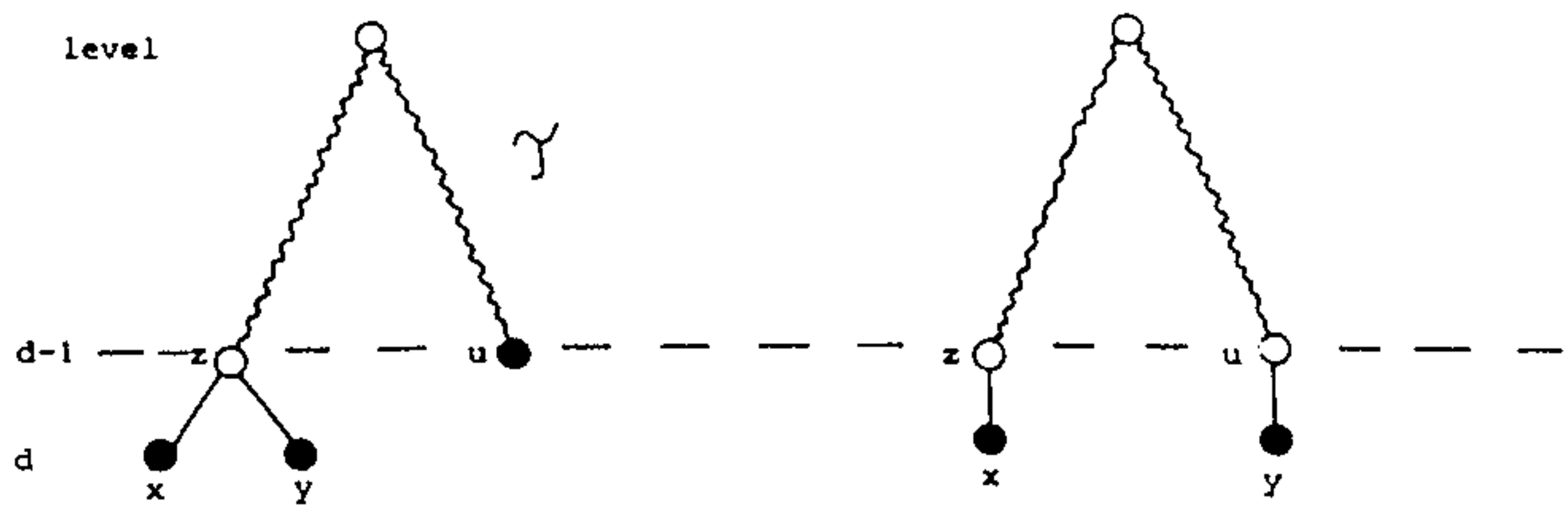

Figure 5. Illustration of transformation II.

Therefore we get

$$\text{Case A:} \quad v_{d-1} = 2^{d-1} - (n - 2^d),$$

$$w(\hat{\tau}) = n \cdot 2^{d-1} - 4^{d-1}.$$

$$\text{Case B:} \quad v_{d-1} = 0$$

$$w(\hat{\tau}) = n \cdot 2^d - 4^d.$$

But in Case A, $0 \leqq n - 2^d \leqq 2^{d-1}$ holds, which yields

$$2^d \leqq n \leqq 2^d + 2^{d-1},$$

whereas in Case B, $2^{d-1} \leqq n - 2^d \leqq 2^d$ implies

$$2^{d-1} + 2^d \leqq n \leqq 2^{d+1}.$$

Thus we get for n with

$$2^d \leqq n \leqq 2^d + 2^{d-1} \quad \text{(Case A):} \quad w(\hat{\tau}) = 2^{d-1} \cdot n - 4^{d-1} =: w_A(n).$$

$$2^d + 2^{d-1} \leqq n \leqq 2^{d+1} \quad \text{(Case B):} \quad w(\hat{\tau}) = 2^d \cdot n - 4^d =: w_B(n).$$

In case $n = 2^d + 2^{d-1}$ (i.e., every node at level $d - 1$ has exactly one successor) both expressions $w_A(n)$ and $w_B(n)$ have the same value.

These two expressions $w_A(n)$ and $w_B(n)$ are tight lower bounds on $w(\hat{\tau})$ for a tree $\hat{\tau}$ with $n - 1$ nodes. To conclude the proof, we have to show that $w_A(n) \geqq \frac{2}{9}n^2$ and $w_B(n) \geqq \frac{2}{9}n^2$. Since both expressions are linear functions of n it suffices to consider the boundary cases $n = 2^d$, $n = 2^d + 2^{d-1}$ and $n = 2^{d+1}$.

In the first case, $n = 2^d$, we get

$$w_A(n) = 2^{d-1}n - 4^{d-1} = \frac{n}{2}n - \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \geqq \frac{2}{9}n^2.$$

In the second case, $n = 2^d + 2^{d-1}$, we have $2^d = \frac{2}{3}n$. Therefore

$$w_A(n) = w_B(n) = 2^d n - 4^d = \tfrac{2}{3}n^2 - (\tfrac{2}{3}n)^2 = \tfrac{2}{9}n^2,$$