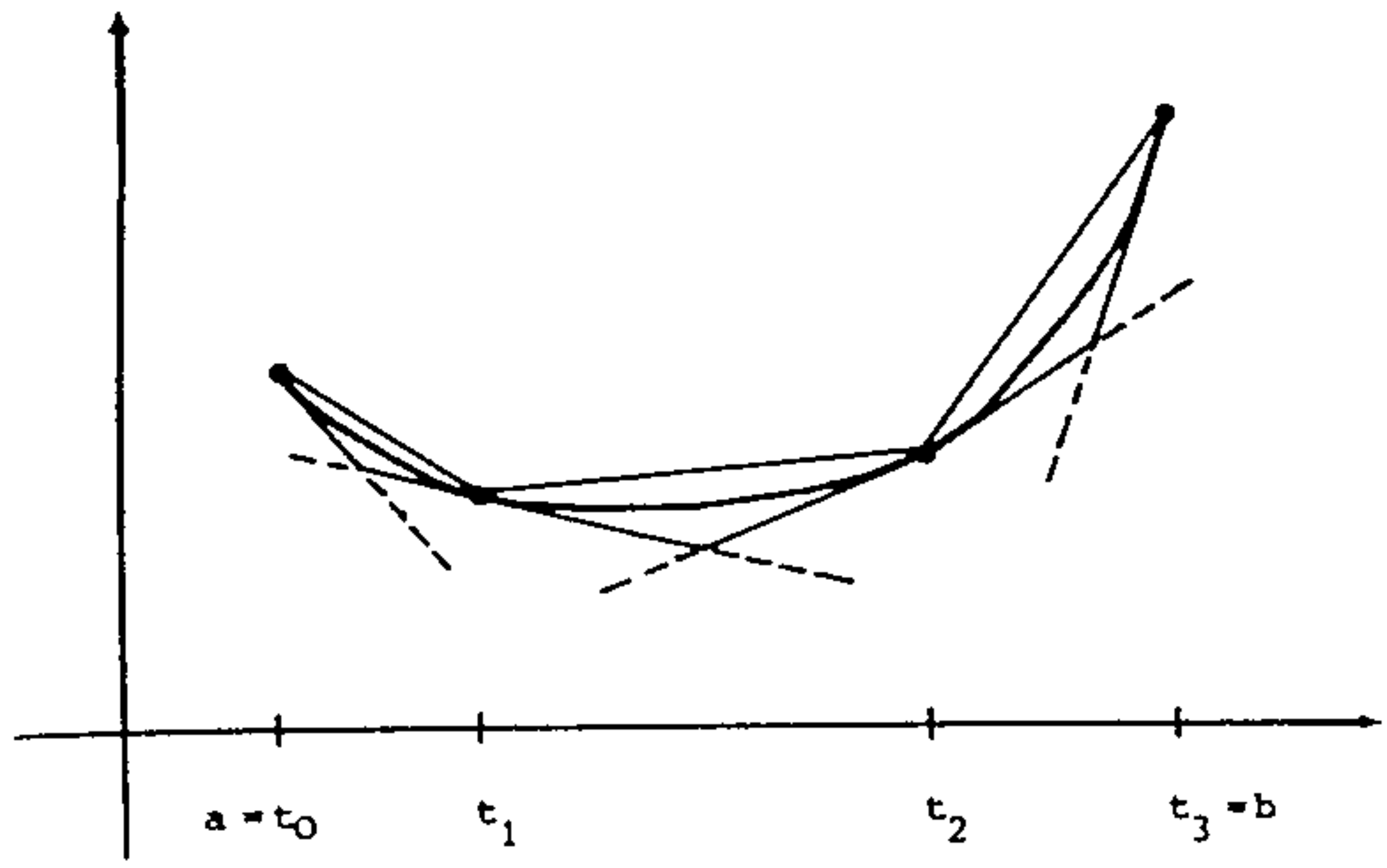

Figure 1. A convex function $h(t)$ with lower approximation $l(t)$ and upper approximation $u(t)$.

It follows from the definitions that

- $l(t)$ and $u(t)$ are piecewise linear and convex
- for all $t \in [a, b]$: $l(t) \leqq h(t) \leqq u(t)$.

In any interval $[t_i, t_{i+1}]$ the maximal difference between $u(t)$ and $l(t)$ is attained at the point t_i^*, defined by

$$h(t_i) + h_i^+(t_i^* - t_i) = h(t_{i+1}) + h_{i+1}^-(t_i^* - t_{i+1}),$$

i.e., t_i^* is uniquely defined, if $h_i^+ < h_{i+1}^-$, namely, by

$$t_i^* = \frac{h(t_{i+1}) - h(t_i) + h_i^+ t_i - h_{i+1}^- t_{i+1}}{h_i^+ - h_{i+1}^-}.$$

If $h_i^+ = h_{i+1}^-$, then $l(t) = h(t) = u(t)$ for all $t_i \leqq t \leqq t_{i+1}$. Therefore the maximal error $E := \max_{a \leq t \leq b}(|h(t) - u(t)|, |h(t) - l(t)|)$ is bounded by

$$\max_{0 \leq i \leq n-1} \{u(t_i^*) - l(t_i^*)\} = \max_{0 \leq i \leq n-1} E_i,$$

where

$$E_i = \left[\frac{h(t_{i+1}) - h(t_i)}{t_{i+1} - t_i} - h_i^+\right] \cdot (t_i^* - t_i). \tag{1}$$

The following lemma (cf. Thakur [20, Lemma 2]) shows a relation between the length of an interval $I = [t_i, t_{i+1}]$, the one-sided derivatives in the endpoints of I and the error $\max_{t \in I}(u(t) - l(t))$.