flip graph can be covered by 1-skeletons of $\lceil n/2 - 2 \rceil$-polytopes.

IV.

Regular Triangulations
Partial Triangulations

Triangulations by Liftings

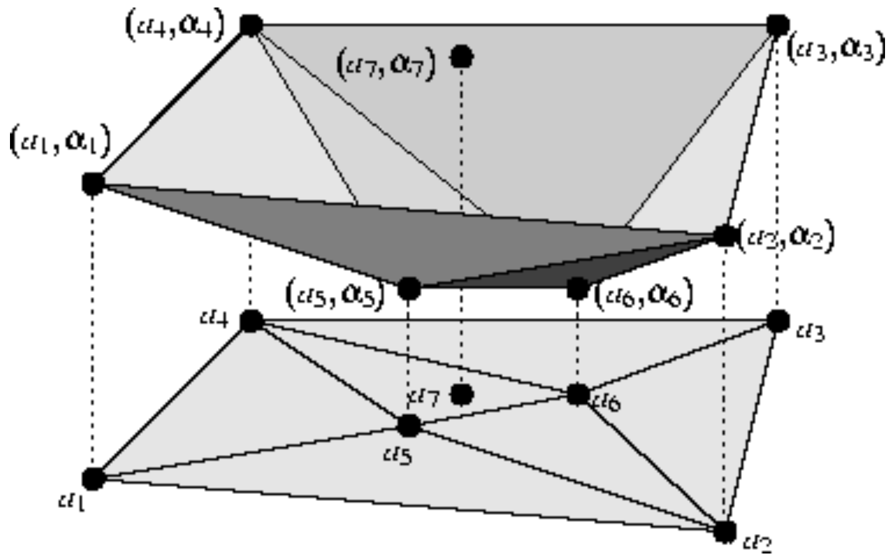
One way of obtaining a triangulation of P is (i) to assign to every point $p = (x_p, y_p)$ a height ω_p, (ii) to take the convex hull of the lifted points $\{(x_p, y_p, \omega_p) \,|\, p \in P\}$ and (ii) to project back the lower convex hull of these lifted points.

(assuming that no four lifted points are coplanar)

(1) In this way, we might skip some points in P (we never skip extreme points in P). We call this a partial triangulation T of P, i.e. a triangulation of $P' = \mathsf{V}(T)$, $\mathsf{extr}(P) \subseteq P' \subseteq P$; $\mathcal{T}_{\mathsf{part}}(P)$ denotes the set of partial triangulations of P.

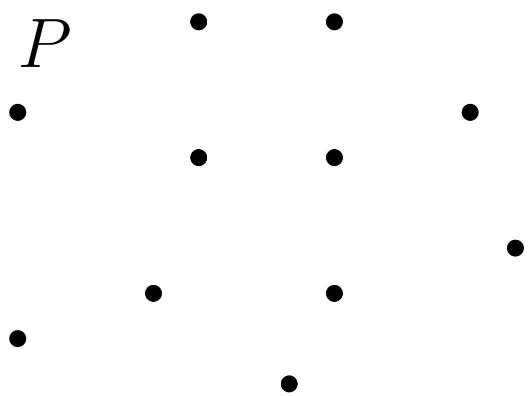
(2) Not all triangulations of P can necessarily be obtained via liftings. Partial triangulations we can obtain via liftings are called regular triangulations; $\mathcal{T}_{\mathsf{reg}}(P)$ denotes the set of regular triangulations of P.

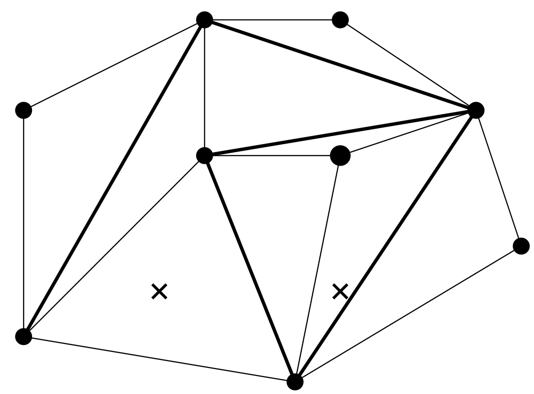
Lifting — Regular Triangulations



from [De Loera, Rambau, Santos, 2010]

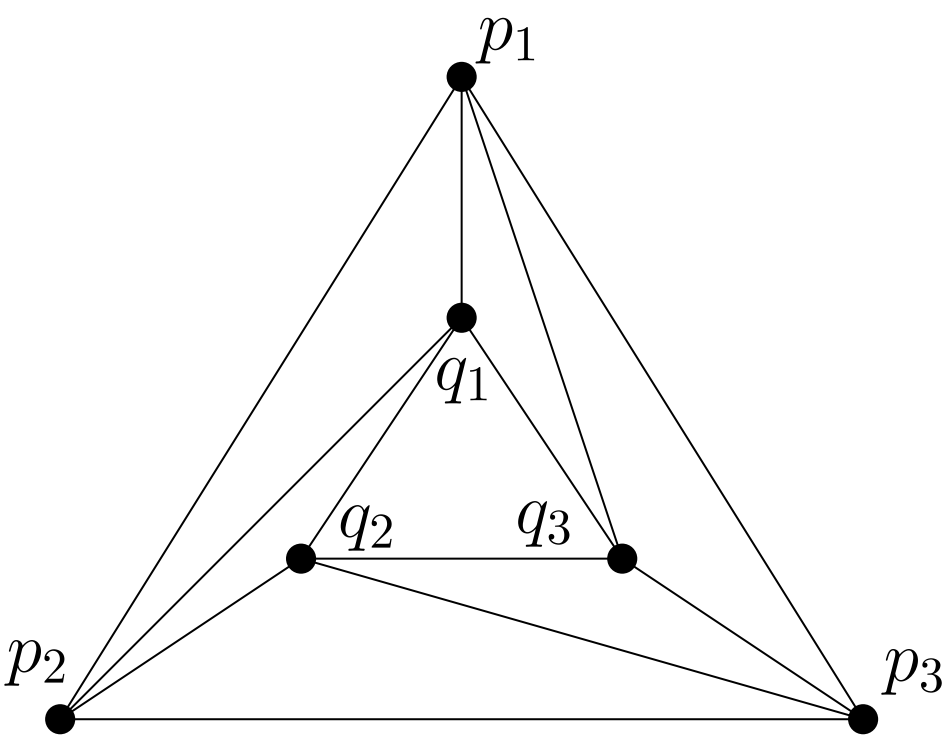
A Partial Triangulations of P





. . . with two points skipped.

A Nonregular Triangulation — Mother of Examples



Assuming $\omega_{q_1} = \omega_{q_2} = \omega_{q_3}$,
p_1 must be higher than p_2,
p_2 must be higher than p_3,
p_3 must be higher than p_1
— contradiction.

Bistellar Flips

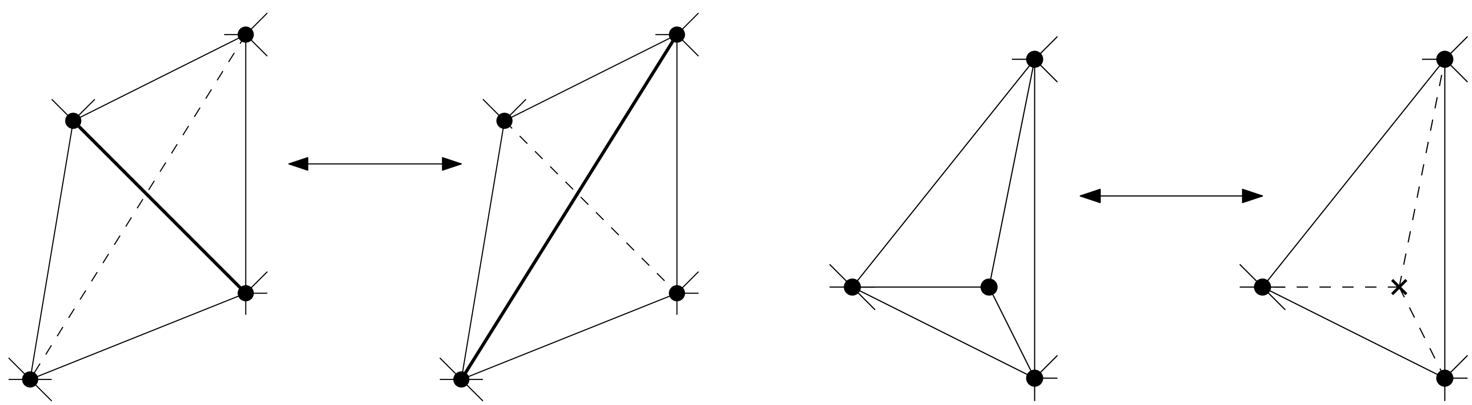
Given a partial triangulation T of P, with vertices $\mathsf{V}(P)$ and edges $\mathsf{E}(P)$, for certain elements $x \in \mathsf{E}(T) \cup P$ we consider bistellar flips on T resulting in a triangulation $T[x]$.

- $x = e \in \mathsf{E}(T)$ is flippable: $T[e]$ as before (edge flip, $(2,2)$-flip).
- $x = p \in \mathsf{V}(T)$, an inner point of degree 3: $T[p]$ is obtained by removal of p with its edges (point removal flip, $(3,1)$-flip).
- $x = p \in P \setminus \mathsf{V}(T)$: $T[p]$ is obtained by inserting p with three incident edges in T (point insertion flip, $(1,3)$-flip).

Gives (bistellar) flip graph of partial triangulations $(\mathcal{T}_{\text{part}}(P), \mathcal{B}(P))$ and (bistellar) flip graph of regular triangulations $(\mathcal{T}_{\text{reg}}(P), \mathcal{B}_{\text{reg}}(P))$.

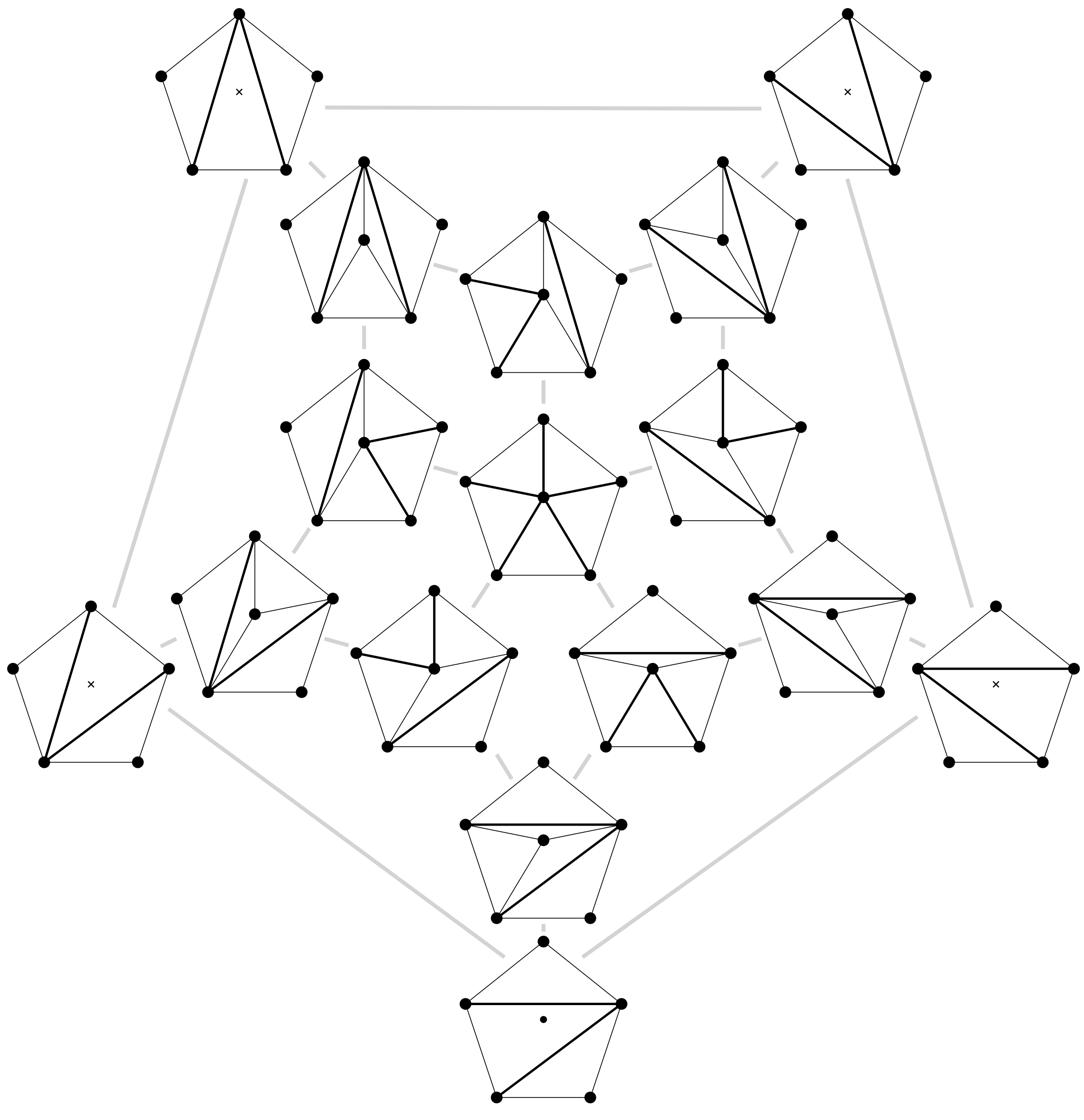
Bistellar Flips

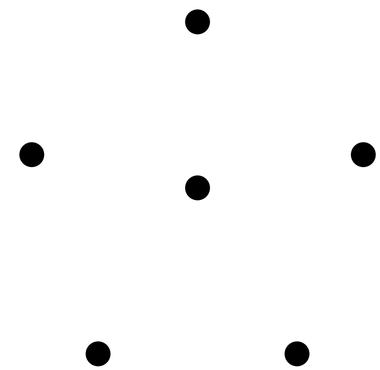
A bistellar flips switches from a bottom view of a tetrahedron in $\mathbb{R}^3$ to the bottom view. Or describes the change of the lower convex hull projection of a lifting, as the height function changes.



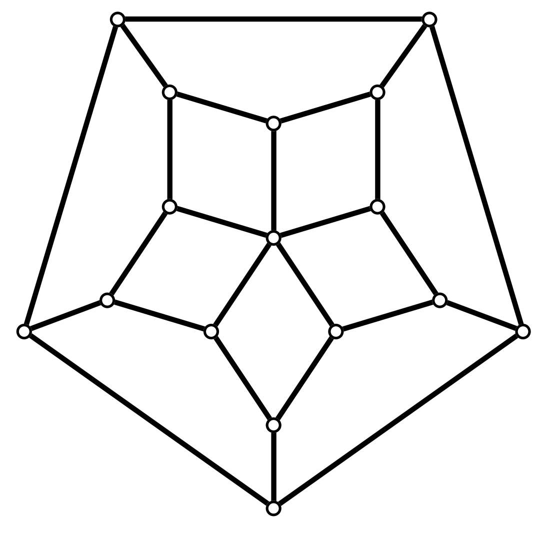
edge flip

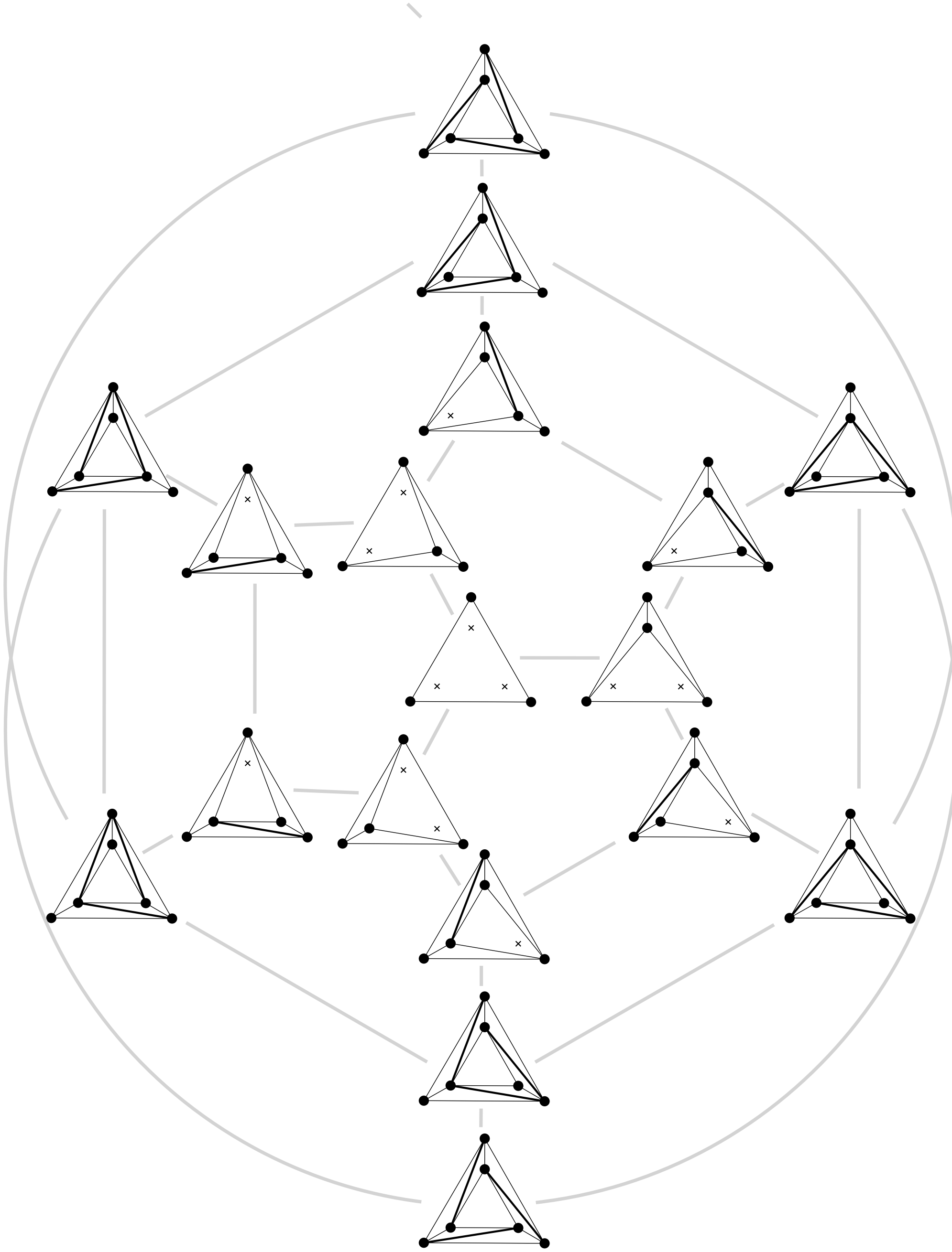
point removal ($\rightarrow$)/ insertion ($\leftarrow$) flip

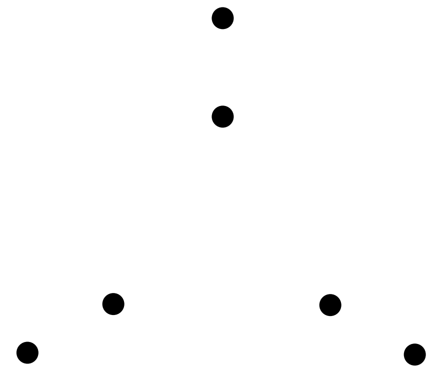




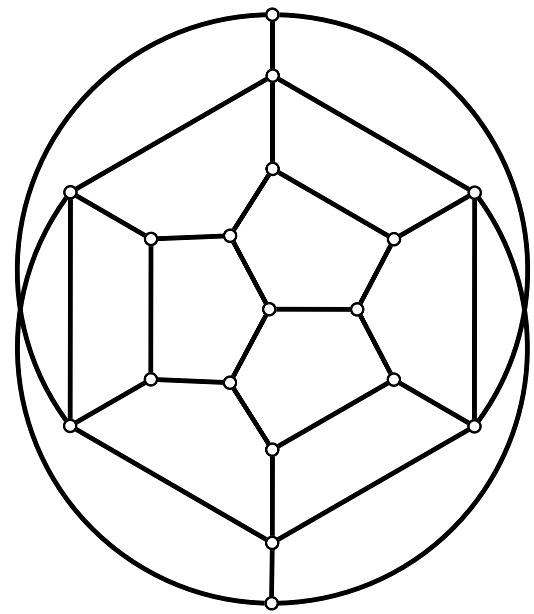
all regular



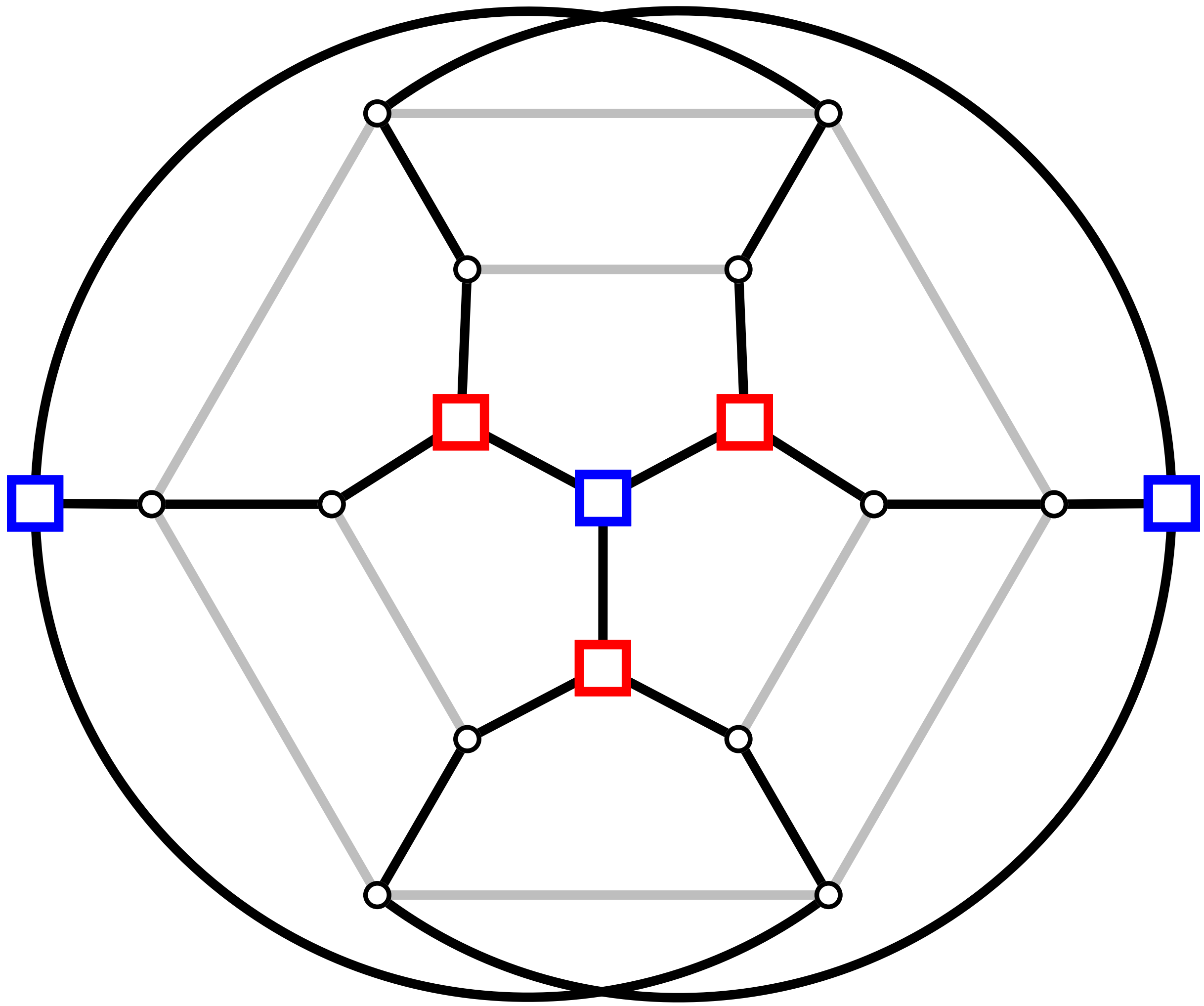




some not regular



not planar



Number of Bistellar Flips in a Partial Triangulation of P

Let T be a triangulation of $\mathsf{extr}(P)$ (as partial triangulation of P), with $h := |\mathsf{extr}(P)|$. In the triangulation T of $\mathsf{extr}(P)$ all inner edges are flippable, there are $h-3$ edge flips. We have another $(n-h)$ point insertion flips, which make altogether $(h-3)+(n-h) = n-3$ bistellar flips possible for T.

Proposition. [De Loera, Santos, Urrutia, 1999] Every partial triangulation of P allows at least $n-3$ bistellar flips.

"Ideally," we can expect $(n-3)$-connectivity in bistellar flip graphs (regular triangulations, partial triangulations).

Secondary Polytope [Gelfand, Kapranov, Zelevinsky, 1990]

Given a partial triangulation T of P assign to each $p \in P$ the sum, $s_T(p)$, of the areas of all faces incident to p. ($= 0$, if p is skipped.)

The secondary polytope of P is the convex hull of

$$\mathsf{GKZ}(P) := \{(s_T(p))_{p\in P} \in \mathbb{R}^n \,|\, T \in \mathcal{T}_{\text{part}}(P)\} \ .$$

Proposition. The secondary polytope has dimension $n-3$, with its vertices the points of regular triangulations. The 1-*skeleton is the flip graph of regular triangulations* (needs general position).

With Balinski's Theorem, it follows that the bistellar flip graph of regular triangulations is $(n-3)$-vertex connected.

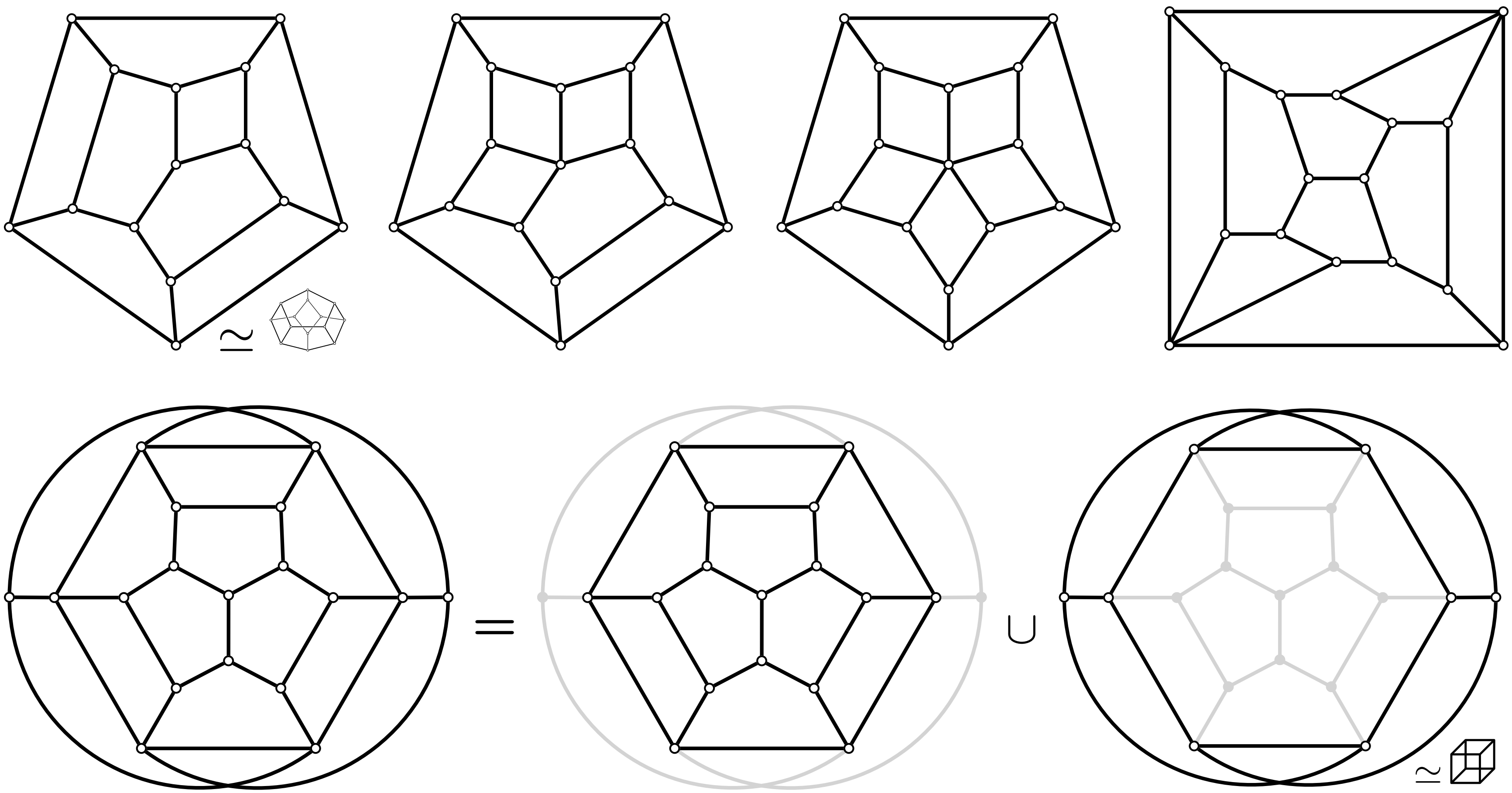
Connectivity & Geometry of Bistellar Graph

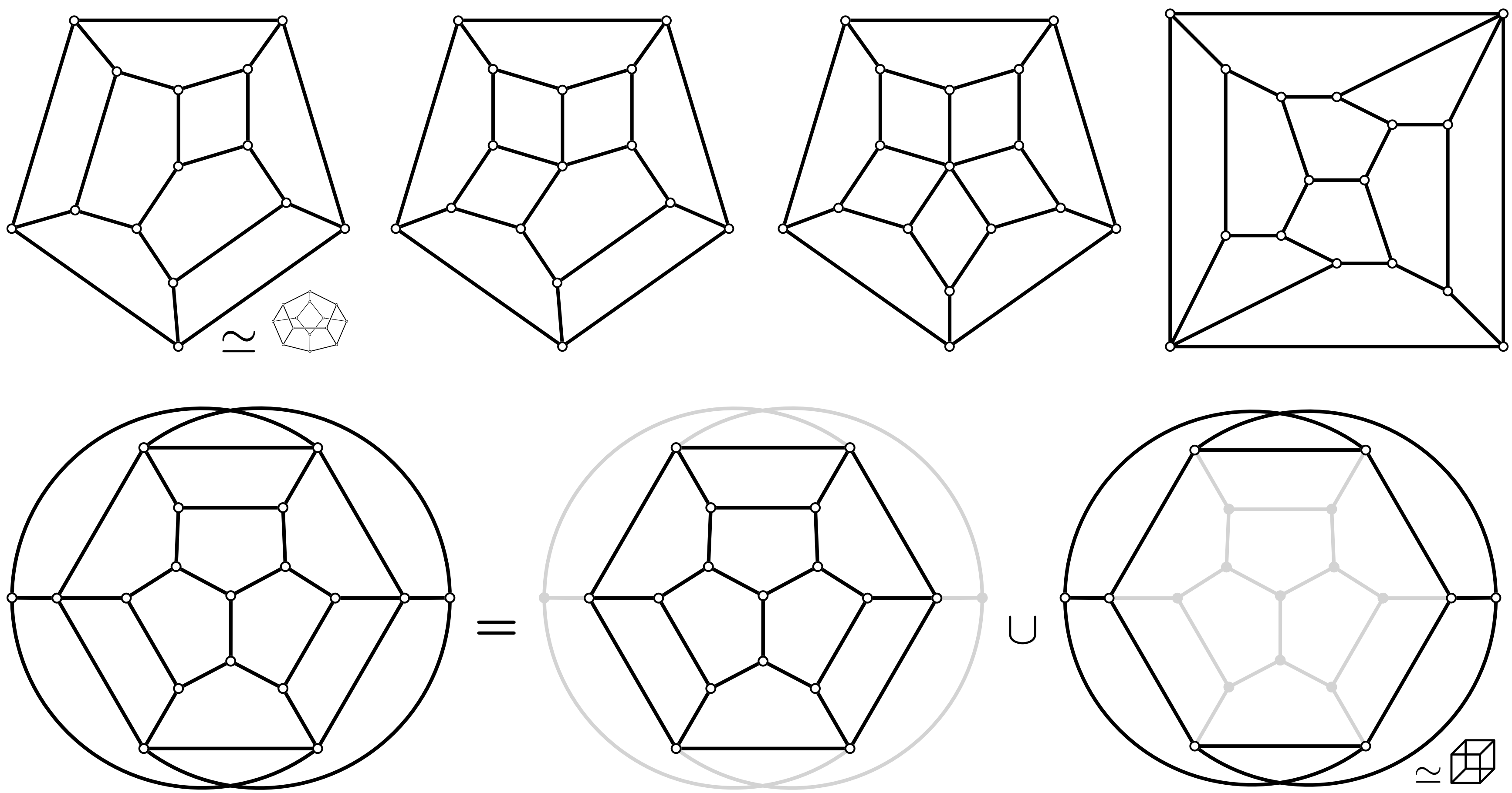
We prove

Theorem. The bistellar flip graph of partial triangulations of P is $(n-3)$-vertex connected.

Theorem. The bistellar flip graph can be covered by 1-skeletons of $(n-3)$-polytopes (products of secondary polytopes).

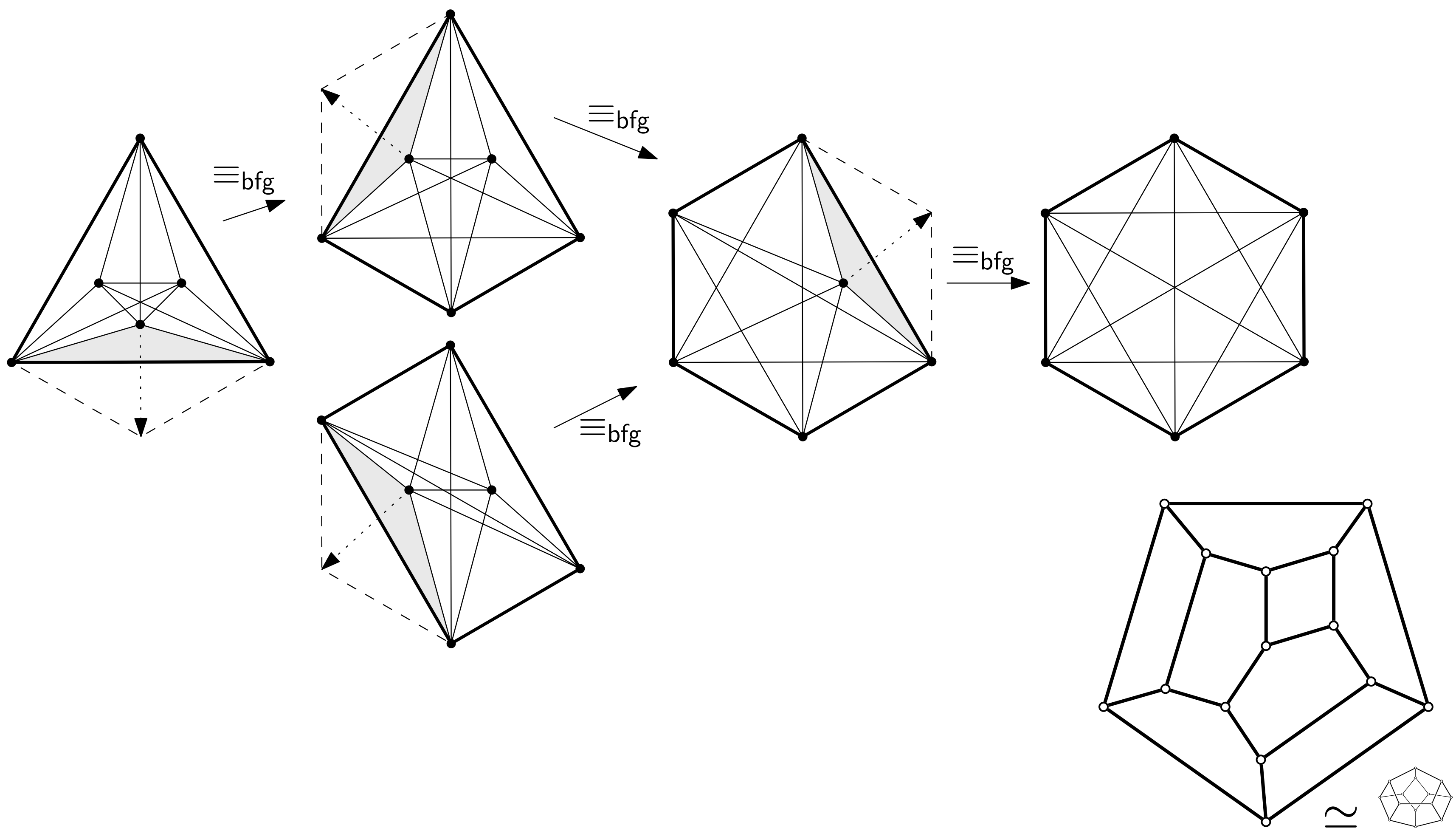
Proofs require a Coarsening Lemma for partial subdivisions and a sufficient condition for regularity of partial triangulations.





All bistellar flip graphs for $n = 6$ (general position, $\exists$ 16 order types).

Many Sets Give the Same Bistellar Flip Graph



Summary

Full Triangulations	$\subseteq$	*Partial Triangulations*	$\supseteq$	*Regular Triangulations*
Edge Flip Graph		Bistellar Flip Graph		Bistellar Flip Graph
$\lceil n/2 - 2 \rceil$-connected		$(n-3)$-connected		$(n-3)$-connected
covered by $\lceil n/2 - 2 \rceil$-polytopes		covered by $(n-3)$-polytopes		"is an" $(n-3)$-polytope
				secondary polytope [Gelfand, Kapranov, Zelevinsky, 1990]

V.
Open Problems

Large Slack Subdivisions

Complexity of computing the largest slack coarsening subdivision for a given subdivision.

(There is always one of slack at least $n/2 - 2$.)

Complexity of computing the largest slack subdivision for a given point set (= largest polytope in flip graph).

(2020 SoCG Geometric Optimization Challenge)

(There is always one of slack at least $2n/3 - 3$ [Sakai,Urrutia 2009/19], and there can be none of slack exceeding $3n/2 - 6$.)

Mixing Time of Random Walk on the Flip Graph

How long does it take until successively randomly flipping edges in a triangulation gets us close to a u.a.r. triangulation of a given point set?

This is known only for the special case of the triangulations of a convex n-gon: [McShine, Tetali, 1997], [Molloy, Reed, Steiger, 1998] show that mixing takes time in $\Omega(n^{3/2}) \cap \mathcal{O}(n^5)$.

[Caputo, Martinelli, Sinclair, Stauffer, 2012] investigate lattice triangulations (not gen. pos.).