Tverberg's theorem

$P \subset \mathbb{R}^d$ n points k-partition $T_1, \ldots T_k$ $\mathrm{conv}(T_1) \cap \ldots \mathrm{conv}(T_k)$ - intersect

$n \geq (d+1)(k-1)+1$

computing the partition - total search problem

$\in PPAD \cap PLS$ because of a reduction to colorful caratheodory Sarkaria

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Mustafa, Barany, Adiprasito - approximate "no-dimensional" version

Ball of radius $\sim \underline{\mathrm{diam}(P) \times \frac{\sqrt{k}}{\sqrt{n}}}$ optimal upto constant

Proof idea: averaging argument

Algorithm: de randomized, but $2^{O(n)}$ time.

Alternate proof. Ball radius $\sim \frac{k}{\sqrt{n}} \mathrm{diam}(P)$

Algorithm : $O(nd \log k)$

Reduction to colorful caratheodory + special derandomization

- colorful Tverberg. Ball $\sim \frac{k}{\sqrt{N}} \max_i (\mathrm{diam}(P_i))$ $P_1, \ldots P_n$ colors

Algorithm $O(ndk)$

Tensor products - outline
averaging argument
putting together

Algorithm
Colorful version, others
Open questions

$u = (u_1, \ldots u_m)$
$v = (v_1, \ldots v_n)$

$u \otimes v = (u v_1, u v_2, \ldots u v_n)$
$(u_1 v_1, u_2 v_1, \ldots u_n v_1, u_2 v_2, \ldots$

Bilinear product

$P = p_1 \ldots p_n$ $Q = q_1 \ldots q_k$ $\sum q_i = 0 \leftarrow$ choose unique comb, but not important here.

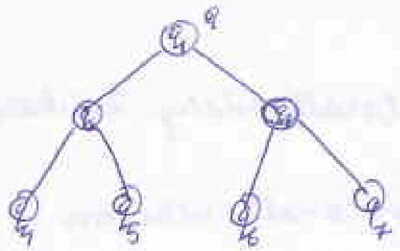
$p_1 \otimes q_1$	$p_2 \otimes q_1$	$\vdots$	
$p_1 \otimes q_2$	$\vdots$		$\underline{\text{Clones of } p_i}$
$\vdots$			
$p_1 \otimes q_k$	$\otimes q_k$	$\vdots$	
P_1	P_2	P_n	$\leftarrow$ centroids are origin

choose $q_1 \dots q_k$?

Build a graph G on k nodes. Each Node $i \longrightarrow q_i$

Each edge $\longrightarrow$ define one co-ordinate

Orient edges arbitrarily



$q_1 \;\dots\; 1 \;\dots\; 1$

$q_3 \;\dots\dots\; -1$

$q_2 \;\dots\; 1 \;\dots\; -1$

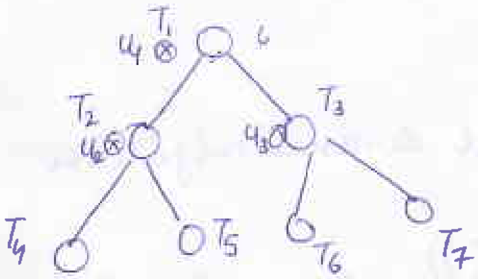
$q_5 \;\dots\; -1 \;\dots$

. . .

$\sum q_i = 0$

$q_i \in \mathbb{R}^{\|G\|}$ $\quad \Delta = \text{max-degree}(G)$

node $q_i \longrightarrow$ set T_i in partition



$u_1, \dots\dots, u_k \in \mathbb{R}^m$

$$\| u_1 \otimes q_1 + u_2 \otimes q_2 + \dots \|^2$$

$$= \|u_1 - u_2\|^2 + \|u_1 - u_3\|^2 + \dots$$

$$= \sum_{q_i q_j} \|u_i - u_j\|^2$$

$P_1 \;\dots\dots\dots\; P_n \in \mathbb{R}^{d\|G\|}$ $\qquad$ colorful-caratheodory like setting

n_i dim no rel^n

$x_1 \in P_1 \;\dots\dots\dots\; x_n \in P_n$

show: traversal exists with centroid close to origin

$$E\left[\|C(x_1, \dots\dots x_n)\|\right]$$ overs all choices of x_i

Fair bit of calculation, Tree structure is handy here.

$$E < \text{diam}(P) \sqrt{\frac{\|G\|}{k(n-1)}} := \gamma$$

$\Rightarrow \exists$ traversal with $C(\dots)$ norm at most γ

$x_1, \ldots\ldots x_n$

$\{p_1, \ldots\ldots\} \otimes q_1, \quad \{p \ldots\ldots\} \otimes q_2, \ldots\ldots \quad \{\ldots\ldots\} \otimes q_k$

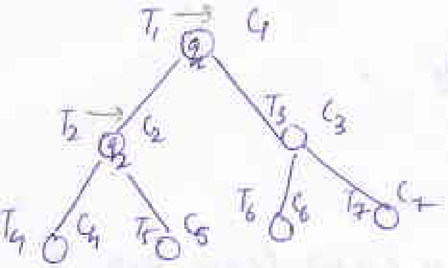
$\downarrow T_1 \qquad \downarrow T_2 \qquad \downarrow T_k$

each T_i equal size

show: $\exists$ ball intersecting convex hulls of each T_i

$C(T_1), \quad C(T_2), \quad C(T_k)$

$c_1 \qquad c_2 \qquad c_k$



$\|\text{centroid}\|^2 < \gamma^2$

$\downarrow$

$$\frac{1}{k^2}\| c_1 \otimes q_1 + c_2 \otimes q_2 + \ldots\ldots c_k \otimes q_k \| < k^2 \gamma^2$$

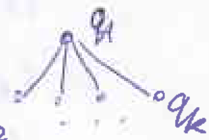
$$\sum_{q_i q_j} \| c_i - c_j \|^2 < k^2 \gamma^2$$

Pick c_1, bound $\|c_1 - c_j\|$ for each centroid

$$\lesssim \frac{\sqrt{k}}{\sqrt{n-1}} \, diam(P) \sqrt{\|G\| \, diam(G)}$$

$\downarrow$

minimized for star graph



~~Hd~~ $\sim \frac{k}{\sqrt{n}} \, diam(P)$

$|T_1| = |T_2| = \ldots\ldots |T_k| = \frac{n}{k}$

choose sizes $r_1, r_2, \ldots\ldots r_k \quad \sum r_i = n$

use above arguments with some modifications

Ball of radius $\sim \frac{\sqrt{n \log k}}{\min_i r_i} \, diam(P) \leftarrow \sim \frac{\sqrt{n} \, diam(P)}{\min_i r_i} \sqrt{\frac{\Delta(G)}{diam(G)}}$

Δ, ~~#~~ $diam(G)$ for a 4-ary tree, balanced

$5 \log_4 k$

Algorithm: derandomize, method of conditional expectations

$$E[\|C(x_1, \ldots\ldots x_n)\|] > E[\|C(x_1, \ldots\ldots x_{n-1}, y_n)\|]$$

k choices for x_n.
pick one that satisfies

$$> E[\|C(x_1, \ldots\ldots x_{n-2}, y_{n-1}, y_n)\|]$$

$$\vdots$$

$$E[C(y_1, \ldots\ldots y_n)] \leftarrow \text{answer}$$

- we do not compute tensor products explicitly
- $E(\ldots)$ can be simplified

Linear option: For each P_i, k clones exist.
each clone takes $O(d)$ time $\rightarrow$ $O(dk)$ time per P_i
$\rightarrow$ Total $O(ndk)$ time

Alternate option: Spend $O(d \log k)$ time per P_i $\rightarrow$ $O(nd \log k)$

- Augment graph $\|G_i\|$ with some information about conditional expectations.
- Follow path to a leaf from the root, spending $O(d)$ time per node
  $\rightarrow$ identify suitable clone
- update tree in similar fashion (local updates required)

information stored: ~~avg~~ to compute expected value of the subtree rooted at that node.

Small differences between balanced & non-balanced case.

balanced case: use ϕG to update neighbor information.

Colorful Tverberg

$C_1, \ldots\ldots, C_n \subseteq \mathbb{R}^d \quad |C_i| = k$

$\underbrace{C_1, \ldots\ldots, C_n}_{\downarrow \text{ partition}}$

$\exists T_1 \ldots T_k$ convex hulls intersecting & T_i is colorful

= 1 pt from each color

$n = d+1 \rightarrow$ default, then true

$k+1 =$ prime

Blagojević, Matschke, Ziegler 2009

No-dim Colorful Tverberg → By SODA paper, non-algorithmic

Ball of radius $\sim \frac{k}{\sqrt{nk}} \max_i \operatorname{diam}(P_i)$ C_i essentially same bound as non-colorful.

- Tensor products, but slightly different

$C = \{x_1 \ldots\ldots x_k\} \rightarrow C' \quad x_1 \otimes q_1 + x_2 \otimes q_2 \ldots\ldots + x_k \otimes q_k$

$x_1 \otimes q_2 + x_2 \otimes q_3 + \ldots \quad x_k \otimes q_1$

Centroid origin $\leftarrow x_1 \otimes q_k + x_2 \otimes q_1$

averaging argument

↓

de-randomized

↓

$O(ndk)$ algorithm, similar fashion

- in each iteration, choose permutation.

Using a tree as before would require a full update of the tree.

Flat-traversal

$P_1, \ldots, P_k \subset \mathbb{R}^d$ $(k-1)$-flat F exists,

F has depth $\frac{|P_i|}{d-k+2}$ for each i.

$k = 1 \rightarrow$ Centerpoint theorem

$k = d \rightarrow$ Ham-Sandwich

Each halfspace that contains F also has "depth" many points from P_i

$P_1, \ldots, P_k$

$\exists$ Ball B of dim $(d-k+1)$ & a $(k-1)$-dimensional subspace L

$B \times L$ has depth $\geq \frac{|P_i|}{m_i}$ for each i.

Radius $\sim \max_i \frac{\text{diam}(P_i)}{\sqrt{m_i}}$

B, L can be determined in $O(dk^2 + Nd)$ $\quad N = \sum |P_i|$

Open questions.

Bounds for regular $\begin{cases} \text{balanced} \\ \text{non-balanced} \end{cases}$, colorful are all sub-optimal

reg-balanced

$\sum_{\text{edge}} \|C_i - C_j\|^2 \leq k^2 \gamma^2$: if one $\|C_i - C_j\|$ is heavy, it dominates. Then C_i is not a good choice of center.

- Alternate centers?
- avoid using centroids always?

reg-unbalanced

- already sub-optimal when plugging in $r_1 = r_2 = \cdots = r_k$

colorful : bound has same issue as reg-balanced

algorithm : can use alternate graphs to simplify terms in expectation slightly, but not completly.

Topological no-dim

- WSPD?