Bipartite Diameter and Other Measures Under Translation

Boris Aronov, **Omrit Filtser**, Matthew J. Katz, and Khadijeh Sheikhan

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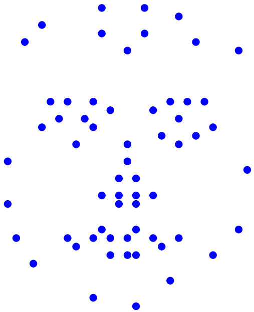
Similarity between two sets of points

Goal: Determining the similarity between two sets of points.



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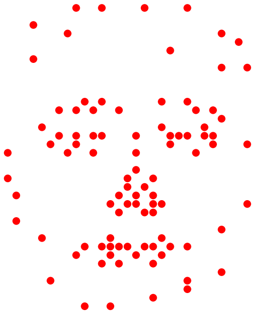

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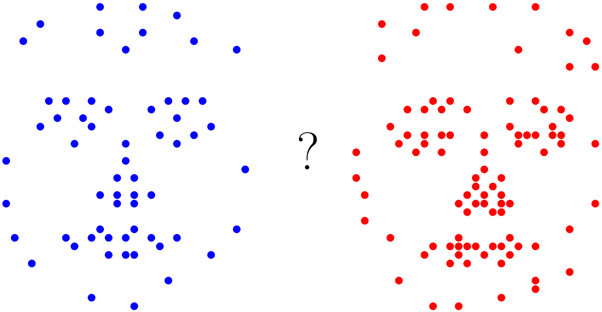

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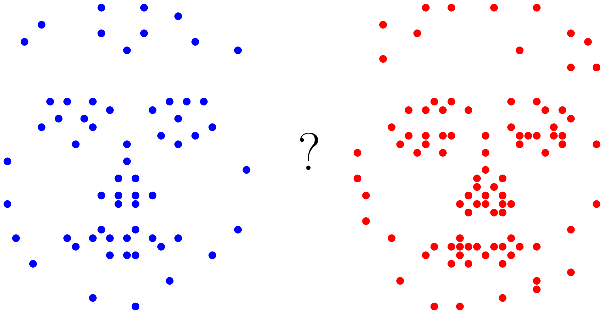
- **Problem**: Sometimes, a bipartite measure is meaningless, unless one of the sets undergoes some <u>transformation</u>.



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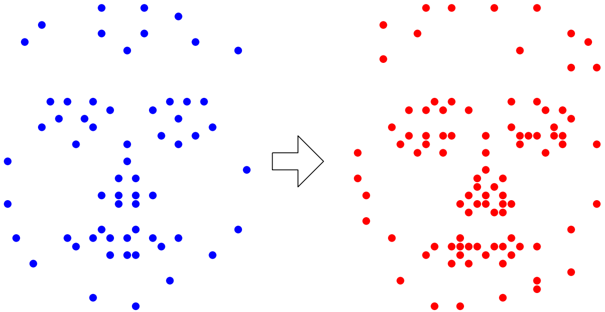
Find a translation which minimizes some bipartite measure.



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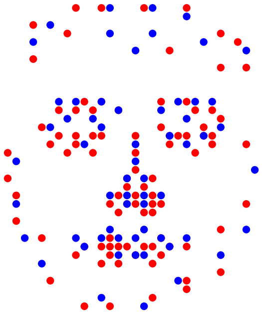

Bipartite measures under translation

$A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ – two sets of points in $\mathbb{R}^d$.

Problem

Find a translation t^* *that minimizes* **some bipartite measure** *of* A *and* $B + t$ *over all translations* t.

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- For the sake of simplicity, we assume that $m = n$.

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- For the sake of simplicity, we assume that $m = n$.
- This class of problems naturally extends to other types of transformations, such as rotations, rigid motions, etc.

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When comparing two sets of points A and B of the same size:

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When comparing two sets of points A and B of the same size:

- **Congruence testing**: decide if there exists a transformation that maps A exactly or approximately into B.

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When comparing two sets of points A and B of the same size:

- **Congruence testing**.
- **RMS distance**: minimize the sum of squares of distances in a perfect matching between A and B.

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When comparing two sets of points A and B of different sizes:

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When comparing two sets of points A and B of different sizes:

- **Hausdorff distance**: the maximum of the distances from a point in each of the sets to the nearest point in the other set.
  Huttenlocher,Kedem, Sharir: $\tilde{O}(n^3)$ in 2D.

Some bipartite measure?

When comparing two sets of points A and B of the same size:

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- **Hausdorff distance**: $\tilde{O}(n^3)$ in 2D.
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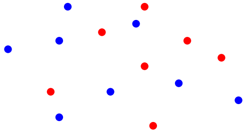
All the above measures (under various geometric transformations) were widely investigated in the literature.

Our results

The main bipartite measures that we consider are:



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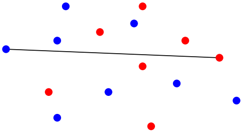
- **diameter** – the distance between the farthest bichromatic pair, i.e. $\max\{\|a-b\| \mid (a,b) \in A \times B\}$.



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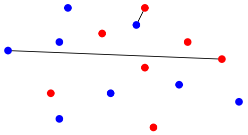
- **diameter** – $\max\{\|a-b\| \mid (a,b) \in A \times B\}$.
- **uniformity** – the difference between the bipartite diameter and the distance between the closest bichromatic pair, i.e. $\text{diam}(A,B) - \min\{\|a-b\| \mid (a,b) \in A \times B\}$.



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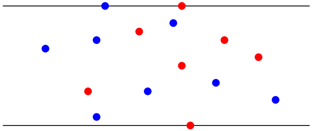
- **union width** – the width of $A \cup B$, where the *width* of a set of points in the plane is the smallest distance between a pair of parallel lines, such that the closed strip between the lines contains the entire set.



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Surprisingly, all of these measures (under translation) were not investigated previously in the literature.

Our results

measure	dimension	running time
diameter	$d = 2$	$O(n \log n)$
	$d = 3$	$O(n \log^2 n)$
	$d > 3$ (fixed)	$O(n^2)$
uniformity	$d = 2$	$O(n^{9/4+\varepsilon})$
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Diameter

$A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ – two sets of points in $\mathbb{R}^d$.

$\mathrm{diam}(A, B) = \max\{\|a - b\| \mid (a, b) \in A \times B\}$

Problem (Bipartite Diameter under Translation)

Find a translation t^* such that for any translation t,
$\mathrm{diam}(A, B + t^*) \leq \mathrm{diam}(A, B + t)$.

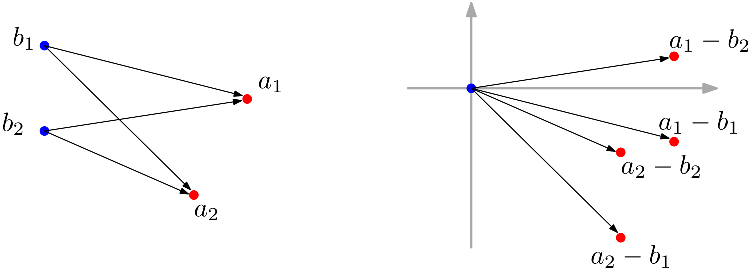
Diameter

$$\mathcal{P} = \{a - b \mid (a, b) \in A \times B\}$$

Diameter

$\mathcal{P} = \{a - b \mid (a, b) \in A \times B\}$

- The set of all possible translations taking a point of B to a point of A.



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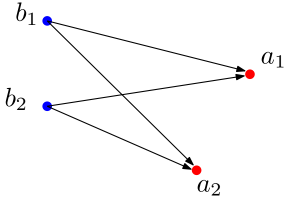
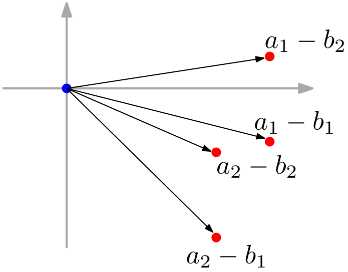
- The set of all possible translations taking a point of B to a point of A.
- Clearly, $|\mathcal{P}| = O(n^2)$.





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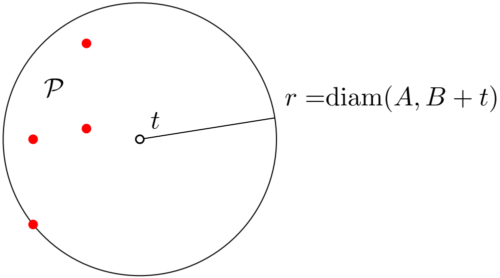
Claim

Given a point t, the radius of the **minimum enclosing ball** *of $\mathcal{P}$ centered at t is equal to* $\text{diam}(A, B + t)$.



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Proof.

This radius is

$$\max_{(a-b)\in\mathcal{P}} \|(a - b) - t\| = \max_{(a,b)\in A\times B} \|a - (b + t)\| = \mathrm{diam}(A, B + t).$$

□

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Given a point t, the radius of the **minimum enclosing ball** *of $\mathcal{P}$ centered at t is equal to* $\text{diam}(A, B + t)$.

Corollary

The optimal translation t^* minimizing the bipartite diameter coincides with the center of the minimum enclosing ball of $\mathcal{P}$.

Diameter: Algorithm (naive implementation)

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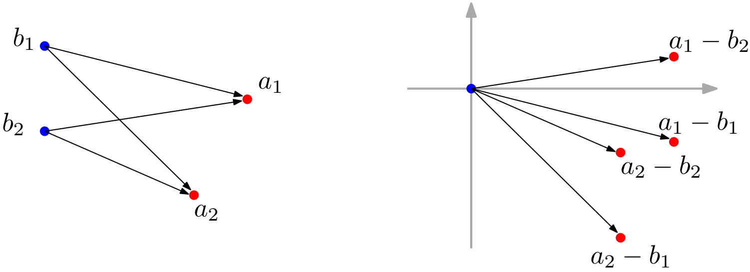
- Compute the set of translations $\mathcal{P}$.



Diameter: Algorithm (naive implementation)

- Compute the set of translations $\mathcal{P}$.
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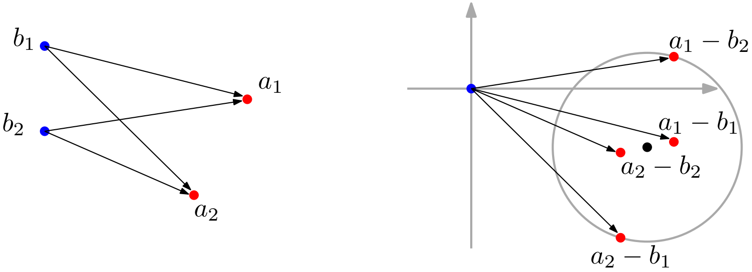

Diameter: Algorithm (naive implementation)

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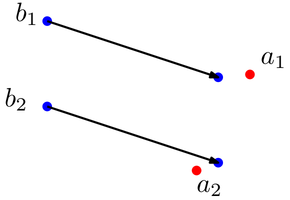
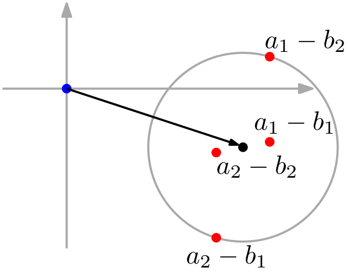
- Translating B by c minimizes the diameter.





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The minimum enclosing ball can be computed in:

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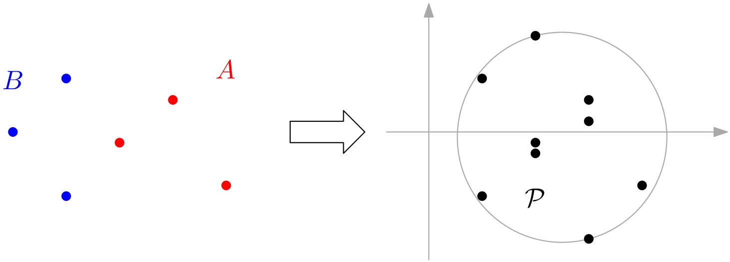
BUT, in 2D and 3D, we can do better!

In fact, computing the minimum enclosing ball of $\mathcal{P}$ in 2D and 3D (*without computing* $\mathcal{P}$ *explicitly*) can be done in **near-linear time**...

Diameter

$\mathcal{P} = \{a - b \mid (a, b) \in A \times B\}$
Goal: compute the minimum enclosing ball of $\mathcal{P}$ implicitly.



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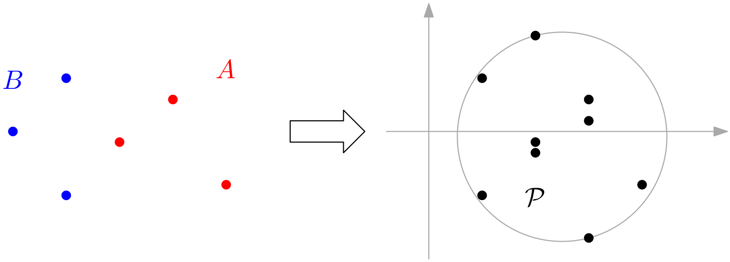

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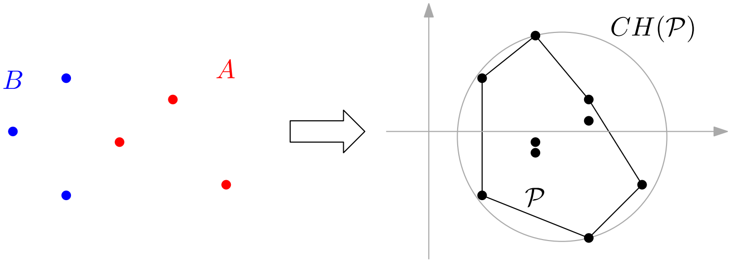

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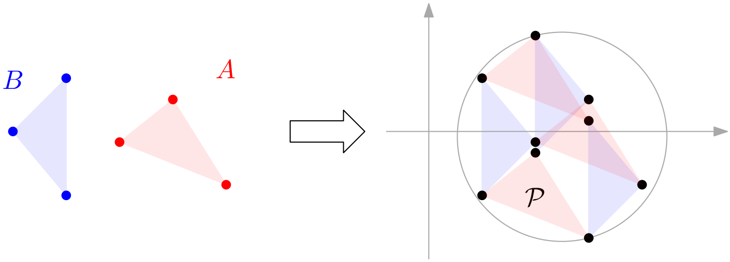
2. $\mathcal{P}$ is the Minkowski sum of A and $-B$, i.e. $\mathcal{P} = A \oplus -B$.



Diameter in 2D

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A fact from the textbook

For points in 2D, the size of $\mathrm{CH}(A \oplus -B)$ is $O(n)$,
and it can be constructed in $O(n)$ time from $\mathrm{CH}(A)$ and $\mathrm{CH}(B)$
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$\Rightarrow$ $O(n \log n)$-time solution for points in 2D!

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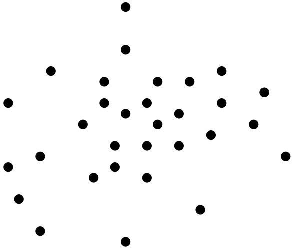
Idea: The minimum enclosing ball is an LP-type problem $\Rightarrow$
adapt Clarkson's ('95) scheme for solving LP-type problems.

Diameter in 3D: Algorithm

X – an empty set of points.
Repeat until the minimum enclosing ball is found:



$\mathcal{P}$ (points)

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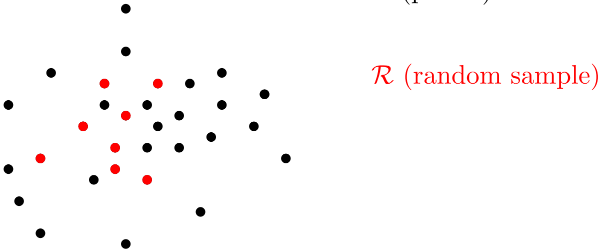
1: Pick a random sample $\mathcal{R}$ of $\mathcal{P}$ of size $4n$.



Diameter in 3D: Algorithm

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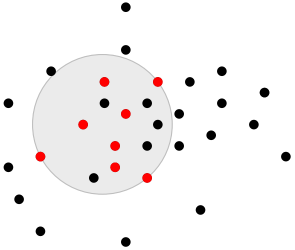
Repeat until the minimum enclosing ball is found:

2: Compute the minimum enclosing ball S of $\mathcal{R} \cup X$.



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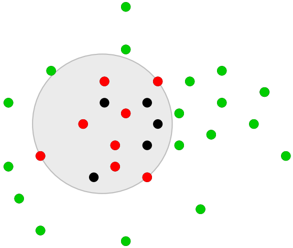
3: Find the set of *violators* V. If $|V| \geq 2n$, go to 1.



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$|V| \geq 2n$, **"bad"** iteration :(

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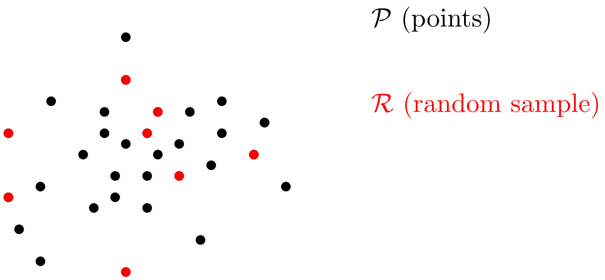

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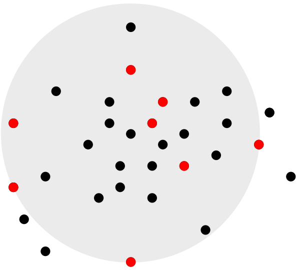

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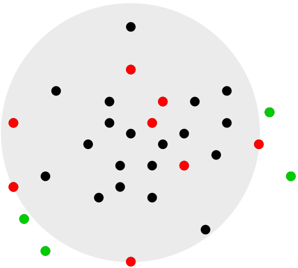

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$|V| < 2n$, **"good"** iteration :)

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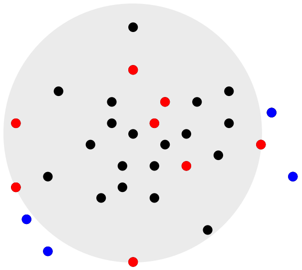
Repeat until the minimum enclosing ball is found:

4: If $V \neq \emptyset$, then $X \leftarrow X \cup V$ and go to 1. Else, return S.



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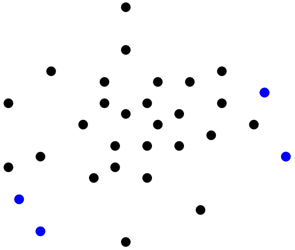

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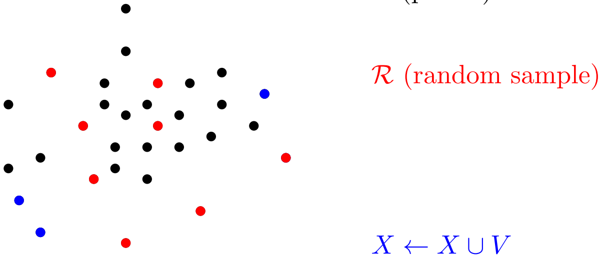

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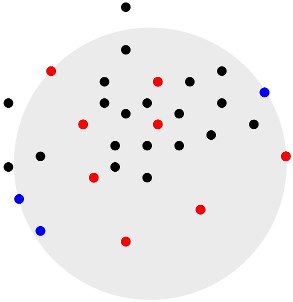

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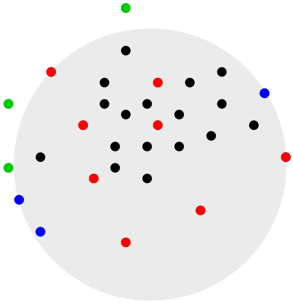
Repeat until the minimum enclosing ball is found:

3: Find the set of *violators* V. If $|V| \geq 2n$, go to 1.



$\mathcal{P}$ (points)

$\mathcal{R}$ (random sample)

V (violators)

$X \leftarrow X \cup V$

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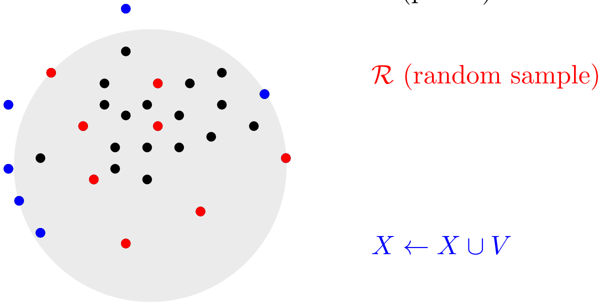

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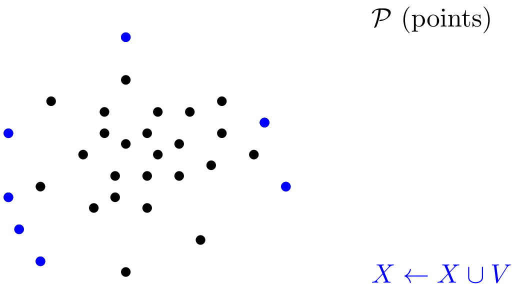

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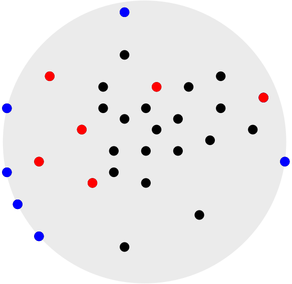

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- Repeatedly pick random points $a \in A$ and $b \in B$ and return $a - b$.

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- Invoke a standard minimum-ball algorithm on $O(n)$ points, requiring $O(n)$ expected time.

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Given two sets A and B, each of n points in $\mathbb{R}^3$, a distance r and a parameter k, report all the pairs of points $a \in A$, $b \in B$ with $\|a - b\| > r$, if there are at most k such pairs.
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$\Rightarrow O(n \log^2 n)$-time solution for points in 3D!

Uniformity

$A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ – two sets of points in $\mathbb{R}^d$.

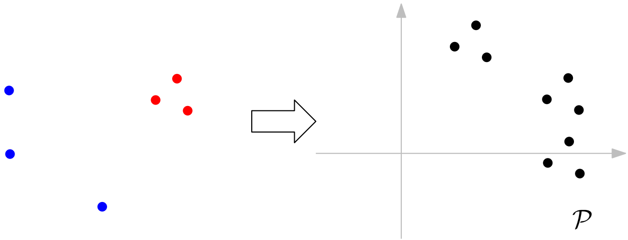
$\text{uni}(A, B) = \text{diam}(A, B) - \min\{\|a - b\| \mid (a, b) \in A \times B\}$

Problem (Uniformity under Translation)

Find a translation t^* such that for any translation t, $\text{uni}(A, B + t^*) \leq \text{uni}(A, B + t)$.

Uniformity

$$\mathcal{P} = \{a - b \mid (a, b) \in A \times B\}$$



Uniformity

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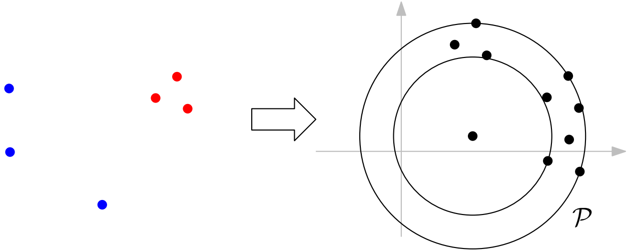
Claim

The optimal translation t^* minimizing the **uniformity** *coincides with the center of the* **minimum-width annulus** *containing $\mathcal{P}$.*



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Thank You!

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  - Our algorithm (for width in 3D under translation) runs in $O(n^2)$ time. Can we do better?