with length $T \cdot 2^{-i}$. Thus Lemma 2.1 yields

$$h^-(\bar{t}_j) - h^+(t_j) > \frac{4\varepsilon}{\bar{t}_j - t_j} = \frac{4\varepsilon 2^i}{T}.$$

Therefore we get for $M > 2$ (i.e., τ nonempty)

$$h^-(b) - h^+(a) \geqq \sum_{\text{leaves of } \tau} [h^-(\bar{t}_j) - h^+(t_j)],$$

since the intervals corresponding to the leaves are disjoint (except for common endpoints); the one-sided derivatives are nondecreasing and $h^-(t) \leqq h^+(t)$ for all t.

Hence, by Lemma 2.2, we get

$$h^-(b) - h^+(a) > \sum_{i \geqq 0} v_i \frac{4\varepsilon \cdot 2^i}{T} = \frac{4\varepsilon}{T} \cdot \sum_{i \geqq 0} 2^i v_i \geqq \frac{8}{9} \frac{\varepsilon}{T} (M - 1)^2.$$

Thus we have shown the following theorem for the interval bisection rule.

THEOREM 2.3: The number M of evaluations of $h(t)$, $h^+(t)$, and $h^-(t)$ needed to obtain an upper and lower ε approximation of $h(t)$ by the sandwich algorithm with interval bisection or with slope bisection is bounded as follows:

$$M \leq \max\left(2, \left\lceil \frac{3}{2} \sqrt{\frac{T}{2\varepsilon} \cdot (h^-(b) - h^+(a))} \right\rceil\right).$$

We can express this conversely by saying: If we use M evaluations of $h(t)$, $h^+(t)$, and $h^-(t)$ and always choose the interval with the largest error to be partitioned next, then the lower and upper approximations $l(t)$ and $u(t)$ fulfill $l(t) \leq h(t) \leq u(t)$ and

$$\max_{a \leq t \leq b} (u(t) - l(t)) \leq \frac{9}{8} \frac{T}{(M - 1)^2} [h^-(b) - h^+(a)]$$

$$= \frac{K}{(M - 1)^2}, \qquad K \text{ constant.}$$

In case of the slope bisection rule, we simply have to exchange the role of $h^-(\bar{t}_j) - h^+(t_j)$ with $\bar{t}_j - t_j$. The proof is then completely analogous: We know that the slope difference at level i is at most $(h^-(\bar{t}_j) - h^+(t_j))/2^i$, and Lemma 2.1 gives us then a lower bound for $\bar{t}_j - t_j$. We then sum $\bar{t}_j - t_j$ over all leaves, which gives $T = b - a$, and the theorem follows in the same way as above.

What remains to be shown is Lemma 2.2.