2.3. The Convergence Rate of the Sandwich Algorithm

We shall analyze in the following the two bisection rules and we show that they guarantee quadratic convergence of $l(t)$ and $u(t)$ to $h(t)$. For the maximum error rule, the proof methods of this article cannot be applied. However, by using different techniques, the quadratic convergence of the Sandwich algorithm with the maximum error rule has been established recently by Rote [16]; cf. also Rote [15].

The treatment of the two bisection rules is very similar. We shall first deal with the interval bisection rule.

The bisection algorithm can be visualized on a binary tree (Figure 3). The root of the tree corresponds to the interval $[a, b]$. Every inner node of the tree corresponds to an interval in which the error bound is not met and has exactly two successors. When the sandwich algorithm terminates, every leaf corresponds to an interval in which the error is not greater than ε.

In each iteration the algorithm picks a leaf of the tree with associated interval $[t_i, \bar{t}_i]$ in which the error exceeds ε. Then we set $\bar{t} = \frac{1}{2}(\bar{t}_i - t_i)$, we compute $h(\bar{t})$ and the one-sided derivatives $h^+(\bar{t})$ and $h^-(\bar{t})$, and we add two successors corresponding to $[t_i, \bar{t}]$ and $[\bar{t}, \bar{t}_i]$ to the tree.

Let $\hat{\tau}$ be the tree at the end of the algorithm. If M is the number of evaluations of $h(t)$ and its one-sided derivatives $h^+(t)$, $h^-(t)$, then $\hat{\tau}$ has M-2 inner nodes and M-1 leaves. When we remove all leaves from $\hat{\tau}$ we get a binary tree τ with M-2 nodes. Let v_i be the number of leaves of this new tree τ at level i. Now we shall make use of the following lemma on binary trees, which will be shown later.

LEMMA 2.2: If a binary tree τ has $n - 1$ nodes ($n \geqq 2$) and the number of its leaves at level i ($i \geqq 0$) is v_i, then

$$w(\tau) := \sum_{i \geqq 0} 2^i v_i \geqq \frac{2}{9} \cdot n^2$$

and this bound is tight for infinitely many n.

If we denote $T := b - a$, any node at level i corresponds to an interval $[t_i, \bar{t}_i]$

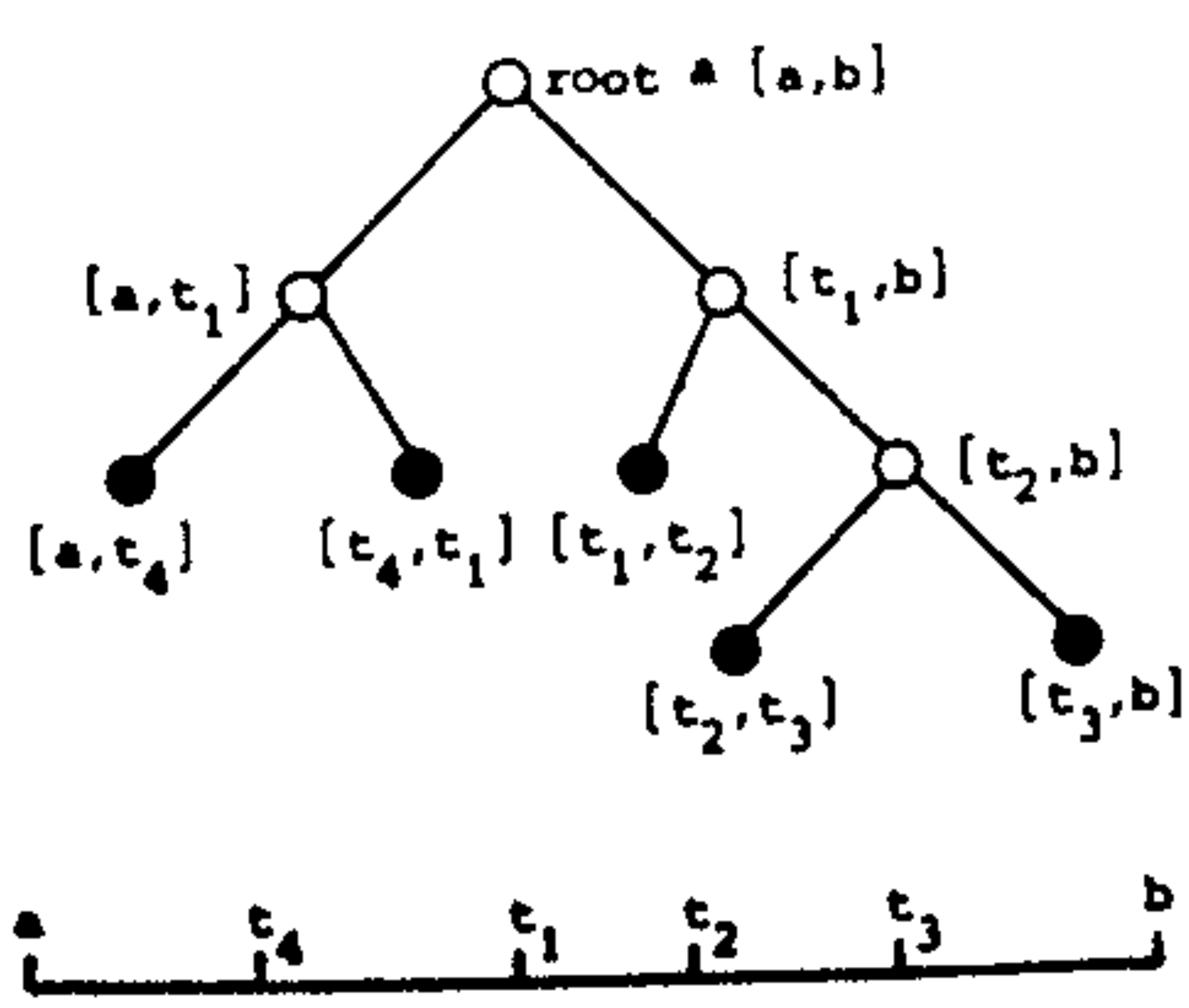

Figure 3. A bisection tree.