Pseudotriangulations in Computational Geometry

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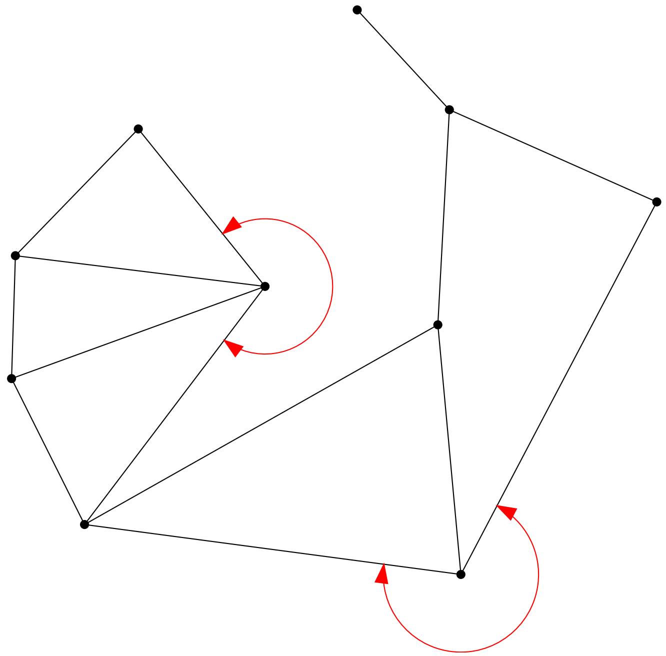
Fall School on Computational Geometry Graduate Program
"Combinatorics, Geometry, and Computing"
October 2–4, 2003, Neustrelitz

Pointed Vertices

A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).



A straight-line graph is pointed if all vertices are pointed.

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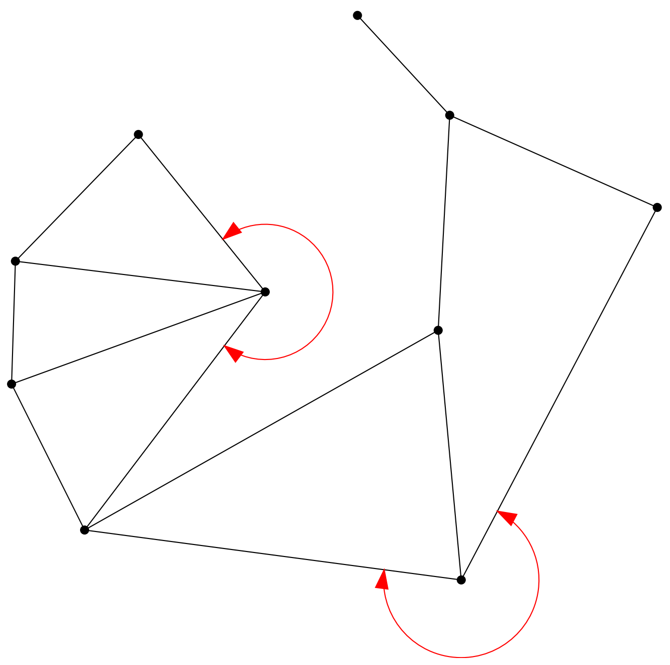
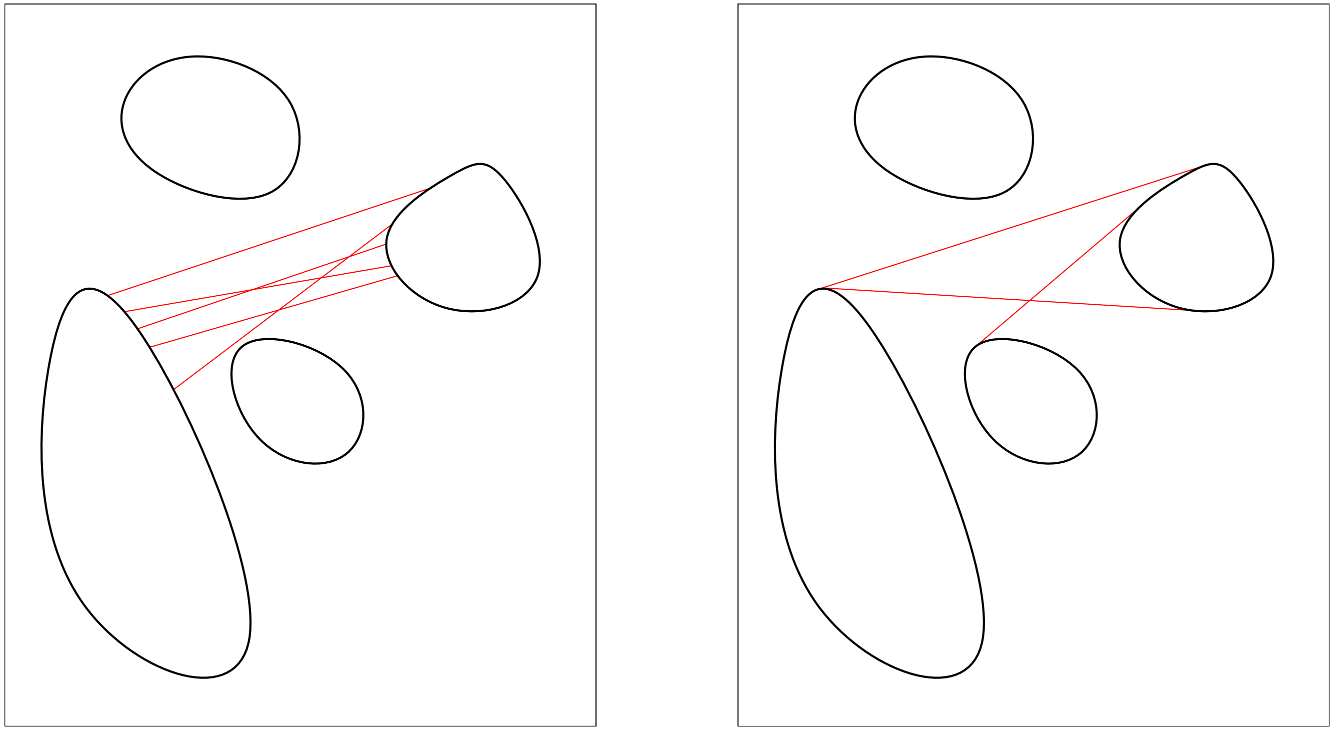


A straight-line graph is pointed if all vertices are pointed.

Where do pointed vertices arise?

Visibility among convex obstacles

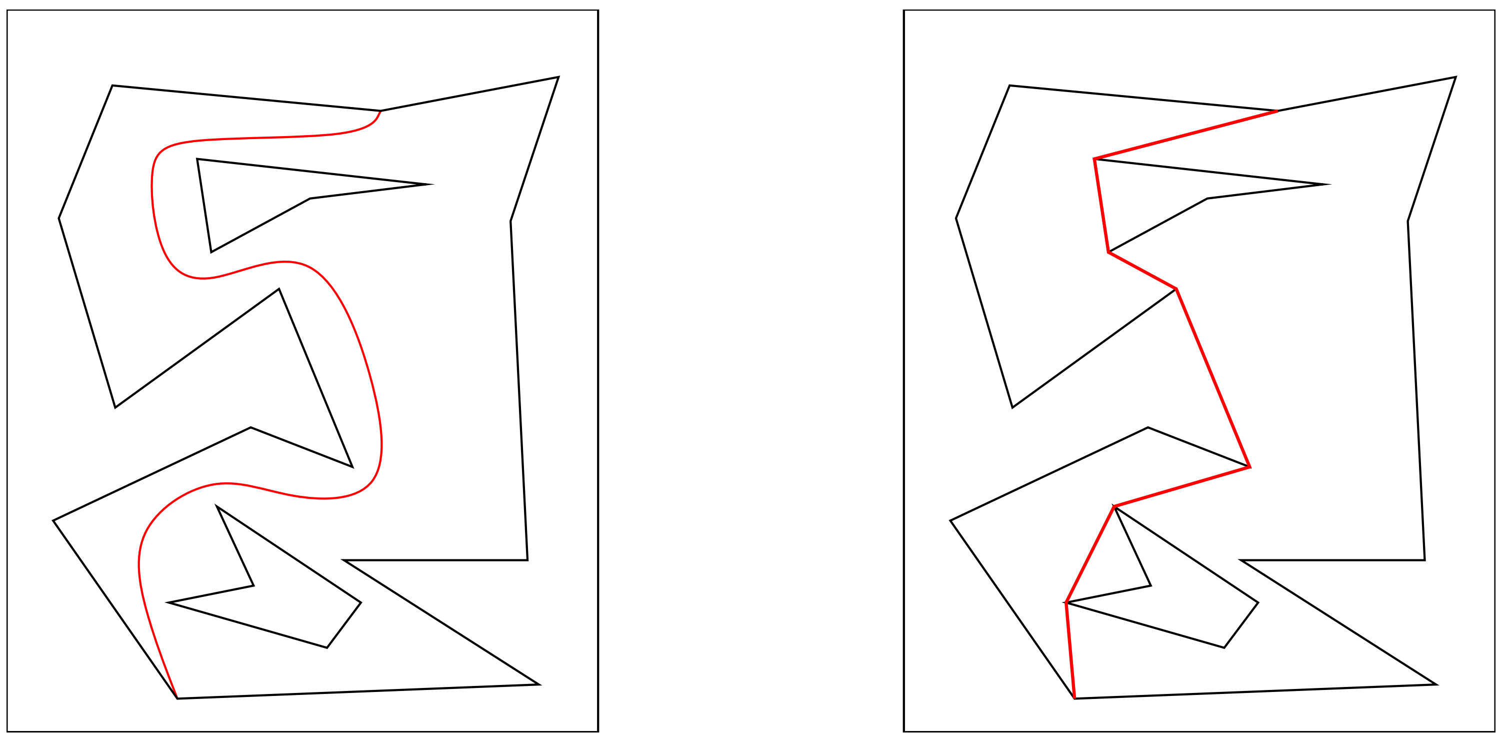
Equivalence classes of *visibility segments*. Extreme segments are *bitangents* of convex obstacles.



[Pocchiola and Vegter 1996]

Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.



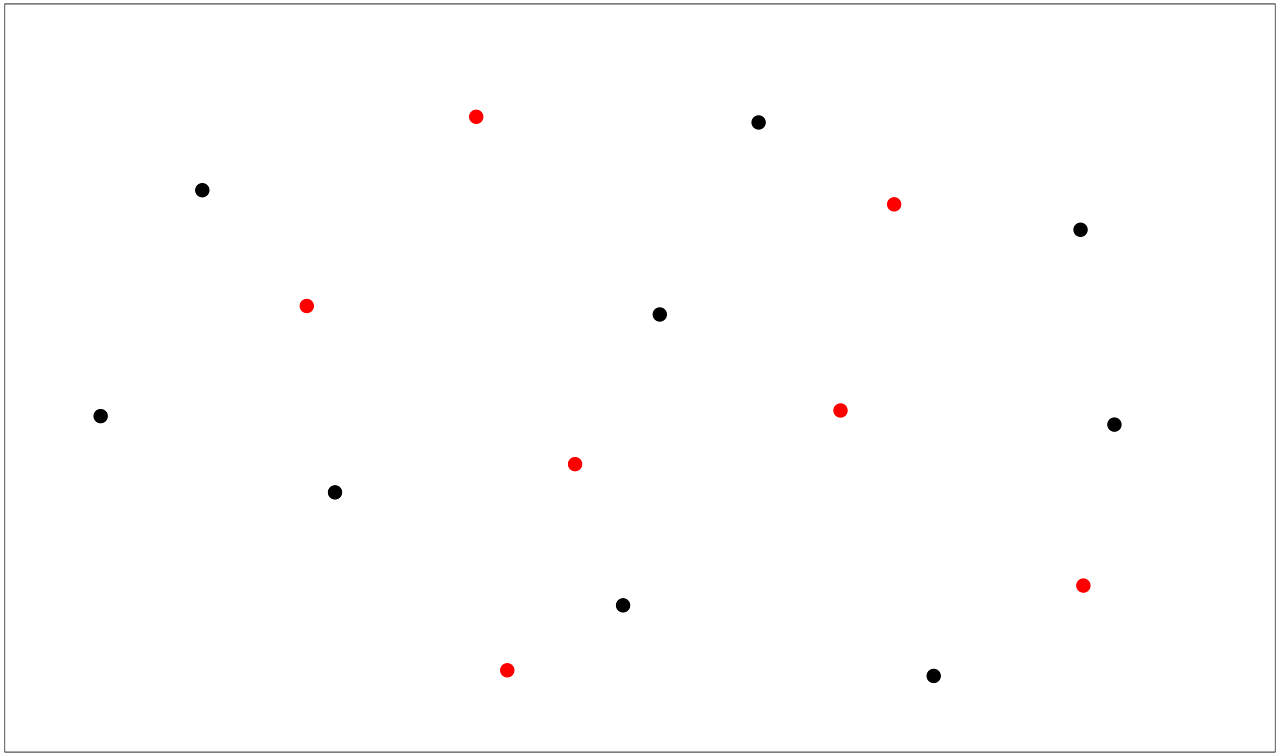
$\longrightarrow$ *geodesic* triangulations of a simple polygon

[Chazelle,Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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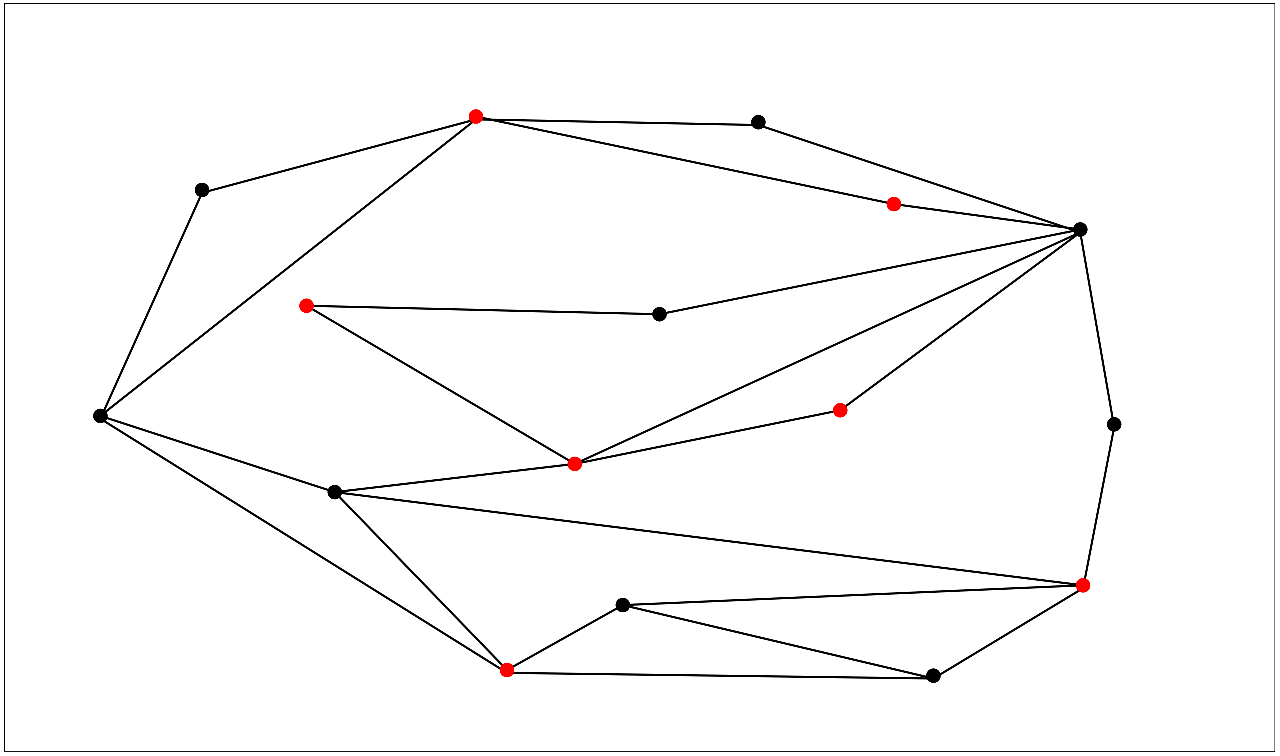

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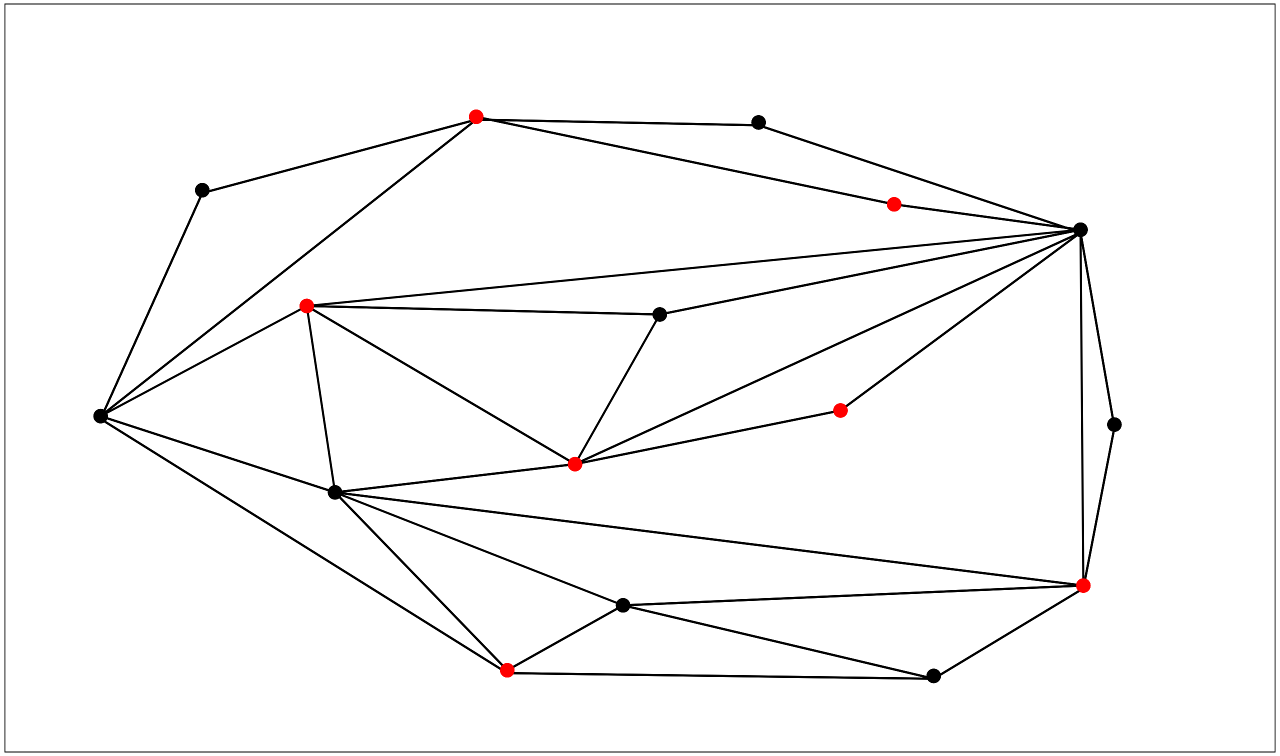

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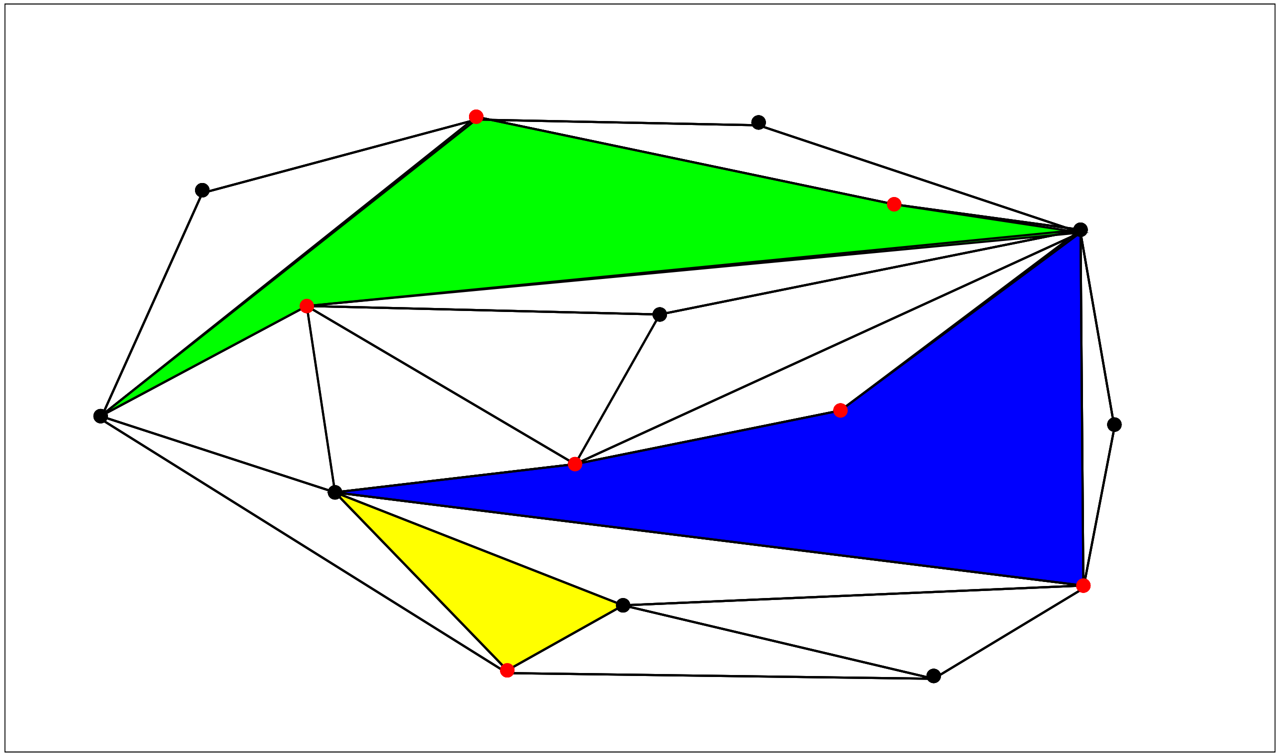
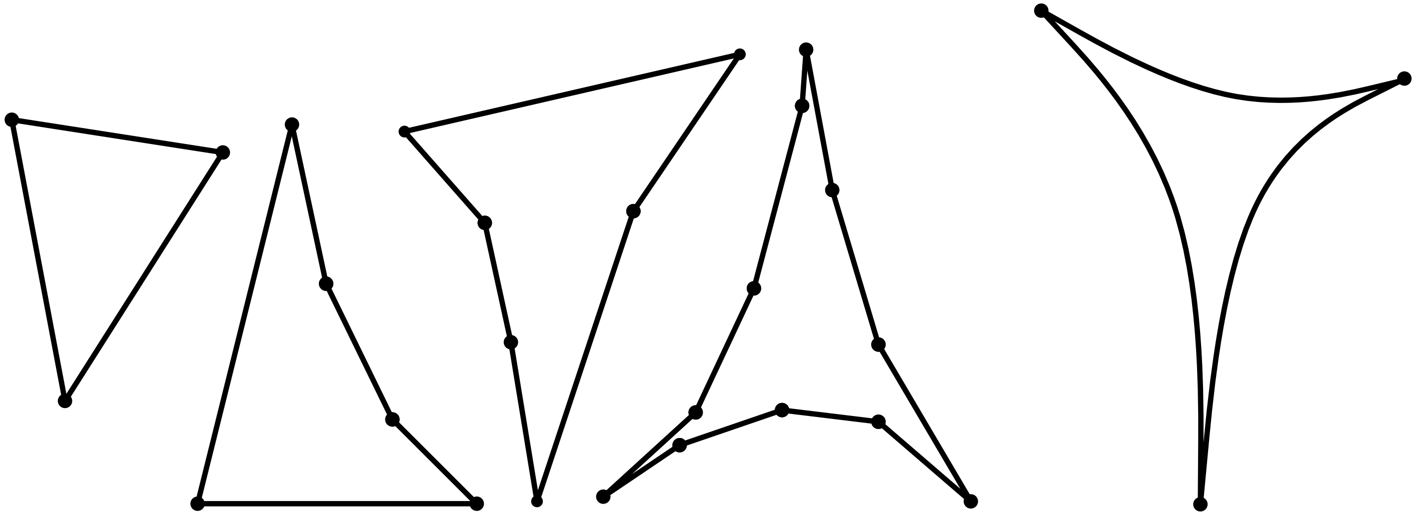


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^\circ$).



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Proof. $(2) \implies (1)$ No edge can be added inside a pseudotriangle without creating a nonpointed vertex.
Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\rightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

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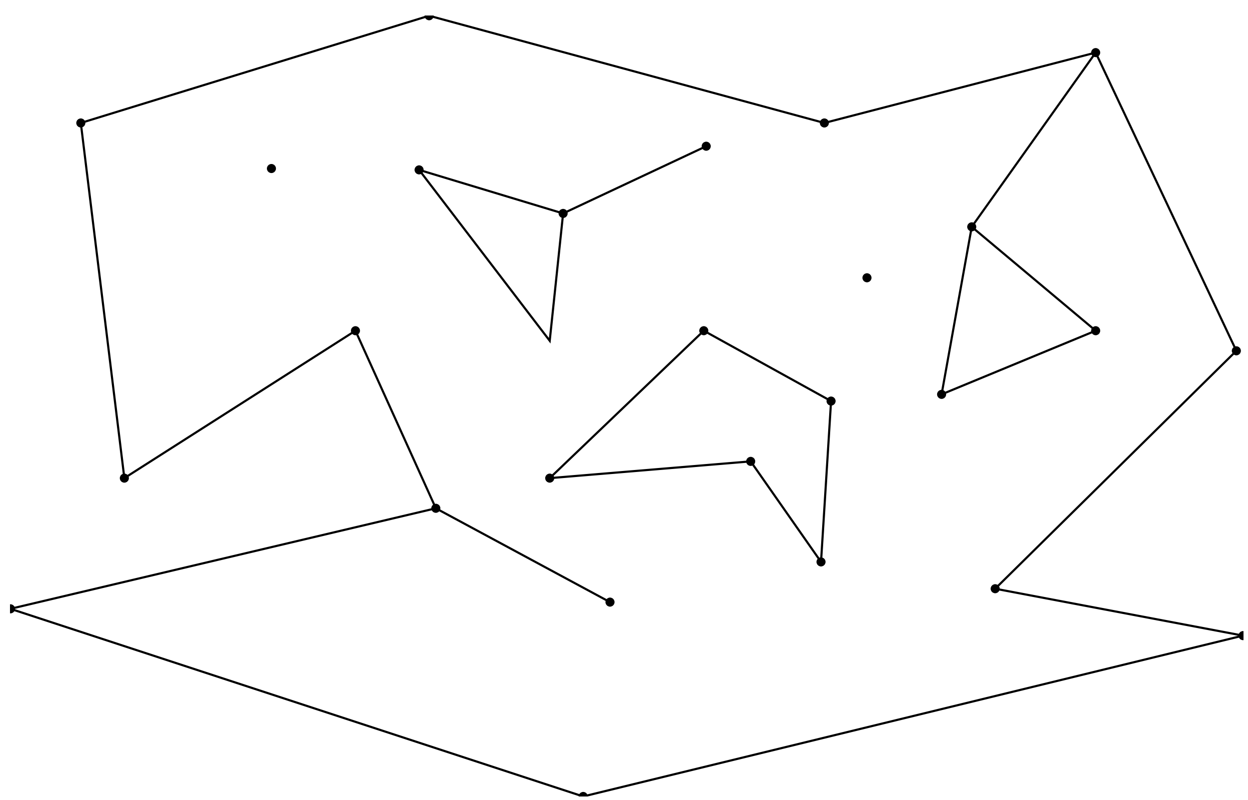
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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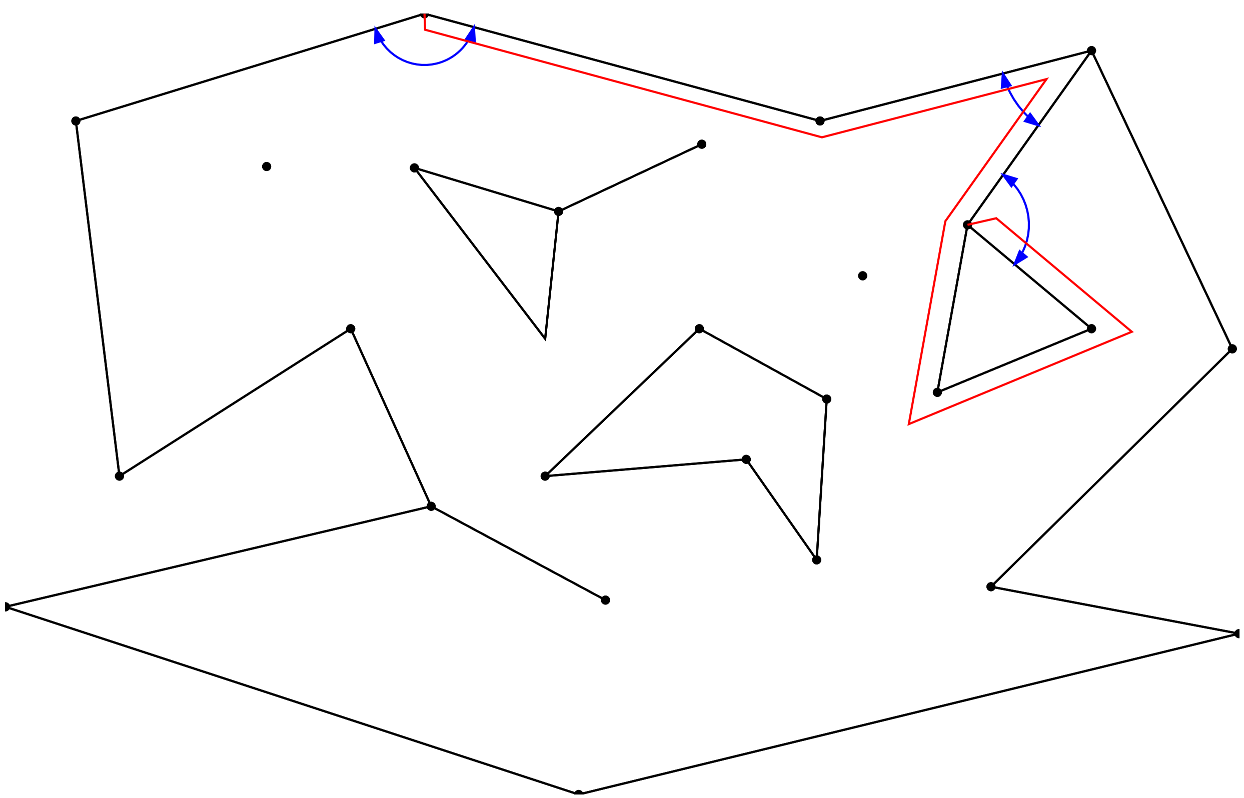

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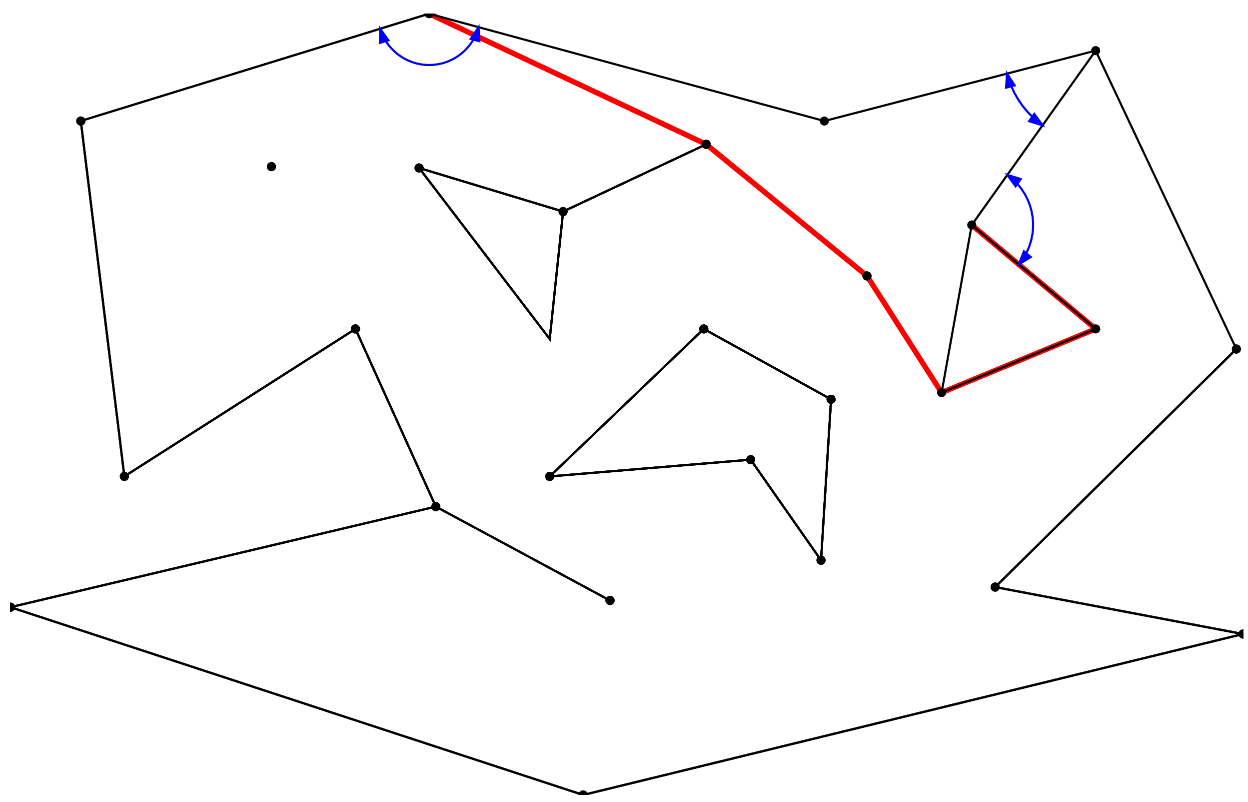
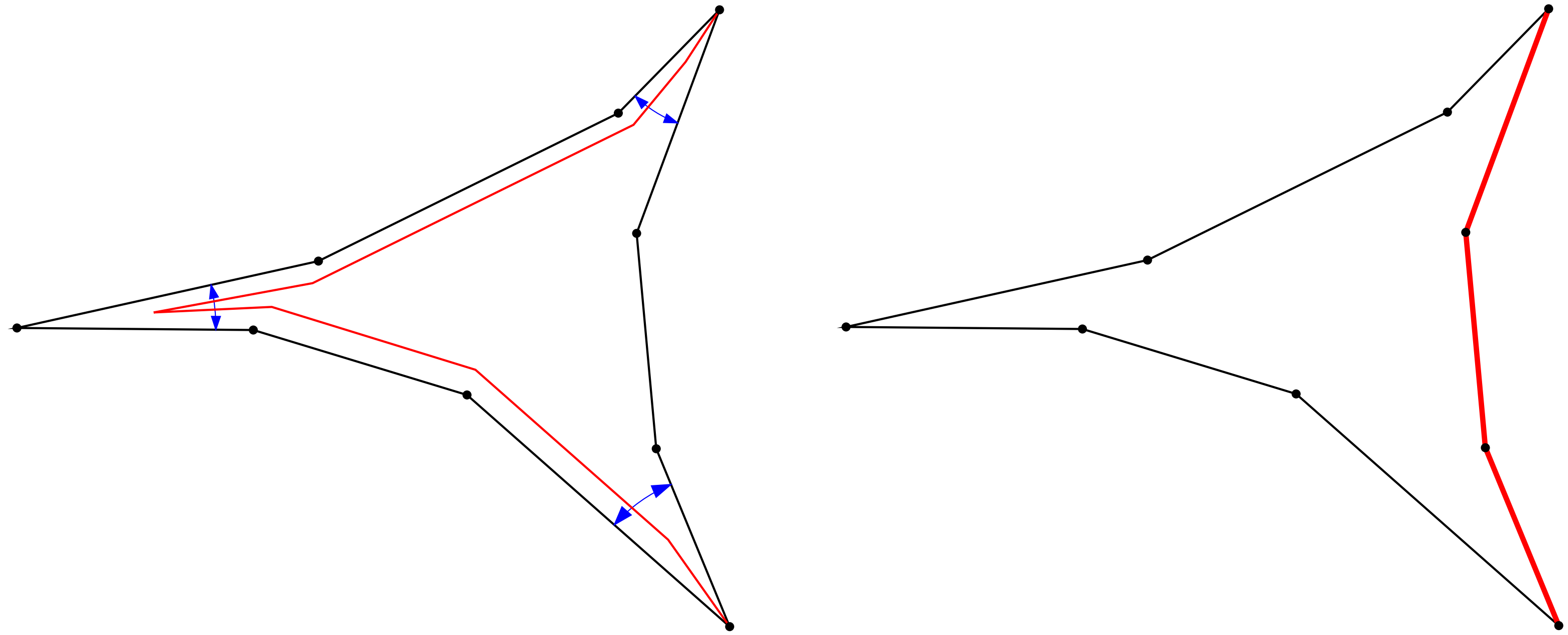


Characterization of pseudotriangulations, continued

A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).

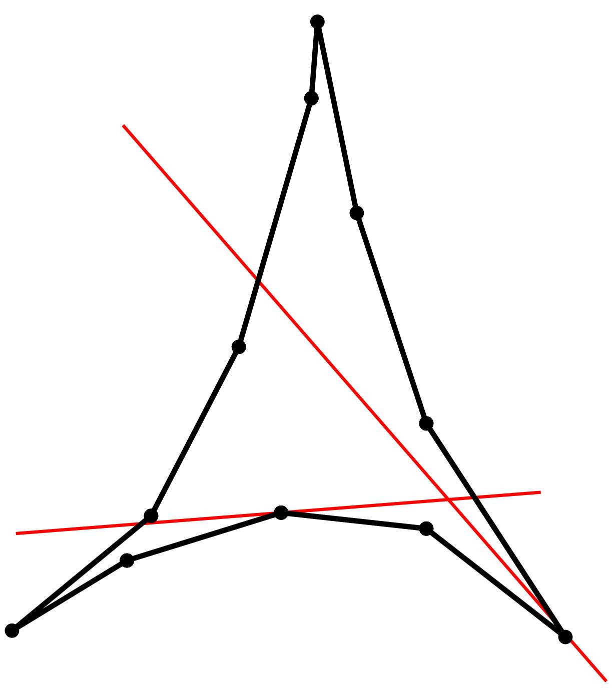


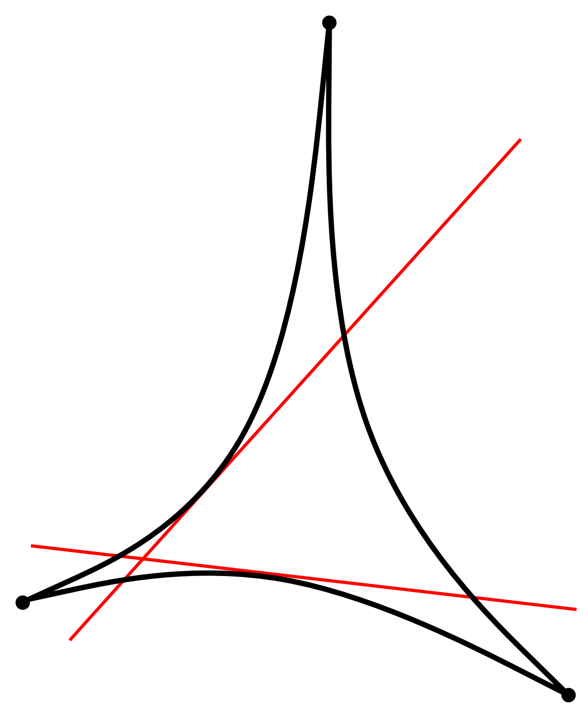
[Rote, C. A. Wang, L. Wang, Xu 2003]

Tangents of pseudotriangles

“Proof. $(2) \implies (1)$ No edge can be added inside a pseudotriangle without creating a nonpointed vertex.”

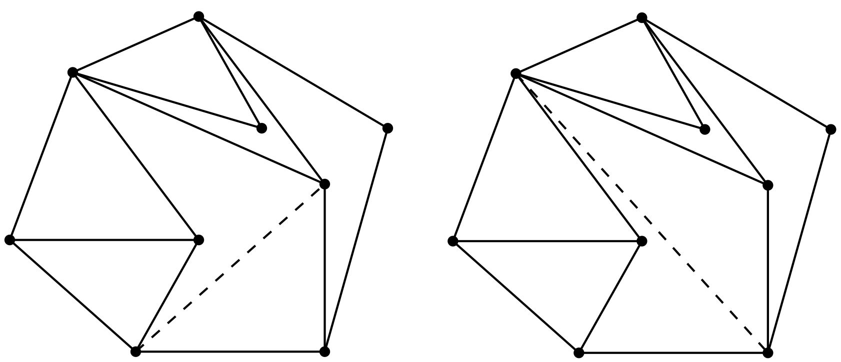
For every direction, there is a unique line which is “tangent” at a reflex vertex or “cuts through” a corner. (See also Exercise 14)





Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique. (See also Exercise 15.)



before after

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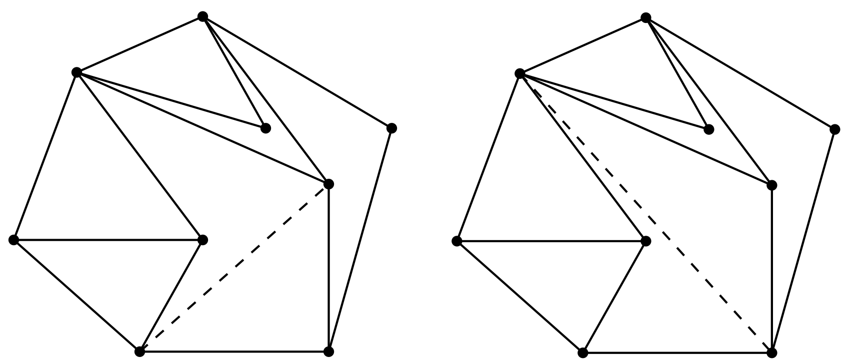

before after

The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

Vertex and face counts

Lemma. *A pseudotriangulation with x nonpointed and y pointed vertices has $e = 3x + 2y - 3$ edges and $2x + y - 2$ pseudotriangles.*

Corollary. *A pointed pseudotriangulation with n vertices has $e = 2n - 3$ edges and $n - 2$ pseudotriangles.*

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$$\underbrace{\sum_t k_t + k_{\text{outer}}}_{2e} - 3|T| = y$$

$$e + 2 = (|T| + 1) + (x + y) \qquad \text{(Euler)}$$

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Corollary. *A pointed graph with* $n \geq 2$ *vertices has at most* $2n - 3$ *edges.*

Pseudotriangulations/ Geodesic Triangulations

Applications:

- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002] (see Exercise 3)
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

2. Pseudotriangulations and Motions
Unfolding of polygons

Theorem. *Every polygonal arc in the plane can be brought into straight position, without self-overlap.*

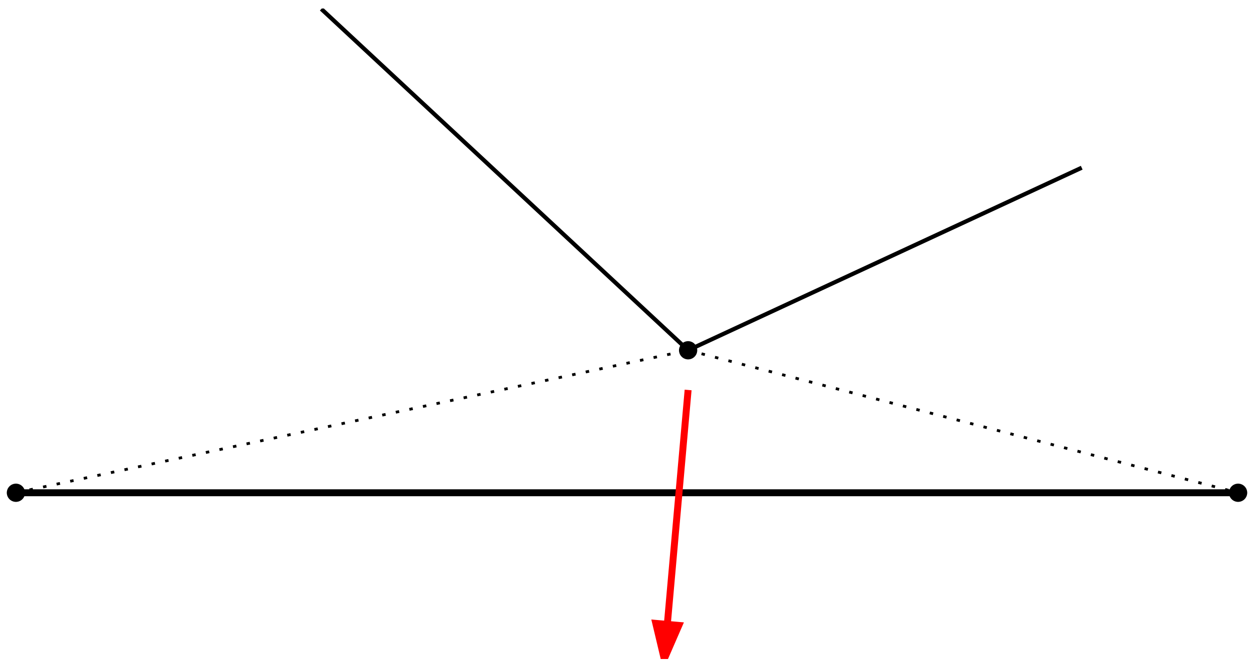
Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2001], [Streinu 2001]

Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot overlap.



Expansive Mechanisms

A *framework* is a set of movable joints (vertices) connected by rigid bars (edges) of fixed length.

Pseudotriangulations with one convex hull edge removed are *expansive mechanisms*: The have one degree of freedom, and their motion is expansive.

Rigid frameworks

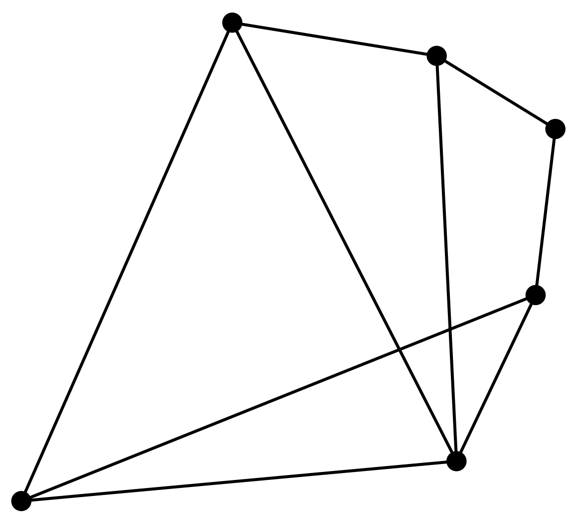
A framework is *rigid* if it allows only translations and rotations of the framework as a whole.

Rigidity is (apart from “exceptional” embeddings) a combinatorial property of the graph: *generic rigidity*.

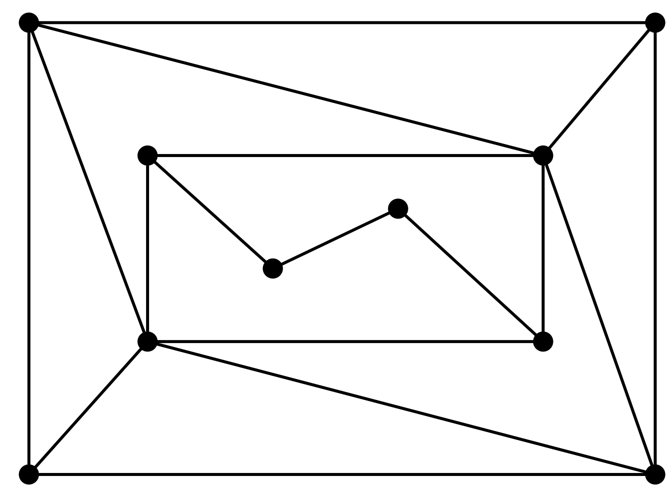
Minimally rigid frameworks

A graph with n vertices is *minimally rigid* in the plane (with respect to $\subseteq$) iff it has the *Laman property*:

- It has $2n-3$ edges.
- Every subset of $k \geq 2$ vertices spans at most $2k-3$ edges.



$n = 6,\ e = 9$



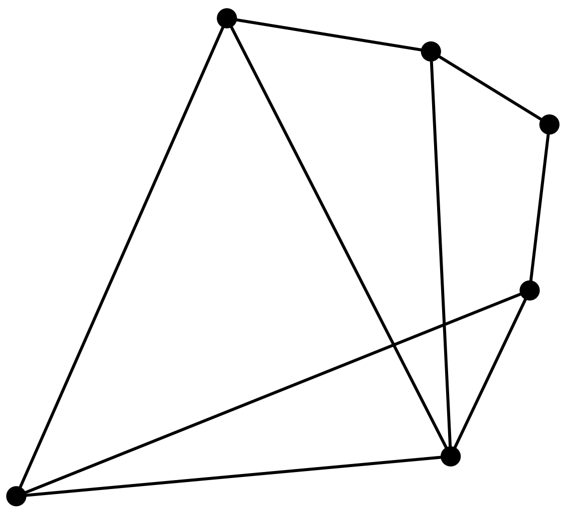
$n = 10,\ e = 17$

[Laman 1961]

Pointed pseudotriangulations are Laman graphs

Theorem. [Streinu 2001] *Every pointed pseudotriangulation has the* Laman property*:*

It has $2n-3$ edges.
Every subset of $k \geq 2$ vertices spans at most $2k-3$ edges.



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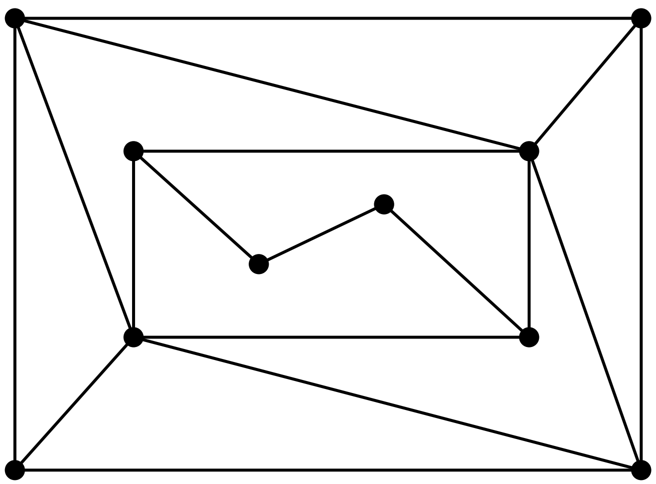

$n = 10,\ e = 17$

Proof: Every subgraph is pointed.

The Laman condition

The Laman property:

- It has $2n-3$ edges.

- Every subset S of $k \geq 2$ vertices spans at most $2k-3$ edges.

The second condition can be rephrased:

- Every subset $\bar{S}$ of $k \leq n-2$ vertices is incident to at least $2k$ edges.

Every planar Laman graph is a pointed pseudotriangulation

Theorem. *Every pointed pseudotriangulation is a Laman graph.*

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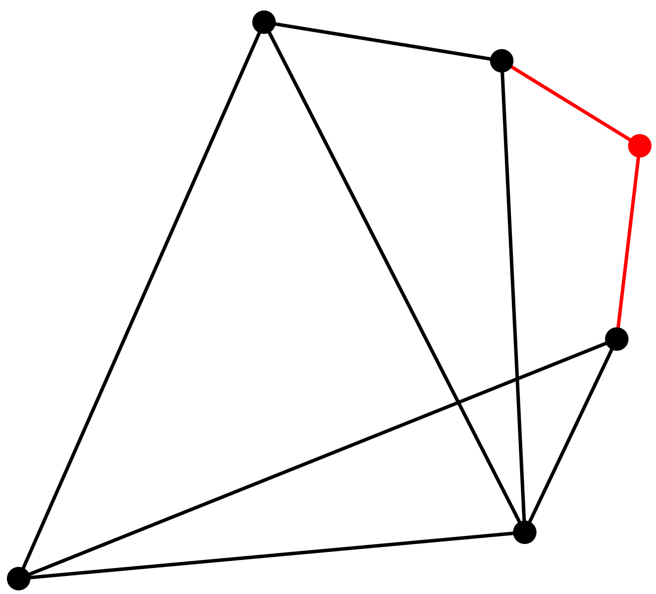
Theorem. *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

Proof I: Induction, using *Henneberg constructions*
Proof II: via Tutte embeddings for directed graphs
[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

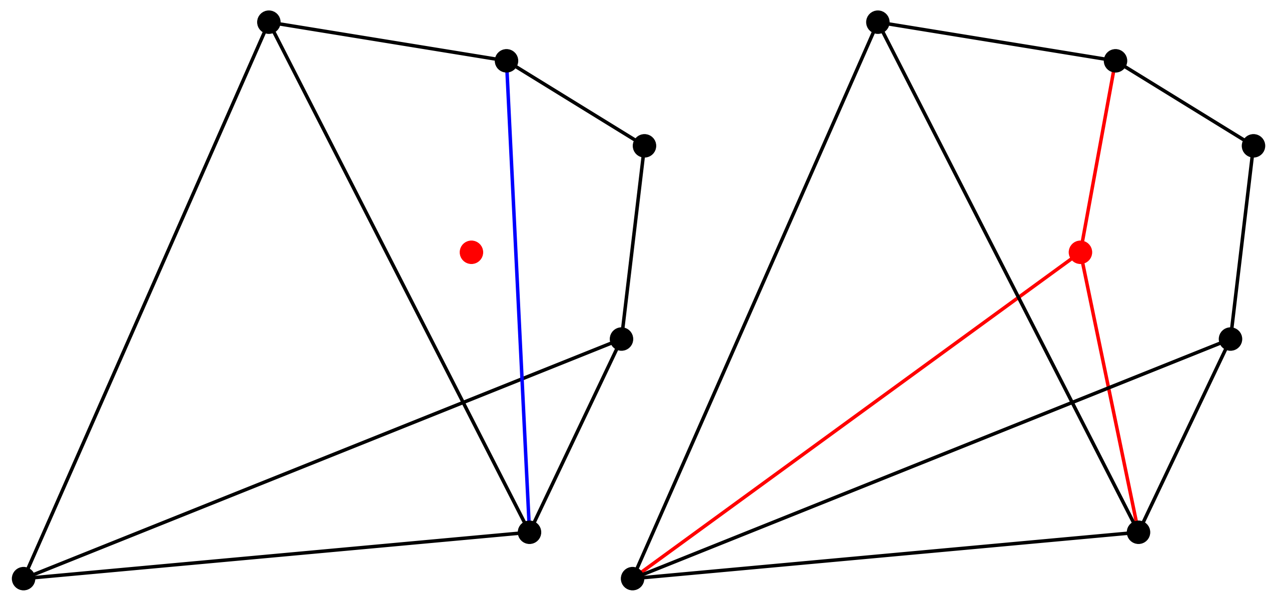
Theorem. *Every rigid planar graph has a realization as a pseudotriangulation.*

[Orden, Santos, B. Servatius, H. Servatius 2003]

Henneberg constructions

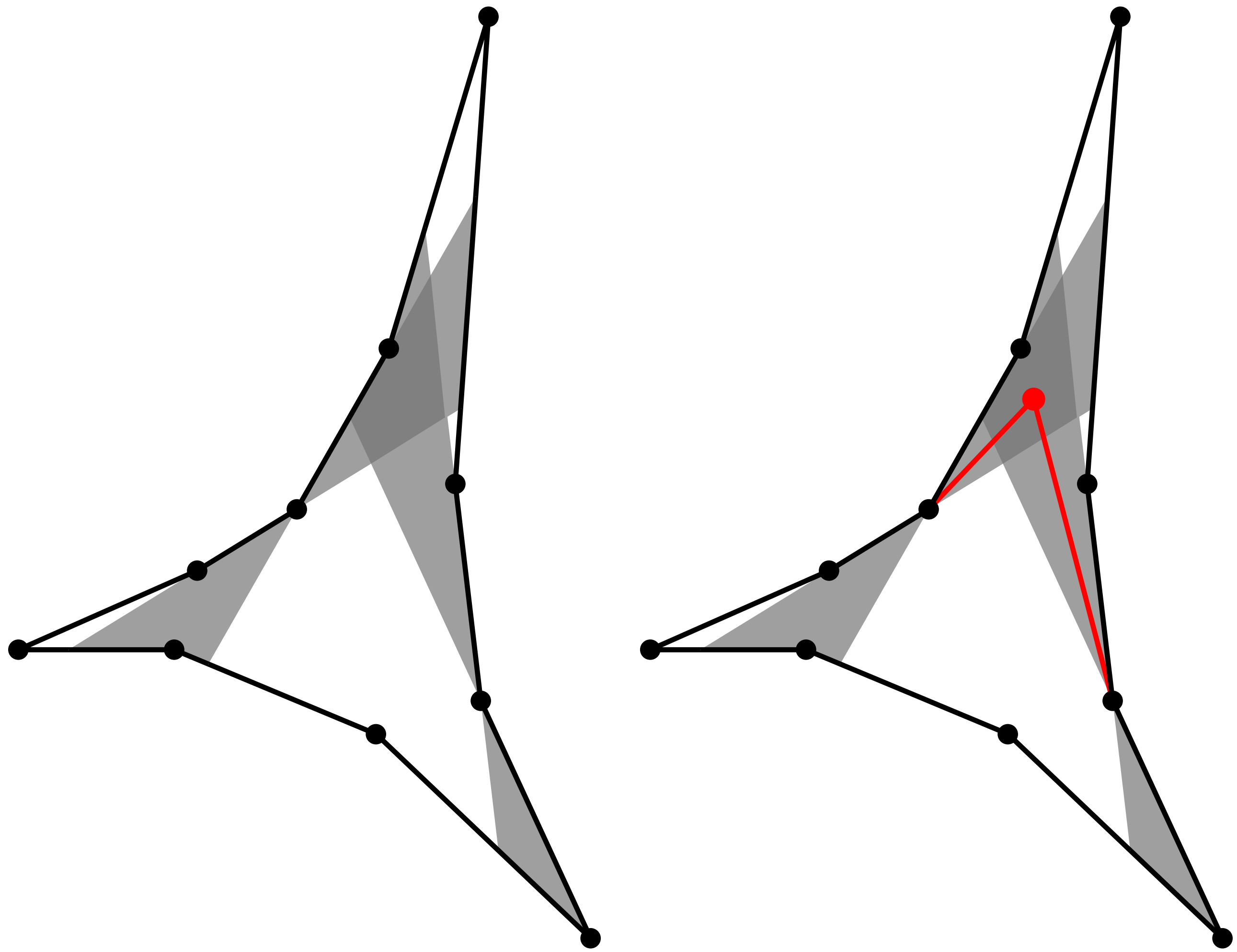


Type I



Type II

Proof I: Henneberg constructions



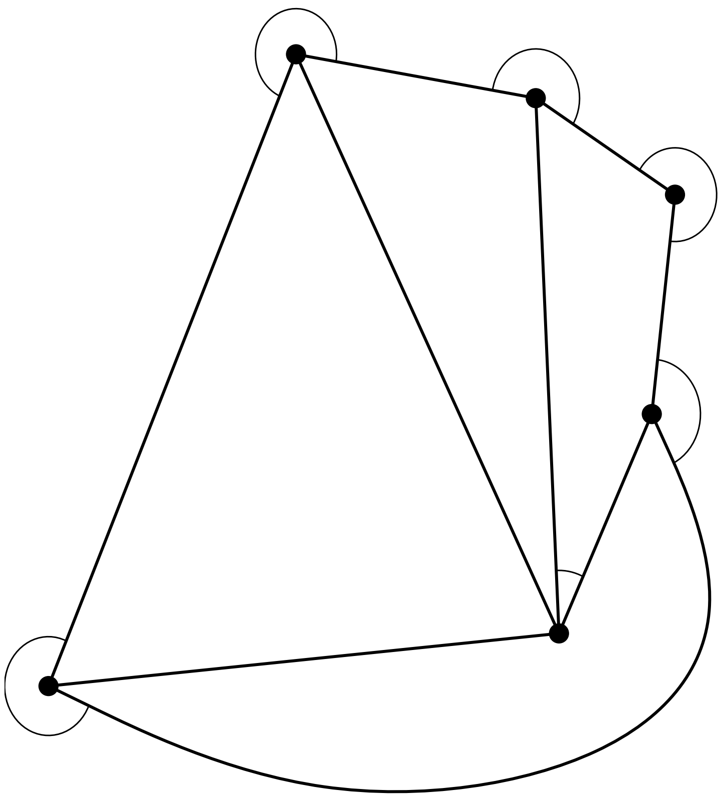
Proof II: embedding Laman graphs via directed Tutte embeddings

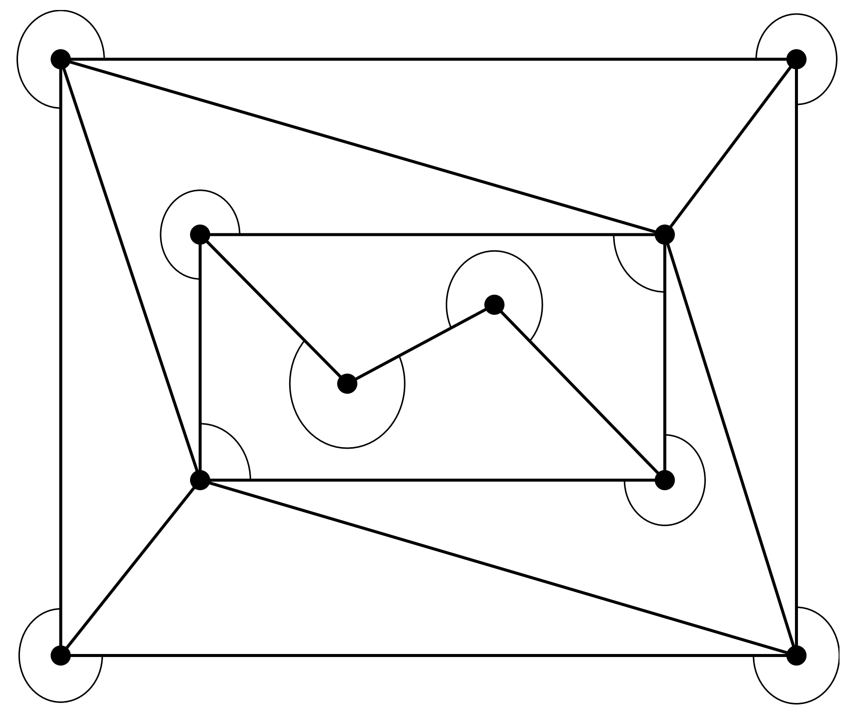
Step 1: Find a *combinatorial pseudotriangulation* (CPT):
Mark every angle of the embedding either as *small* or *big*.

- Every interior face has 3 small angles.
- The outer face has no small angles.
- Every vertex is incident to one big angle.

Step 2: Find a geometric realization of the CPT.

Combinatorial pseudotriangulations





Step 1: Find a combinatorial pseudotriangulation

Bipartite network flow model:
sources = vertices: supply $= 1$.
sinks = faces: demand $= k - 3$ for a k-sided face
arcs = angles: capacity 1. flow=1 $\iff$ angle is big.

Prove that the max-flow min-cut condition is satisfied.

Step 2—Tutte's barycenter method

Fix the vertices of the outer face in convex position. Every interior vertex p_i should lie at the barycenter of its neighbors.

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every vertex } i$$

$\omega_{ij} \geq 0$, but ω need not be symmetric.

Theorem. *If every interior vertex has three vertex disjoint paths to the outer boundary, using arcs with $\omega_{ij} > 0$, the solution is a planar embedding.*

[Tutte 1961, 1964], [Floater and Gotsman 1999],
[Colin de Verdière, Pocchiola, Vegter 2003]

Tutte's barycenter method for 3-connected planar graphs

Theorem. *Every 3-connected planar graph G has a planar straight-line embedding with convex faces. The outer face and the convex shape of the outer face can be chosen arbitrarily.*

Tutte used *symmetric* $\omega_{ij} = \omega_{ji} > 0$.

$\rightarrow$ animation of spider-web embedding (requires Cinderella 2.0 software)

Good embeddings

Consider a directed subgraph of G. A *good* embedding is a set of positions for the vertices with the following properties:

1. The vertices of the outer face form a strictly convex polygon.

2. Every other vertex lies in the relative interior of the convex hull of its out-neighbors.

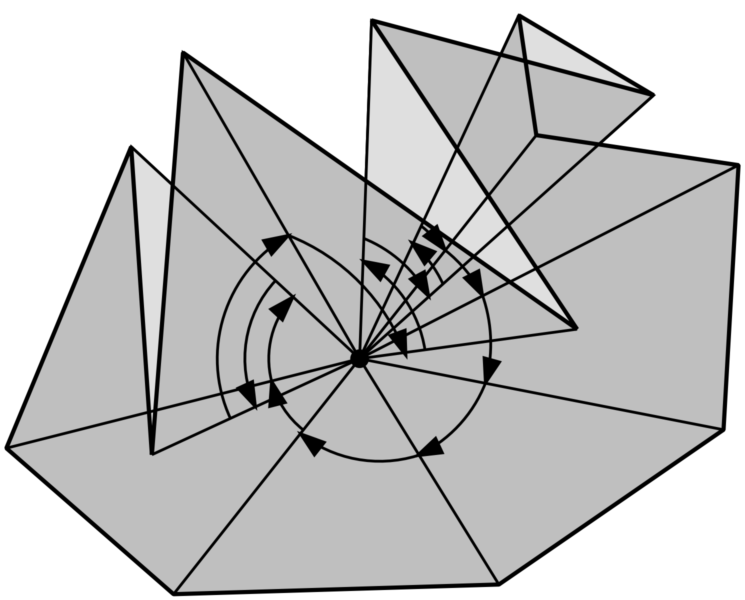
3. No vertex v is degenerate, in the sense that all out-neighbors lie on a line through v.

Lemma. *A good embedding gives rise to a planar straight-line embedding with strictly convex faces.*

Good embeddings are good

Lemma. *A good embedding is non-crossing.*

Proof: Assume that interior faces of G are triangles. (Add edges with $\omega_{ij} = 0$.)
Total angle at b boundary vertices: $\geq (b-2)\pi$.
Total angle around interior vertices: $\geq (n-b) \times 2\pi$.
$2n-b-2$ triangles generate an angle sum of $(2n-b-2)\pi$.



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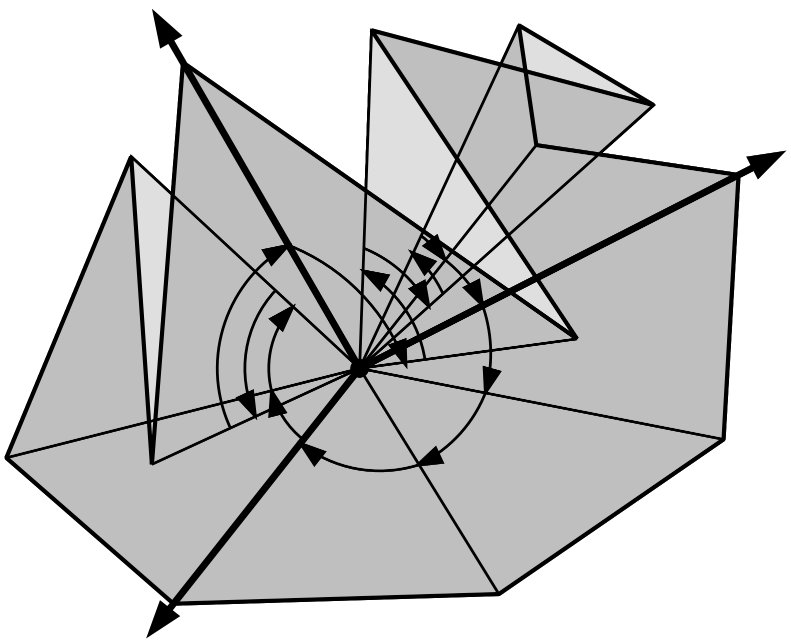

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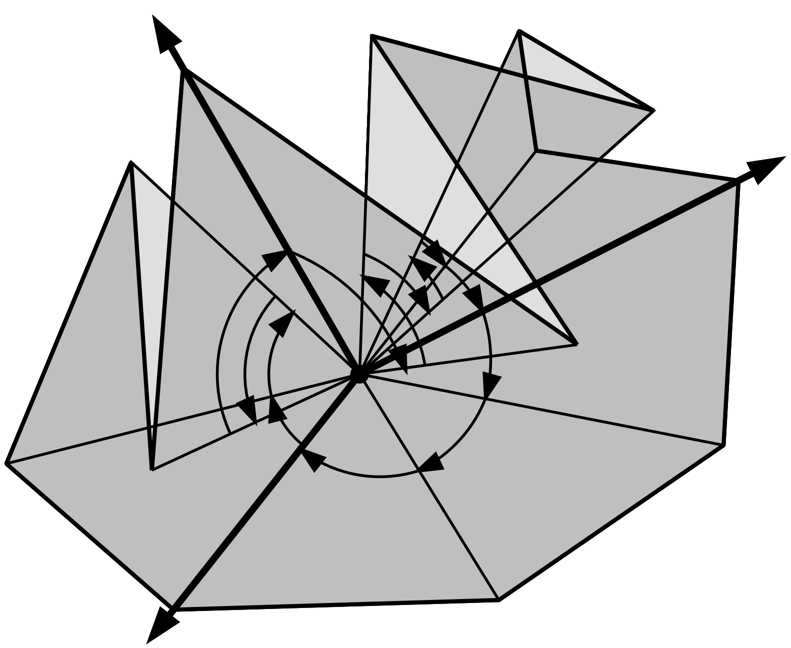
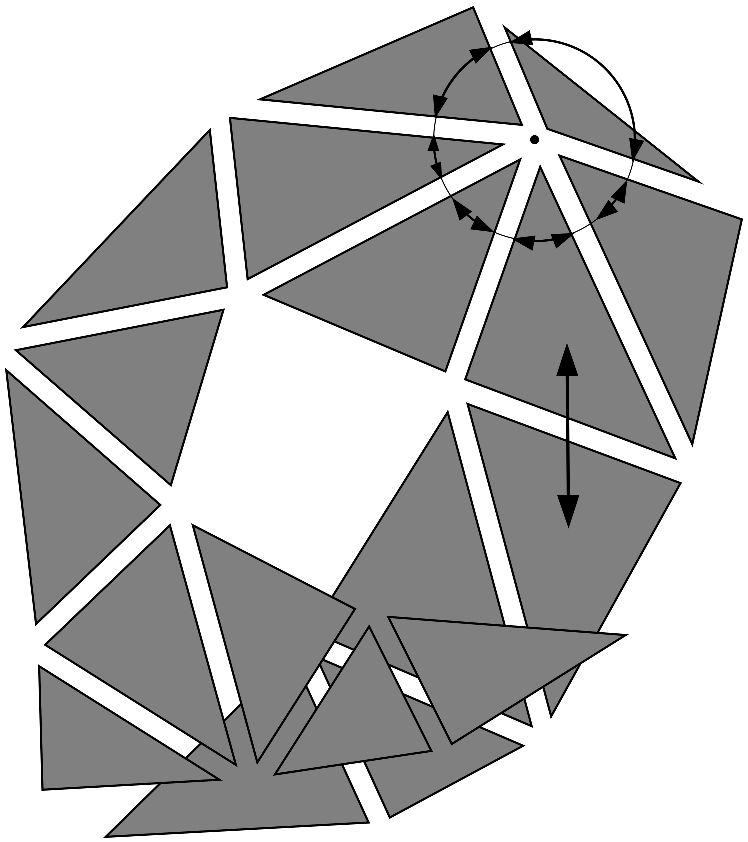


$\rightarrow$ all triangles must be oriented consistently.

Good embeddings are good

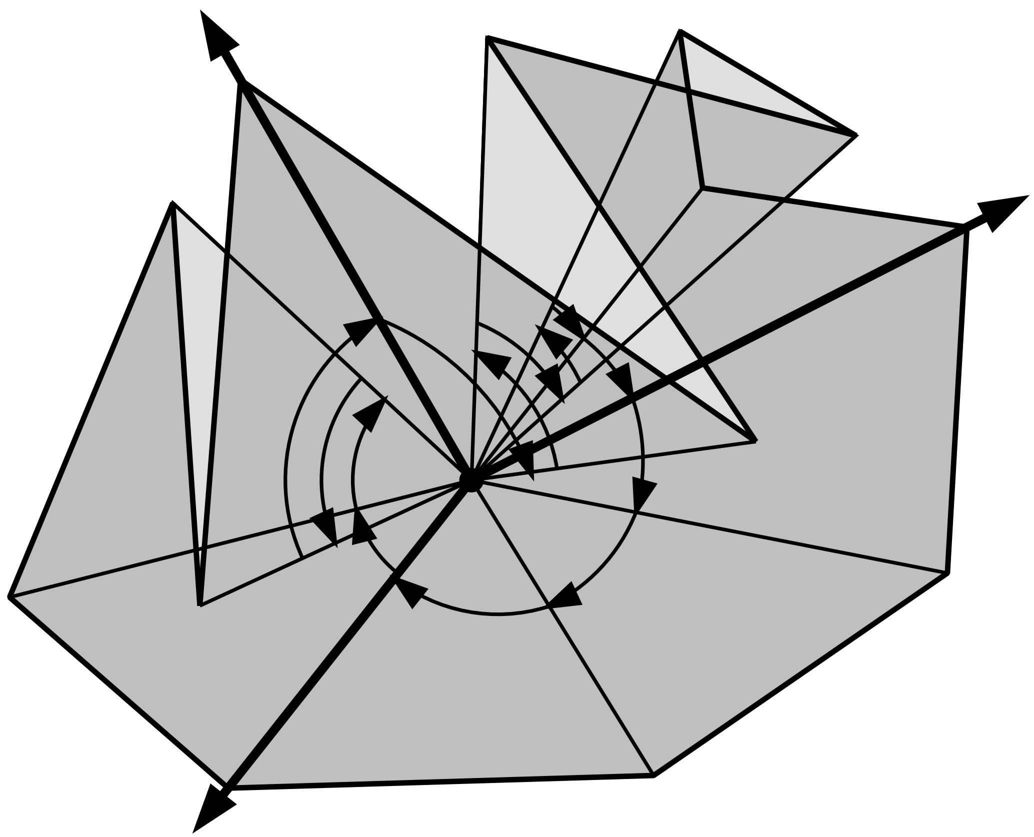
Triangles fit together locally.

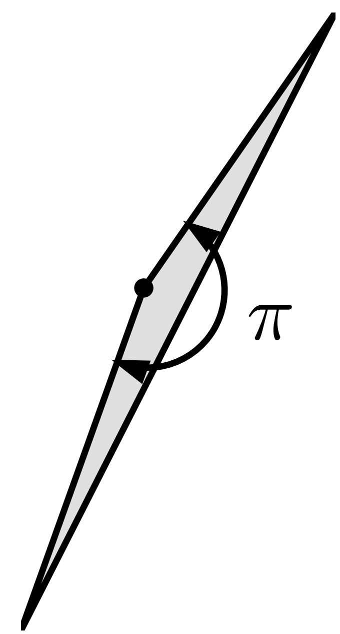


equal covering number on both sides of every edge.

Good embeddings are good

There is no space for triangles with 180° angles.





Equilibrium implies good embedding

The system

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every interior vertex } i \qquad (*)$$

has a unique solution. (Exercise 16)

We have to show that the solution gives rise to a good embedding. The out-neighbors of a vertex i in the directed subgraph are the vertices j with $\omega_{ij} > 0$.

Equilibrium implies good embedding

1. The vertices of the outer face form a convex polygon.
2. Every other vertex lies in the relative interior of the convex hull of its out-neighbors.
3. No vertex p_i is degenerate, in the sense that all out-neighbors p_j lie on a line through p_j.

We have (i) by construction. (ii) follows directly from the system

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every interior vertex } i \qquad (*)$$

We need 3-connectedness and planarity for (iii).

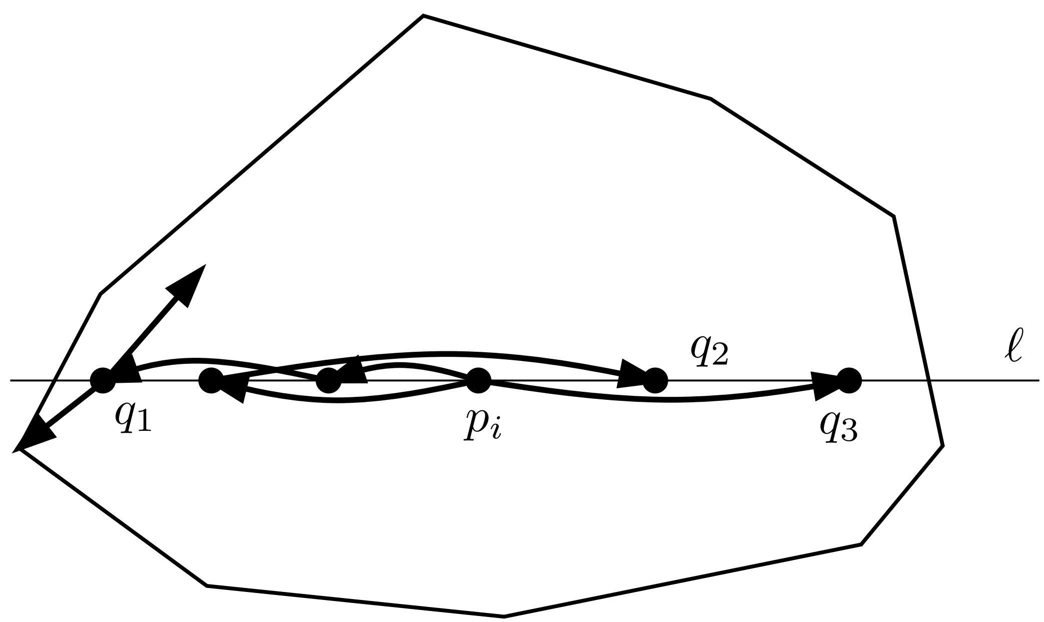
The equilibrium embedding is nondegenerate

Assume that all neighbors of p_i lie on a horizontal line ℓ.

We have 3 *vertex-disjoint paths* from i to the boundary.

$q_1, q_2, q_3 =$ last vertex on each path that lies on ℓ.

By *equilibrium*, q_k must have a neighbor above ℓ and below ℓ.



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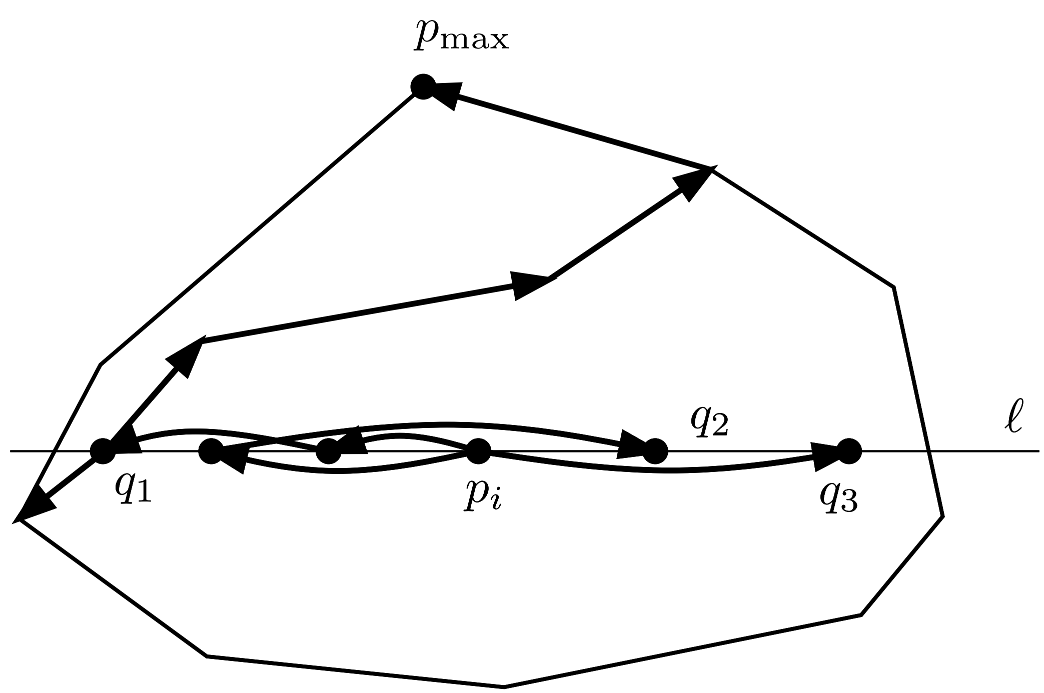
Continue upwards to the boundary and along the boundary to the highest vertex $p_{\max}$



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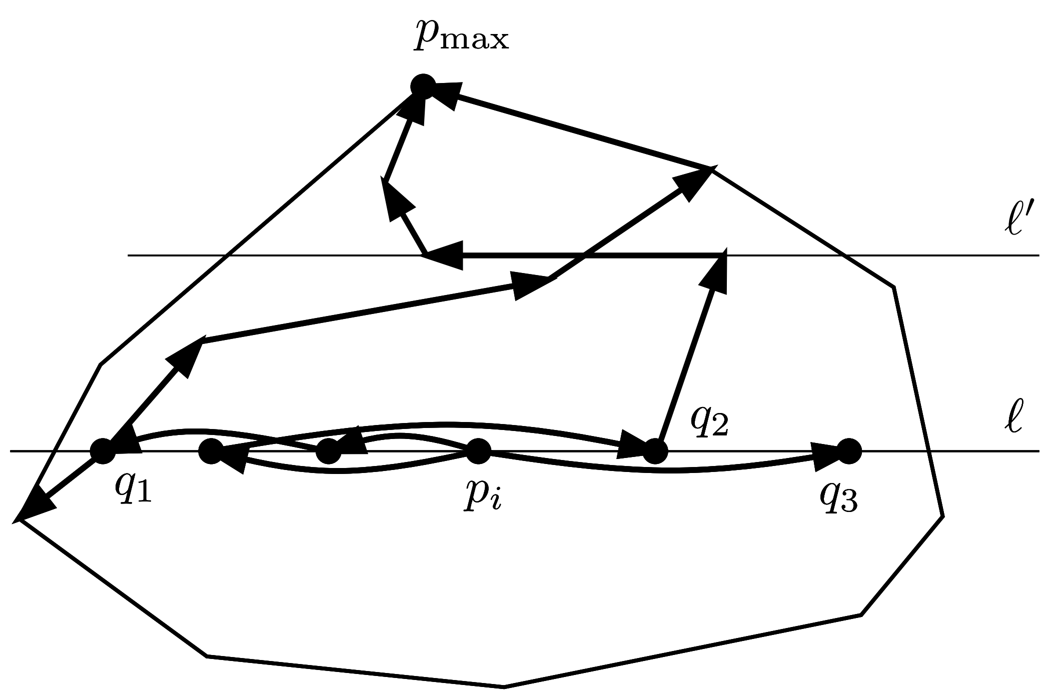
We have 3 *vertex-disjoint paths* from i to the boundary.
$q_1, q_2, q_3 =$ last vertex on each path that lies on ℓ.
By *equilibrium*, q_k must have a neighbor above ℓ and below ℓ.
Continue upwards to the boundary and along the boundary to the highest vertex $p_{\max}$



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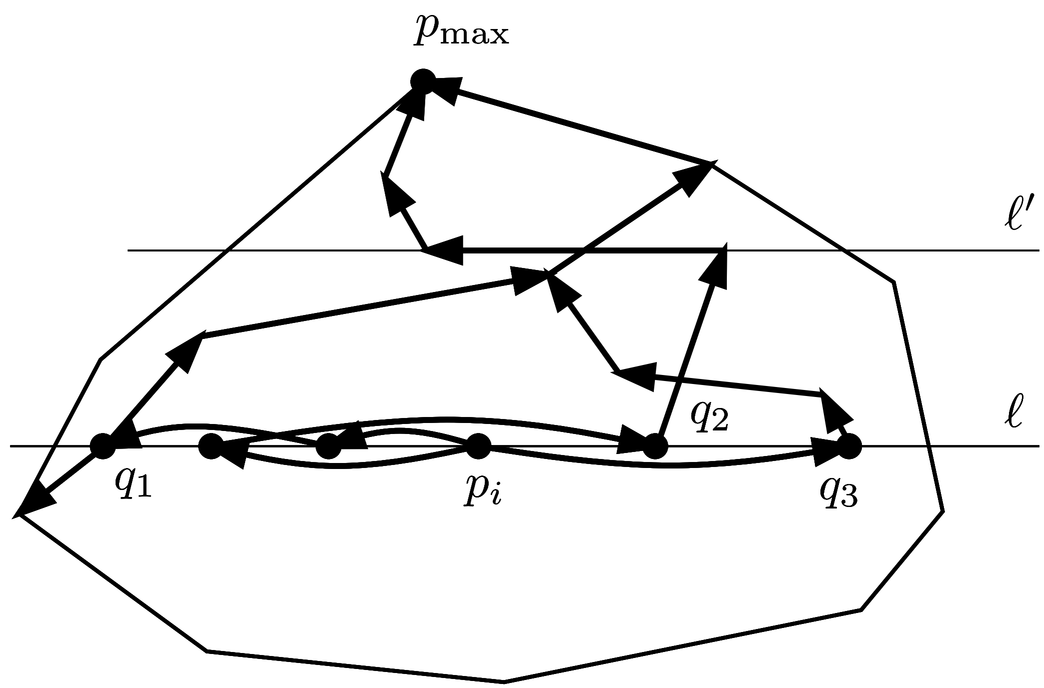

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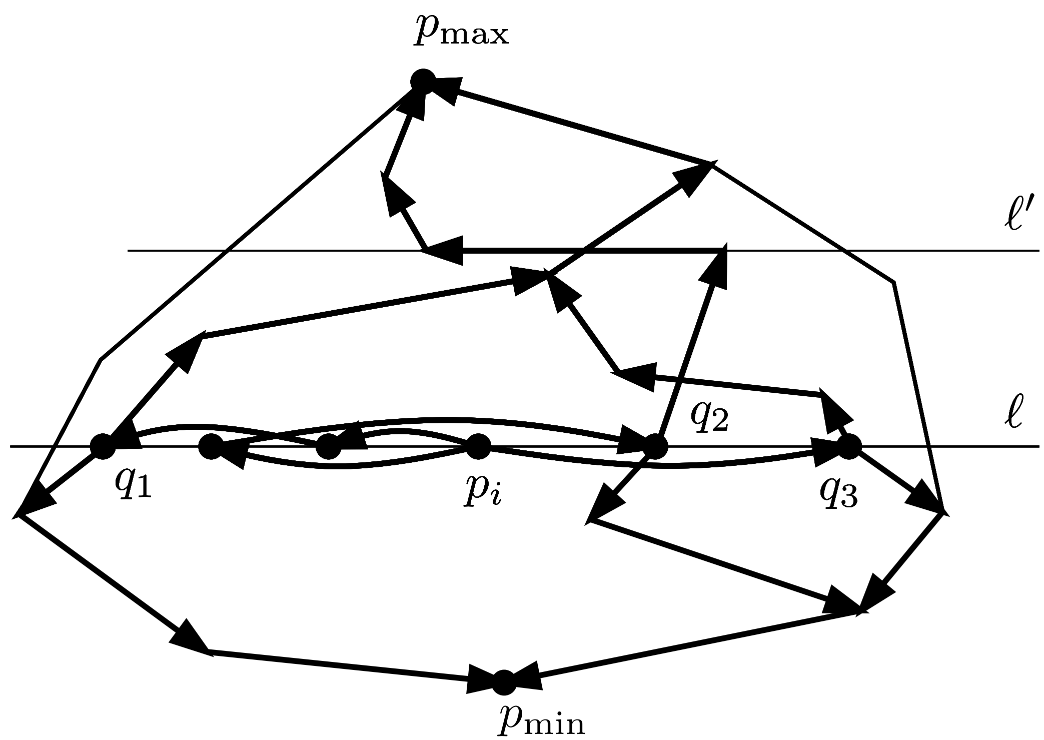
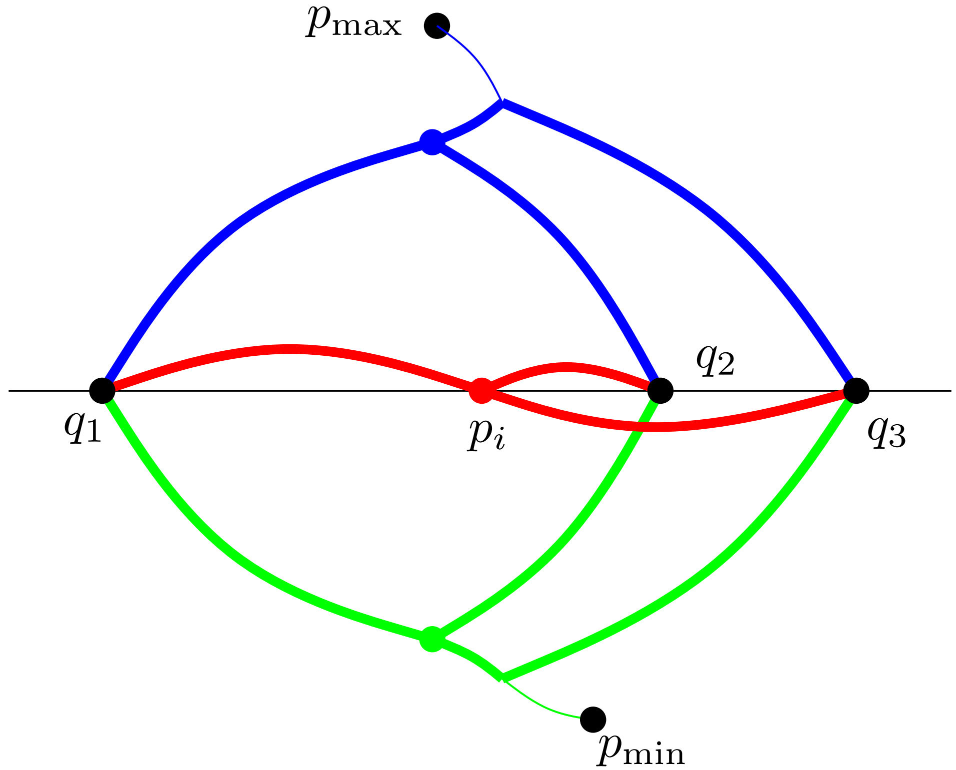
By *equilibrium*, q_k must have a neighbor above ℓ and below ℓ.
Continue upwards to the boundary and along the boundary to the highest vertex $p_{\max}$, and similarly to the lowest vertex.



Using planarity



Three paths from three different vertices q_1, q_2, q_3 to a common vertex $p_{\max}$ always contain three vertex-disjoint paths from q_1, q_2, q_3 to a common vertex (the "Y-lemma").
Together with the three paths from p_i to q_1, q_2, q_3 we get a subdivision of $K_{3,3}$.

Tutte's barycenter method for directed planar graphs

Theorem. *Let D be a partially directed subgraph of a planar graph G with specified outer face.*

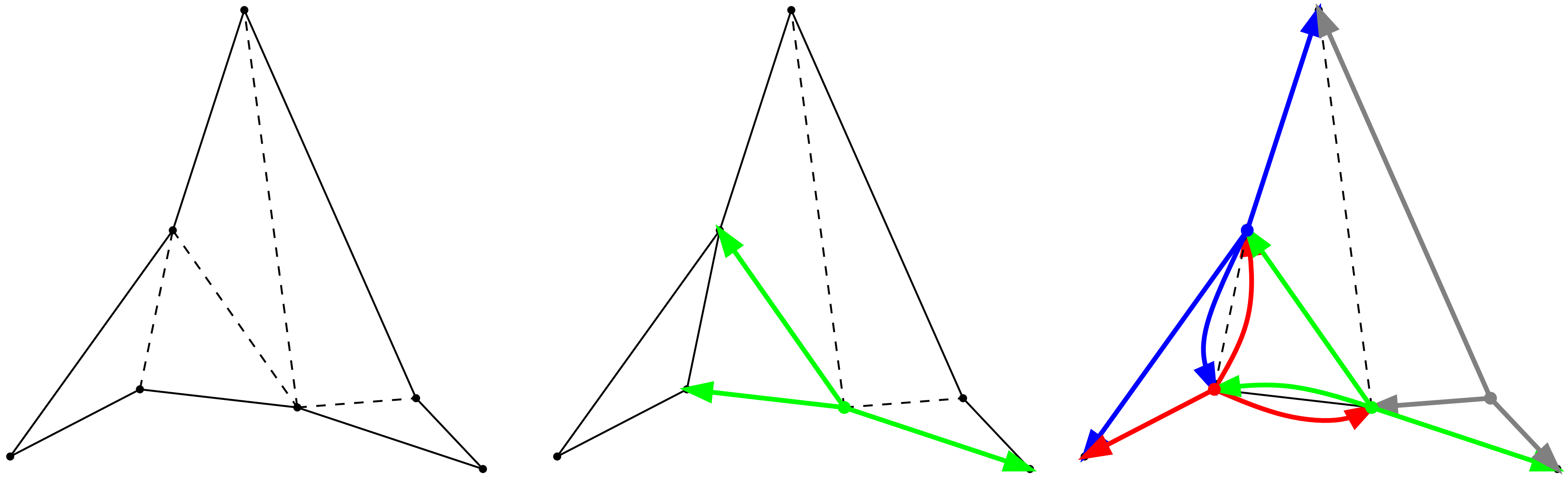
If every interior vertex has three vertex disjoint paths to the outer face, there is a planar embedding where every interior vertex lies in the interior of its out-neighbors. $\square$

Selection of outgoing arcs

3 outgoing arcs for every interior vertex:

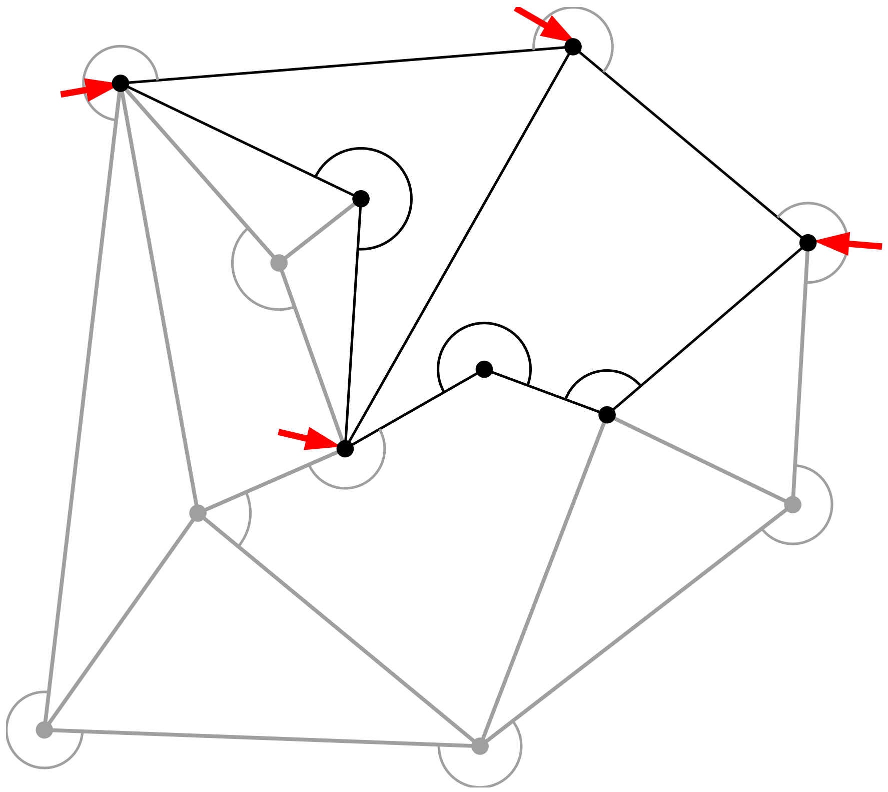
Triangulate each pseudotriangle arbitrarily.
For each reflex vertex, select

- the two incident boundary edges
- an interior edge of the pseudotriangulation



3-connectedness—geometric version

Lemma. *Every induced subgraph of a planar Laman graph with a CPT has at least 3 outside "corners".*



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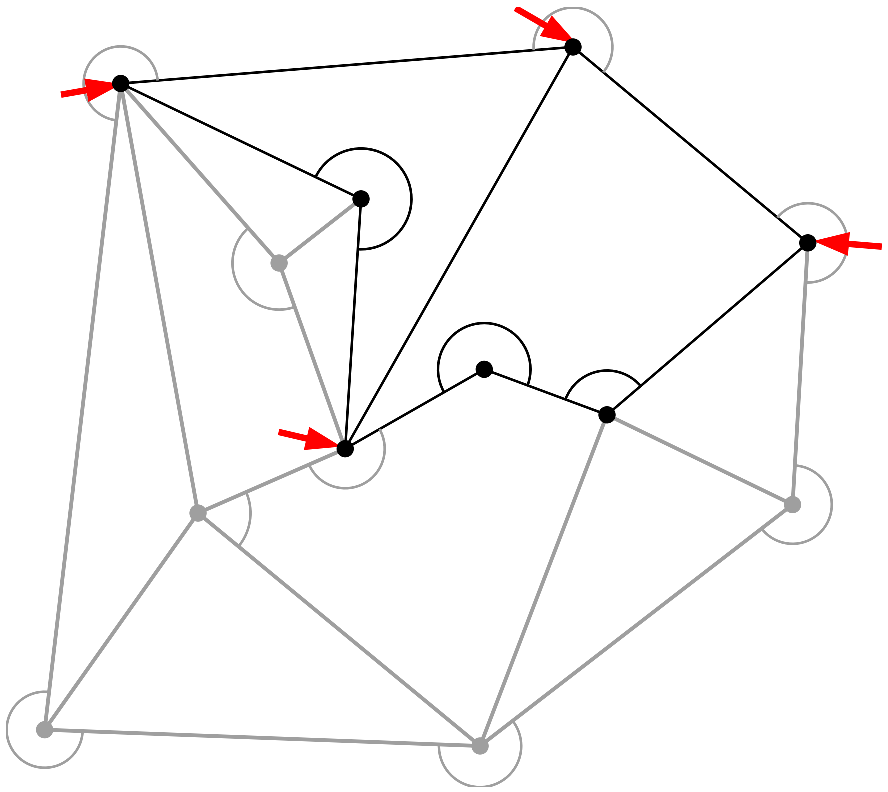
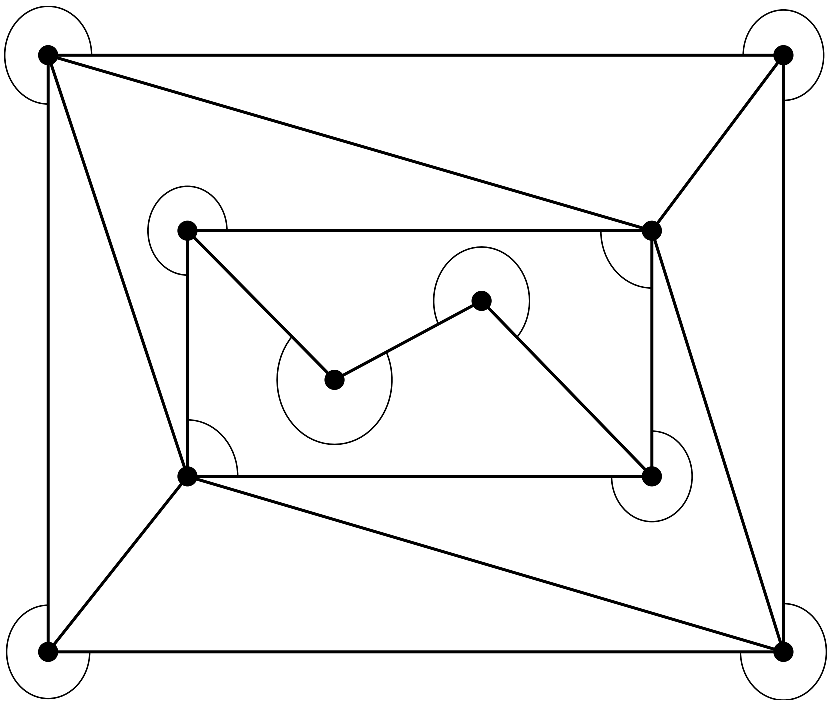
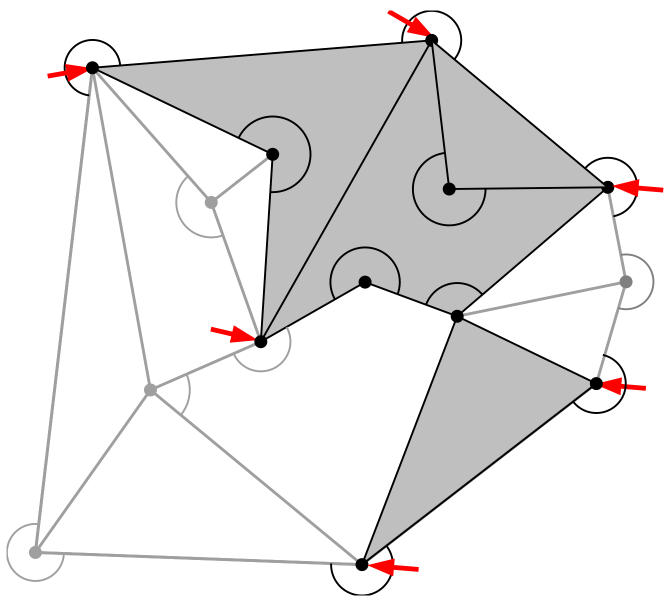




Every subgraph has at least 3 corners



b boundary edges, $b_0 \leq b$ boundary vertices, with c corners.

\# interior angles $= 2e - b$

\# interior small angles $= 3f$

\# interior big angles $= n - c$

Euler: $e + 2 = n + (f + 1)$

$$\implies e = 2n - 3 - (b - c)$$

interior edges and vertices: $e_{\text{int}} = e - b$, $v_{\text{int}} = n - b_0$

Laman: $e_{\text{int}} \geq 2v_{\text{int}}$

$$\implies c \geq 3$$

3-connectedness in the graph

Need to show: Every interior vertex a has three vertex disjoint paths to the outer face.

Apply Menger's theorem: After removing two "blocking vertices" b_1, b_2, there is still a path $a \longrightarrow$boundary.

3-connectedness in the graph

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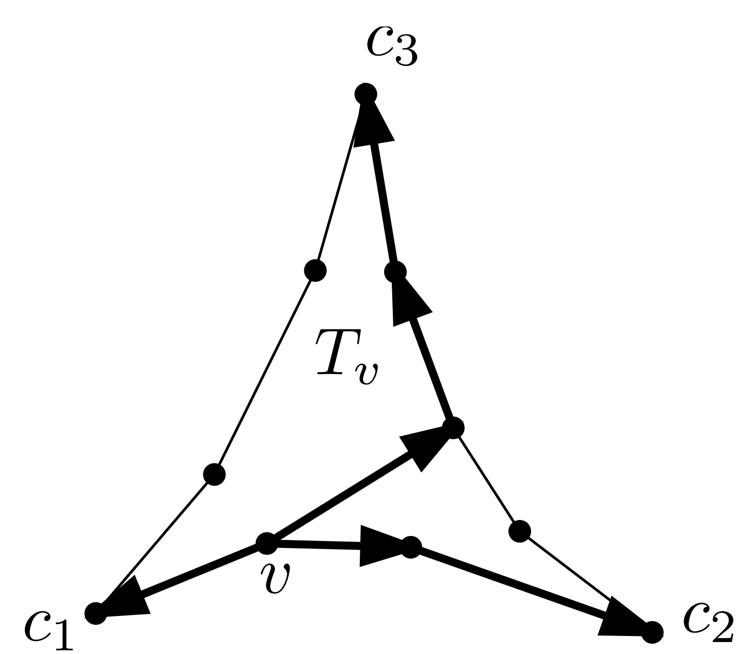
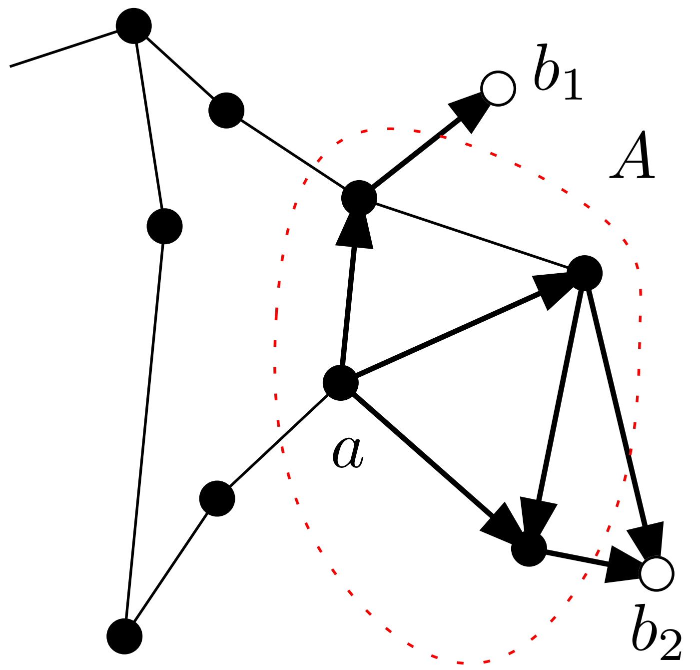
Apply Menger's theorem: After removing two "blocking vertices" b_1, b_2, there is still a path $a \rightarrow$boundary.

Lemma. *An interior vertex v has its big angle in a unique pseudotriangle T_v.*
There are three vertex-disjoint paths $v \rightarrow c_1$, $v \rightarrow c_2$, $v \rightarrow c_3$ to the three corners c_1, c_2, c_3 of T_v.



3-connectedness in the graph

$A :=$ the vertices reachable from a.



i

S

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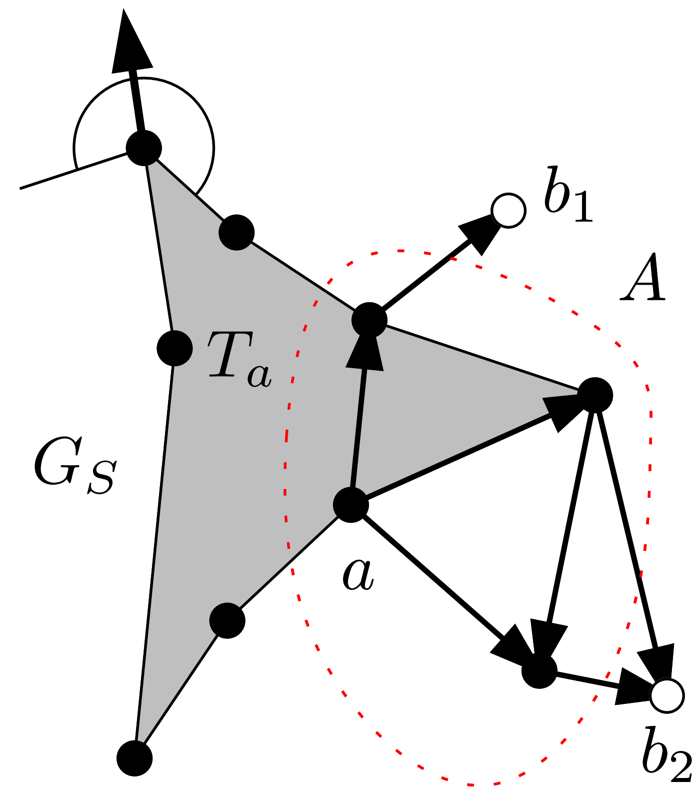
$G_S := \cup\{\, T_v : v \in A \,\}$



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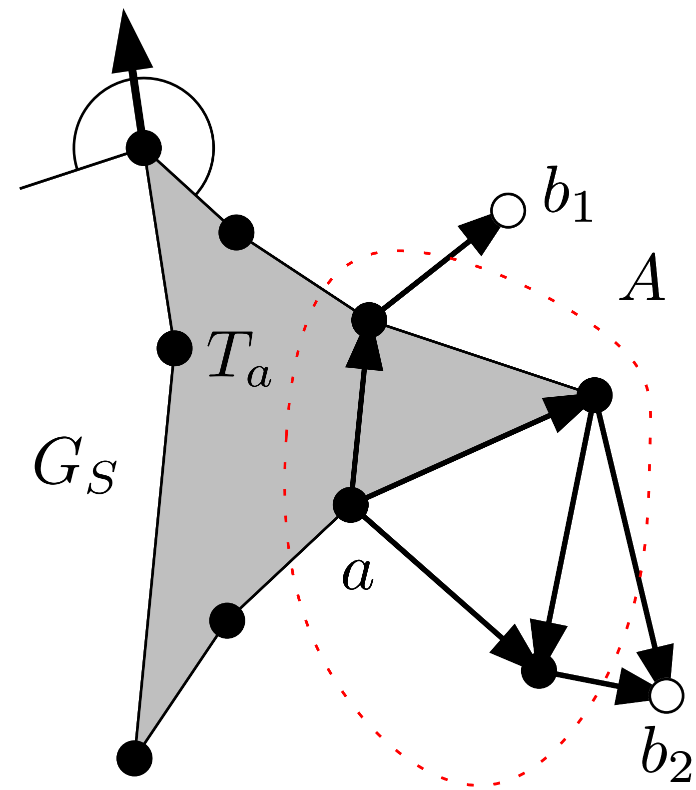
G_S has at least three corners c_1, c_2, c_3.
Find v_1, v_2, v_3 with $c_i \in T_{v_i}$ and paths
$v_1 \longrightarrow c_1$, $v_2 \longrightarrow c_2$, $v_3 \longrightarrow c_3$.



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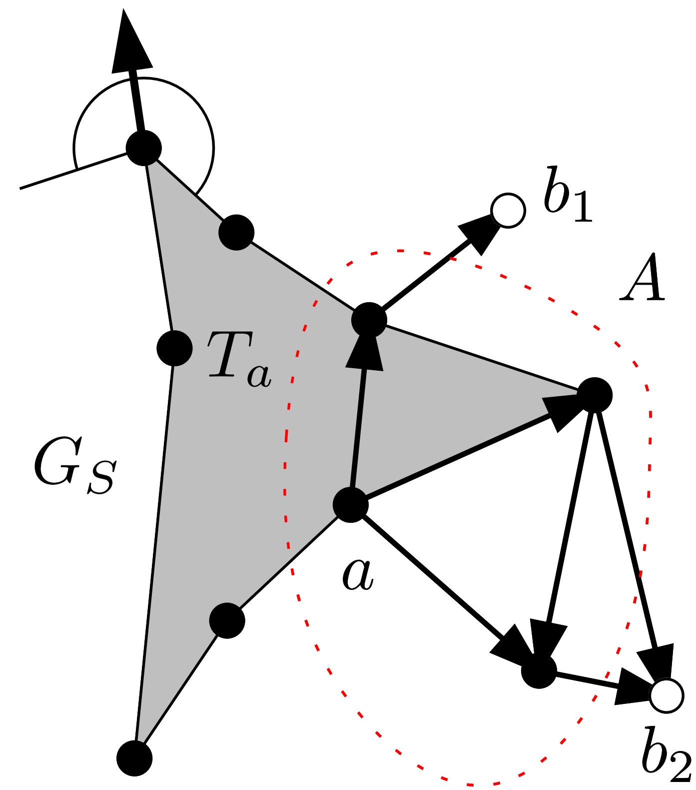
A blocking vertex b_1, b_2 can block only one of these paths. $\Longrightarrow$ some $c_i \in A$.



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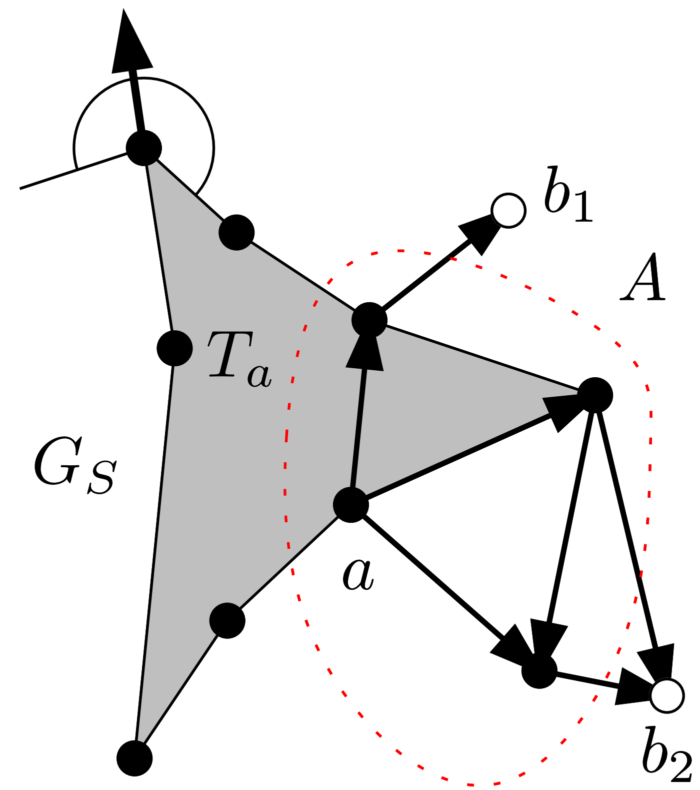
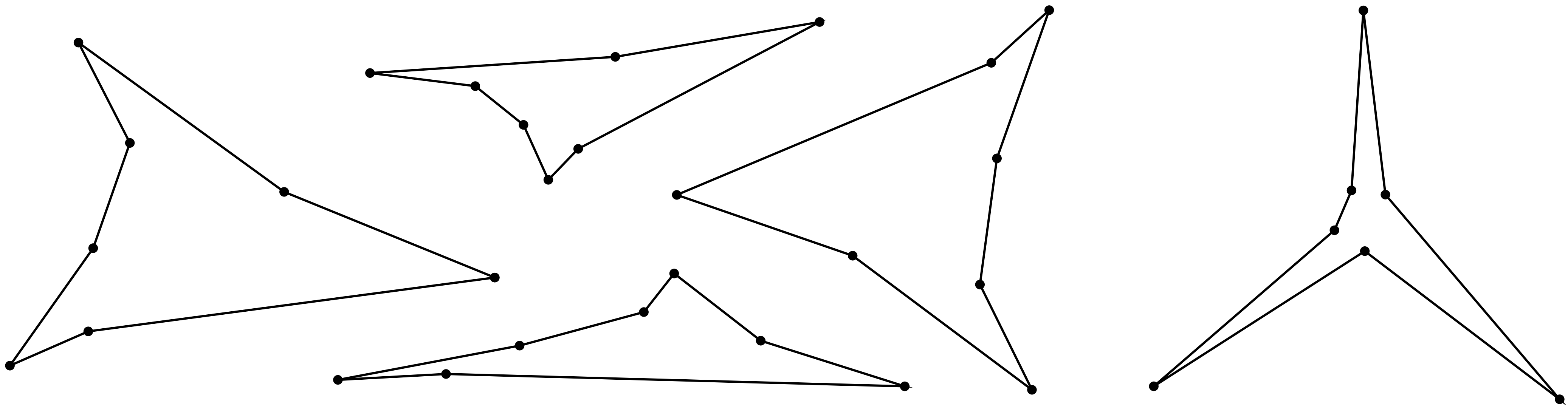


Either c_i lies on the boundary or one can jump out of G_S.

Specifying the shape of pseudotriangles

The shape of every pseudotriangle (and the outer face) can be arbitrarily specified up to affine transformations.



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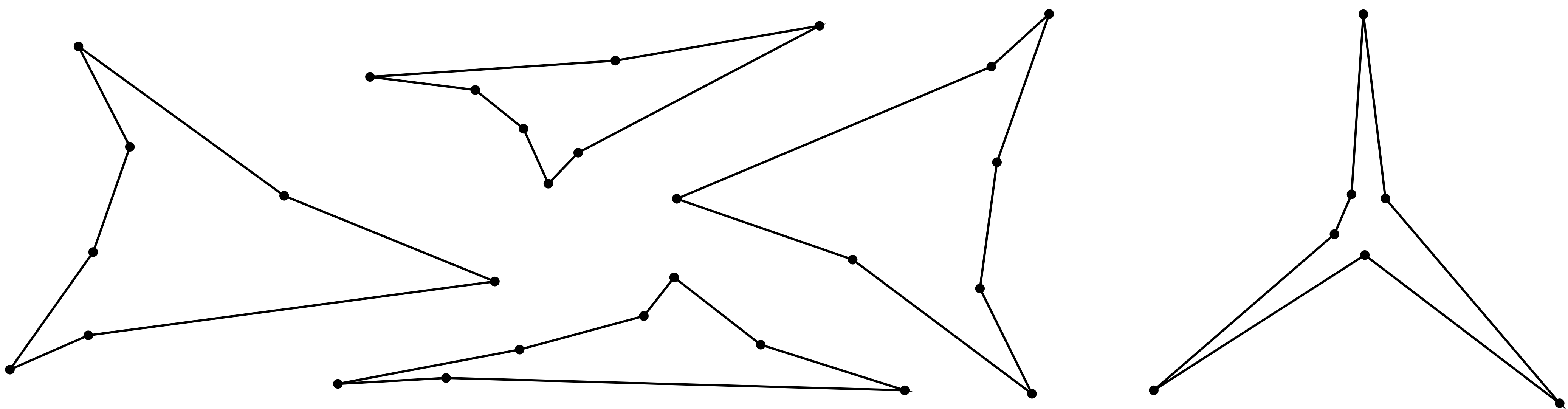
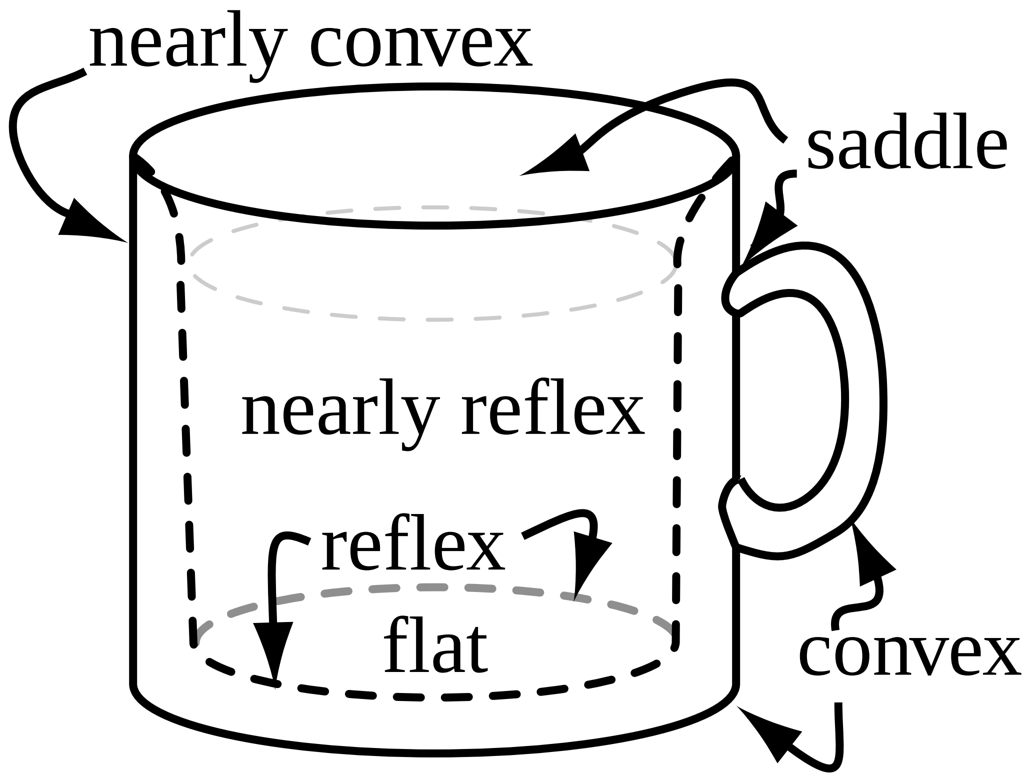


The Tutte embedding with all $\omega_{ij} = 1$ yields rational coordinates with a common denominator which is at most $12^{n/2}$, i. e. with $O(n)$ bits.
OPEN PROBLEM: Can every pseudotriangulation be embedded on a polynomial size grid? On an $O(n) \times O(n)$ grid?

3. Locally convex surfaces
Motivation: the reflex-free hull

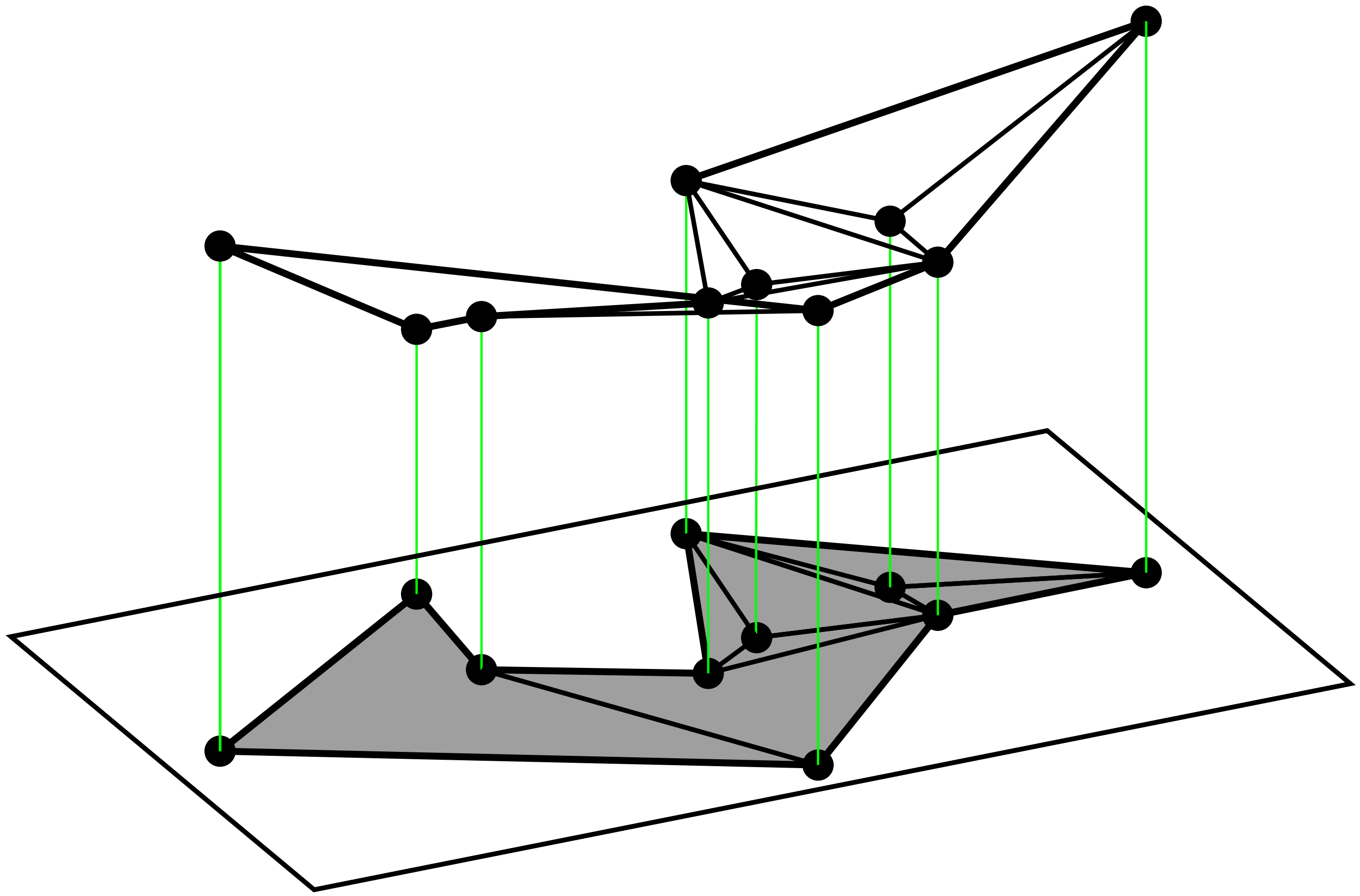


an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]

Locally convex functions

A function over a polygonal domain P is *locally convex* if it is convex on every segment in P.



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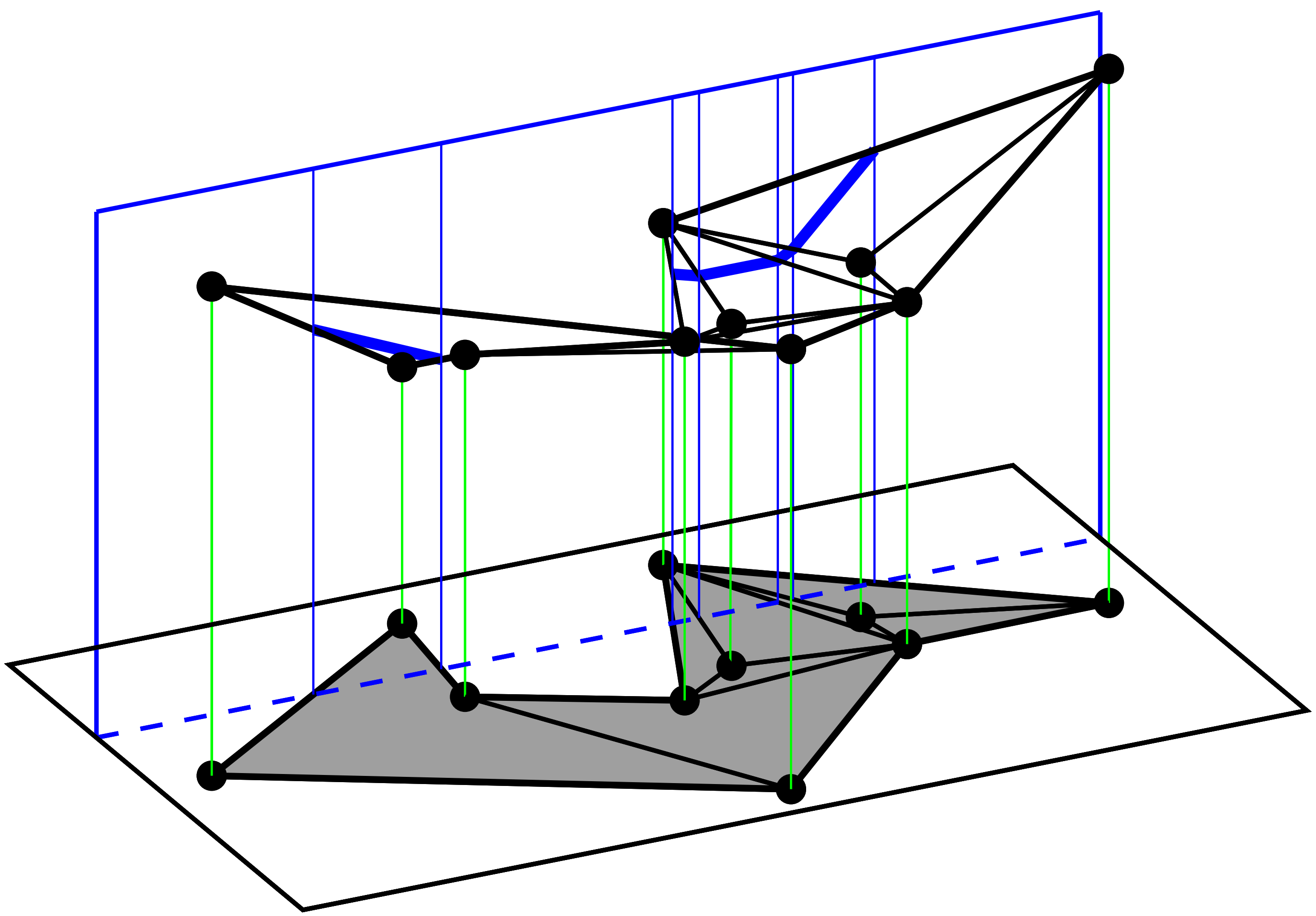


Locally convex functions on a poipogon

A *poipogon* (P, S) is a simple polygon P with some additional vertices inside.

Given a poipogon and a height value h_i for each $p_i \in S$, find the highest locally convex function $f\colon P \to \mathbb{R}$ with $f(p_i) \leq h_i$.

If P is convex, this is the lower convex hull of the three-dimensional point set (p_i, h_i).

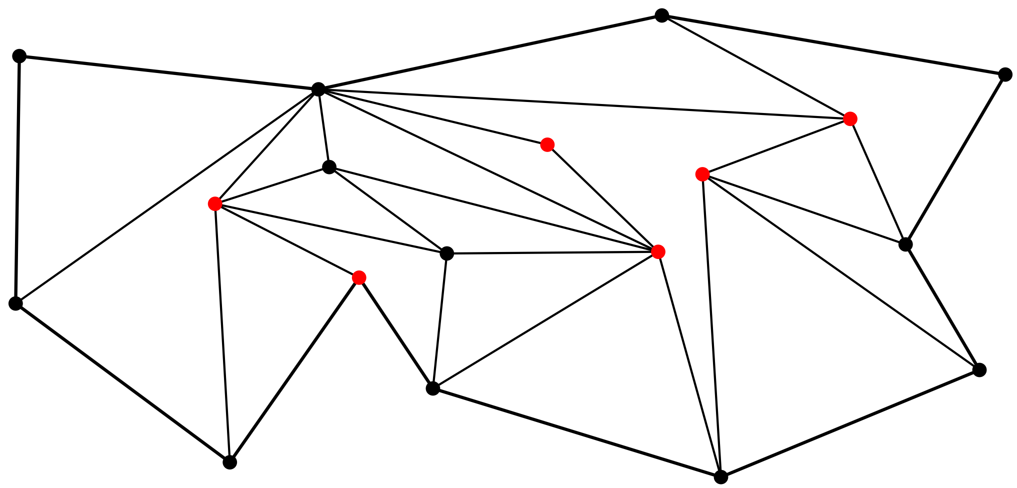
In general, the result is a piecewise linear function defined on a pseudotriangulation of (P, S). (Interior vertices may be missing.)

$\to$ *regular pseudotriangulations*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

The surface theorem

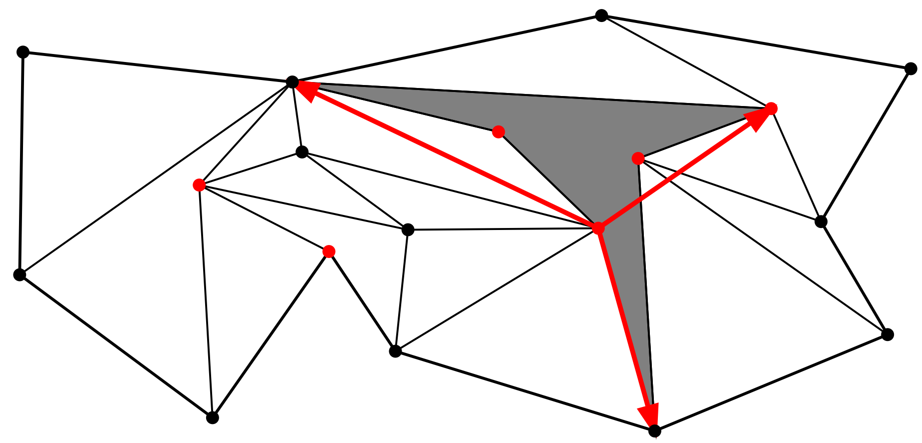
In a pseudotriangulation T of (P, S), a vertex is *complete* if it is a corner in all pseudotriangulations to which it belongs.



Theorem. *For any given set of heights h_i for the complete vertices, there is a unique piecewise linear function on the pseudotriangulation with the complete vertices. The function depends monotonically on the given heights.*

In a triangulation, all vertices are complete.

Proof of the surface theorem



Each incomplete vertex p_i is a convex combination of the three corners of the pseudotriangle in which its large angle lies:

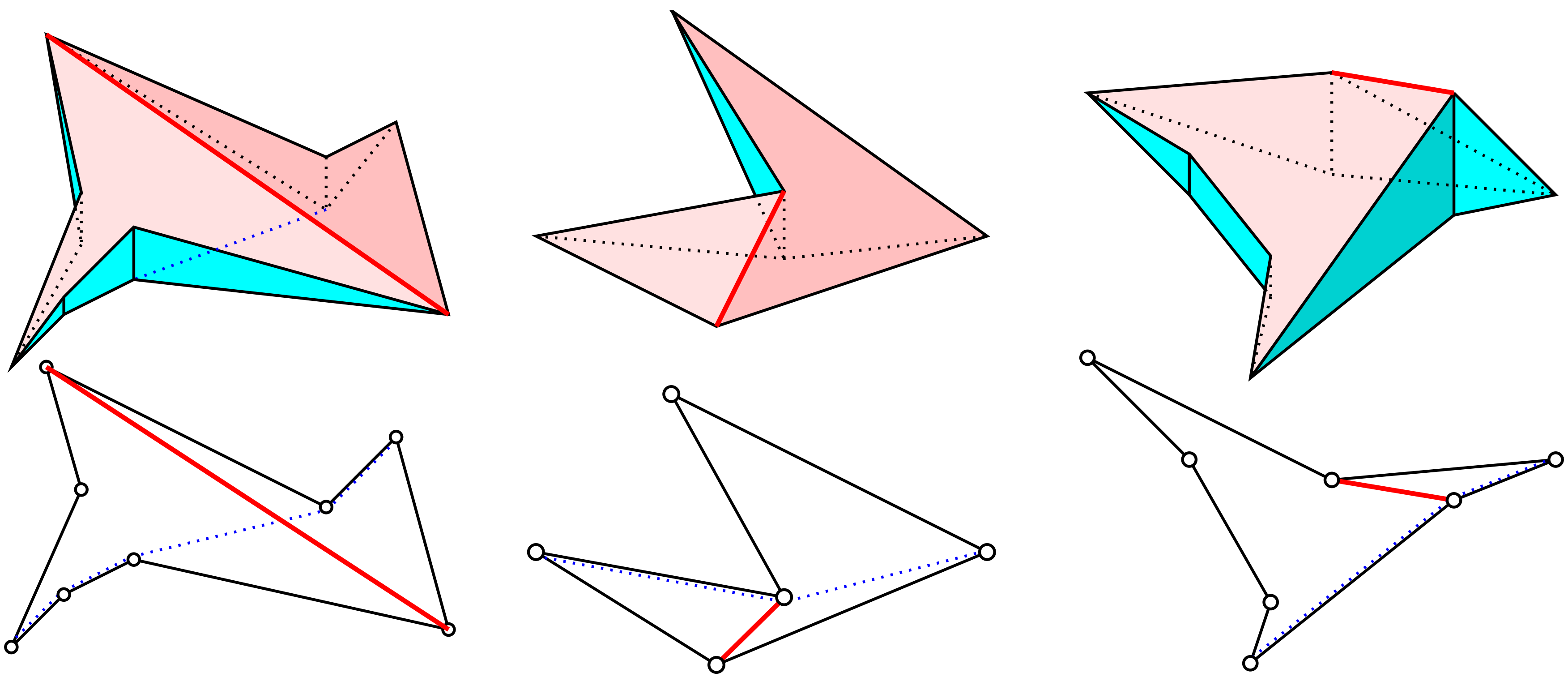
$$p_i = \alpha p_j + \beta p_k + \gamma p_l, \text{ with } \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma > 0.$$

$$\rightarrow h_i = \alpha h_j + \beta h_k + \gamma h_l$$

The coefficient matrix of this mapping $M\colon (h_1, \ldots, h_n) \mapsto (h'_1, \ldots, h'_n)$ is a stochastic matrix. M is a monotone function. There is always a unique solution. (Exercise 16)

Flipping to optimality

Find an edge where convexity is violated, and flip it.



convexifying flips

a planarizing flip

A flip has a non-local effect on the whole surface.
The surface moves down monotonically.

Realization as a polytope

There exists a convex polytope whose vertices are in one-to-one correspondence with the regular pseudotriangulations of a poipogon, and whose edges represent flips.

For a simple polygon (without interior points), all pseudotriangulations are regular.

4. Expansive motions and the polytope of pointed pseudotriangulations
Infinitesimal Motion

n vertices $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

4. Expansive motions and the polytope of pointed pseudotriangulations
Infinitesimal Motion

n vertices $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

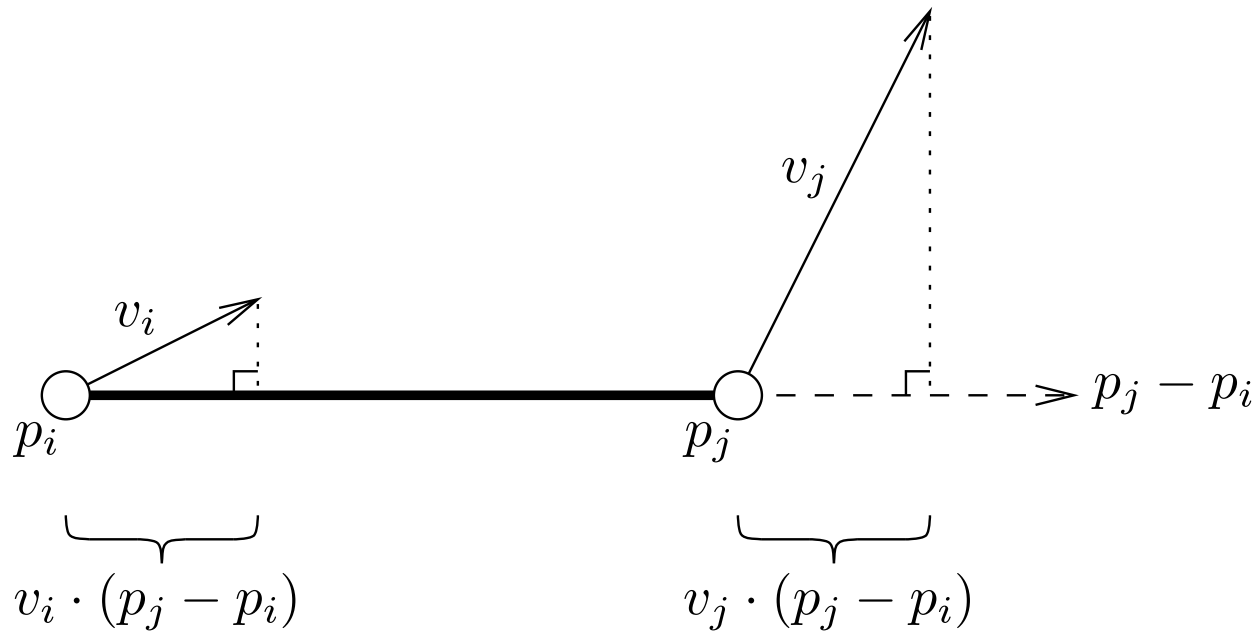
2. *infinitesimal motion* (local motion)

$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

Velocity vectors $v_1, \ldots, v_n$.

Expansion

$$\tfrac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



expansion (or *strain*) $\exp_{ij}$ of the segment ij

The rigidity map

$$M\colon (v_1, \ldots, v_n) \mapsto (\exp_{ij})_{ij \in E}$$

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The rigidity matrix:

$$M = \underbrace{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}_{2|V|} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} E$$

Expansive Motions

$\exp_{ij} = 0$ for all *bars* ij

(preservation of length)

$\exp_{ij} \geq 0$ for all other pairs (*struts*) ij

(expansiveness)

The unfolding theorem
Proof outline

1. Prove that expansive motions *exist*.

2. Select an expansive motion and provide a global motion.

The unfolding theorem
Proof outline

1. Prove that expansive motions *exist*. [2 PROOFS]

2. Select an expansive motion and provide a global motion.

Proof Outline

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

[Connelly, Demaine, Rote 2000]

The expansion cone

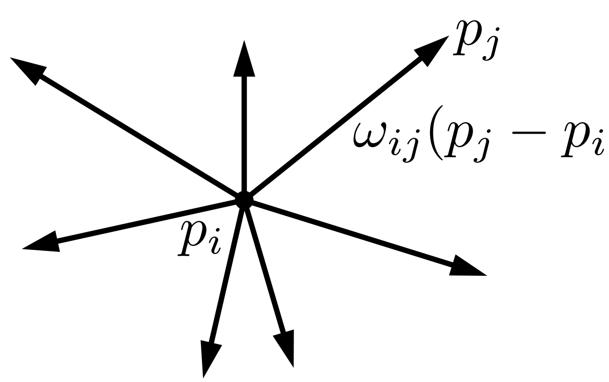
The set of expansive motions forms a convex polyhedral cone $\bar{X}_0$ in $\mathbb{R}^{2n}$, defined by homogeneous linear equations and inequalities of the form

$$\langle v_i - v_j, p_i - p_j \rangle \left\{ \begin{array}{c} = \\ \geq \end{array} \right\} 0$$

Bars, struts, frameworks, stresses

Assign a *stress* $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ to each edge.

Equilibrium of forces in vertex i:



$$\sum_j \omega_{ij}(p_j - p_i) = 0$$

$\omega_{ij} \leq 0$ for struts: Struts can only push.
$\omega_{ij} \in \mathbb{R}$ for bars: Bars can push or pull.

Motions and stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$\omega_{ij} \in \mathbb{R}, \quad \text{for a bar } ij$$
$$\omega_{ij} \leq 0, \quad \text{for a strut } ij$$

Motions and stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 \\ > 0 \end{cases}$$

$$\left[\ Mv \begin{cases} = 0 \\ > 0 \end{cases} \ \right]$$

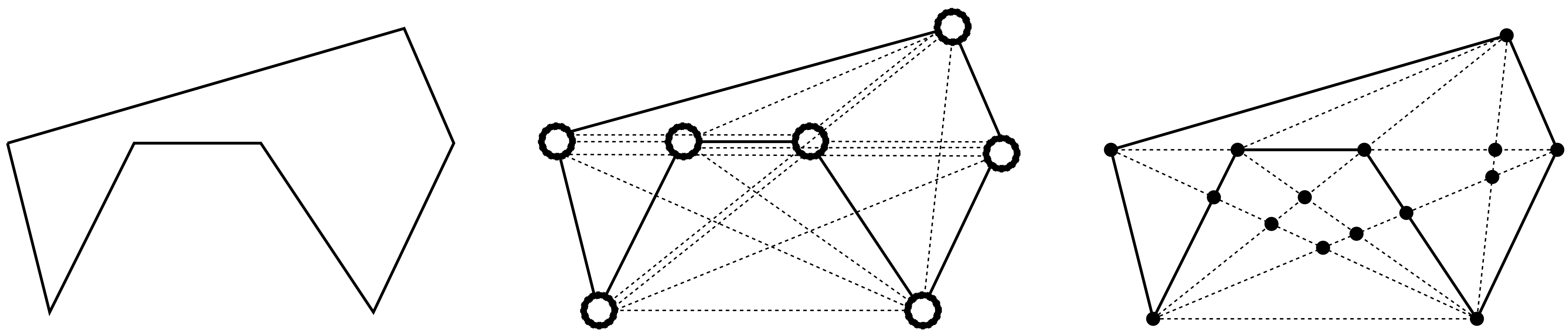
$$\sum_j \omega_{ij}(p_j - p_i) = 0, \text{ for all } i$$

$$[\ M^{\mathrm{T}}\omega = 0\ ]$$

$$\omega_{ij} \in \mathbb{R}, \quad \text{for a bar } ij$$

$$\omega_{ij} \leq 0, \quad \text{for a strut } ij$$

Making the framework planar



- subdivide edges at intersection points
- collapse multiple edges

The Maxwell-Cremona Correspondence [1864/1872]

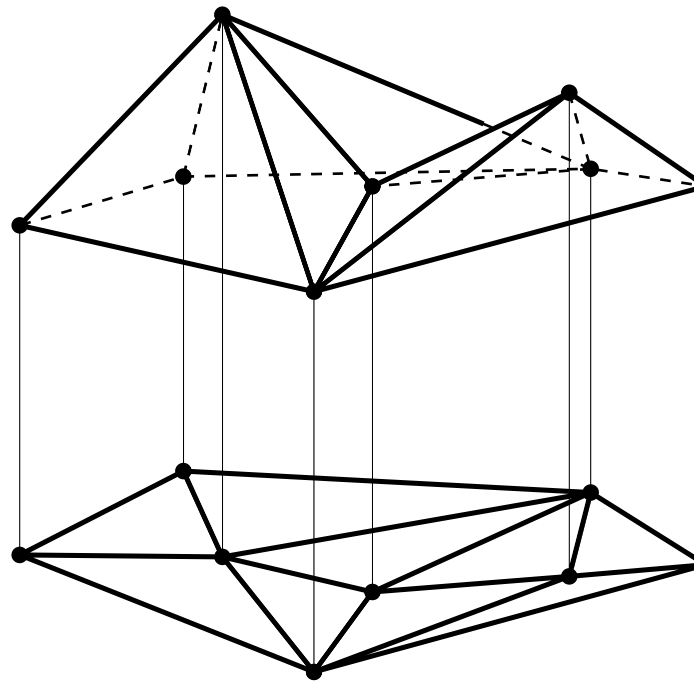
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

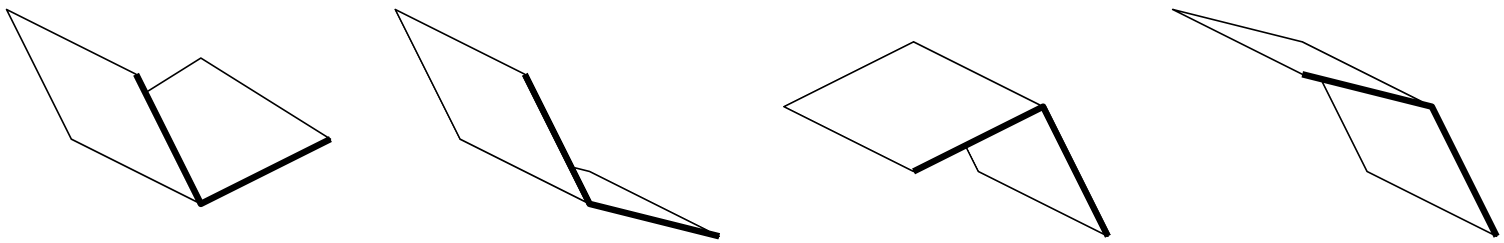
reciprocal diagram

$\Updownarrow$ one-to-one correspondence

3-d lifting (polyhedral terrain)



Valley and mountain folds

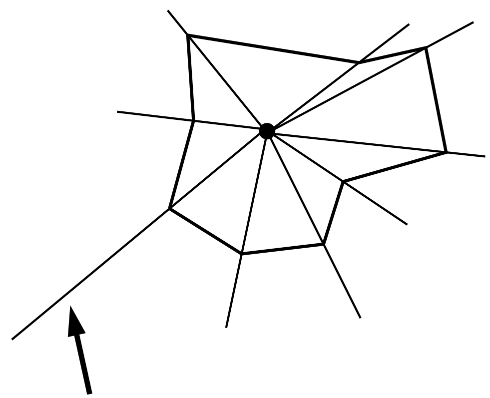


$\omega_{ij} > 0$ | $\omega_{ij} < 0$

valley | mountain

bar or strut | bar

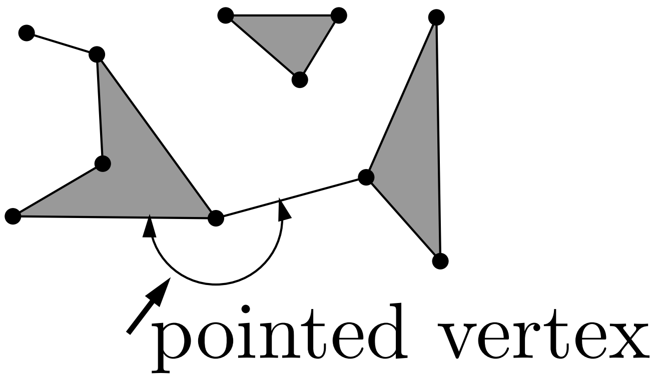
Look a the highest peak!



Every polygon has > 3 convex vertices

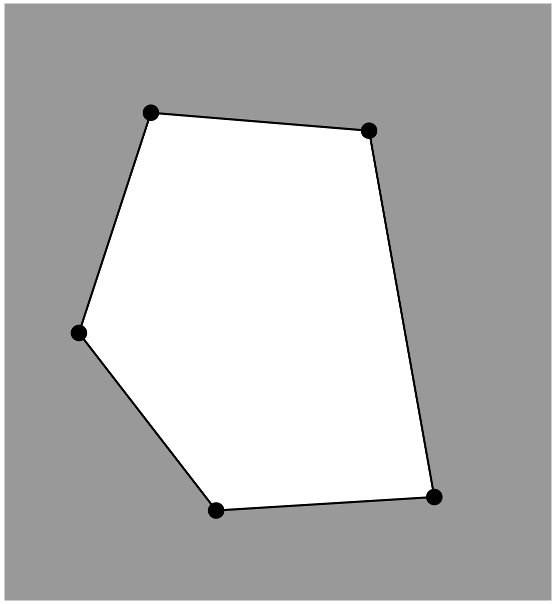
$\longrightarrow$ 3 mountain folds $\longrightarrow$ 3 bars.

The general case



There is at least one vertex with angle $> \pi$.

The only remaining possibility



a convex polygon

□

Constructing a global motion

[Connelly, Demaine, Rote 2000]

- Define a point $v := v(p)$ in the *interior* of the expansion cone, by a suitable non-linear convex objective function.
- $v(p)$ depends smoothly on p.
- Solve the differential equation $\dot{p} = v(p)$

Constructing a global motion

[Connelly, Demaine, Rote 2000]

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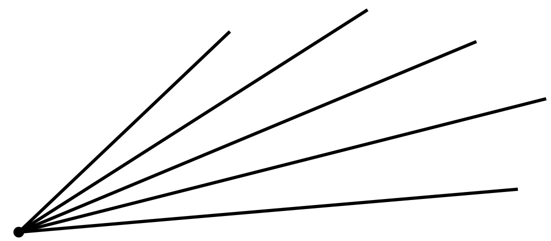
Alternative approach: Select an *extreme ray* of the expansion cone.

Streinu [2000]:
Extreme rays correspond to pseudotriangulations.

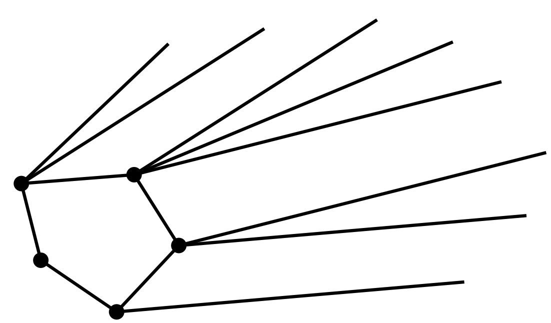
Cones and polytopes

[Rote, Santos, Streinu 2002]

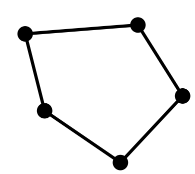
- The *expansion cone*
  $\bar{X}_0 = \{ \exp_{ij} \geq 0 \}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{ \exp_{ij} \geq f_{ij} \}$



- The *PPT polytope*
  $X_f = \{ \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



The PPT polytope

Theorem. *For every set* S *of points in general position, there is a convex* $(2n-3)$*-dimensional polytope whose vertices correspond to the pointed pseudotriangulations of* S.

Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Pin a vertex and a direction. ("tie-down")

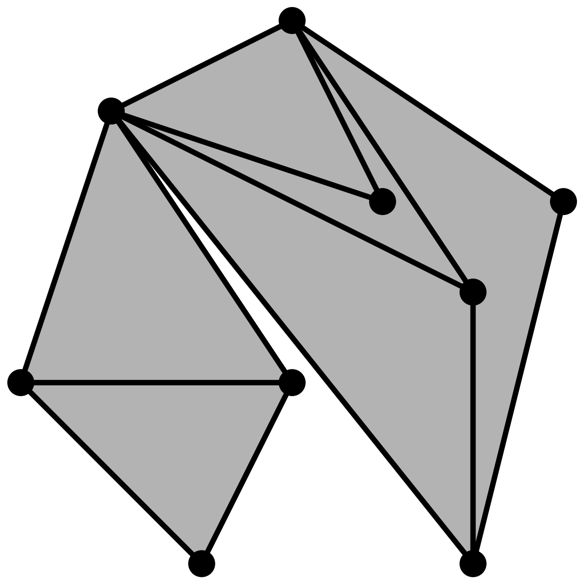
$$v_1 = 0$$

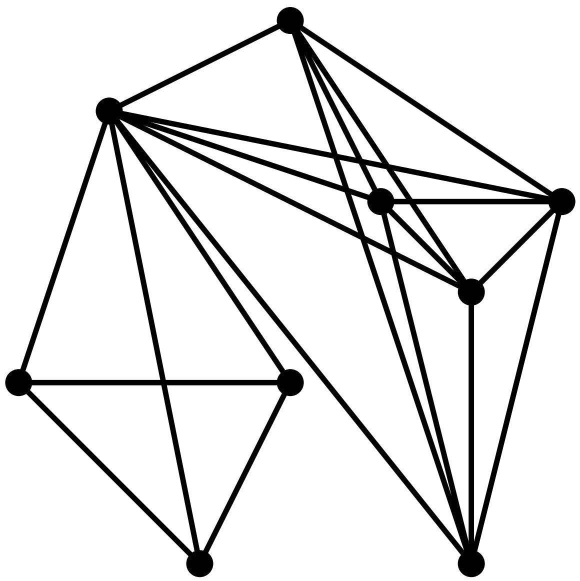
$$v_2 \parallel p_2 - p_1$$

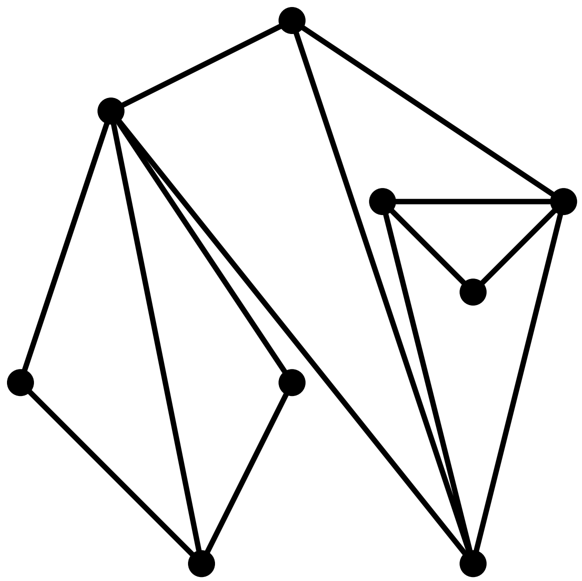
This eliminates 3 degrees of freedom.

Extreme rays of the expansion cone

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]
Rigid substructures can be identified.



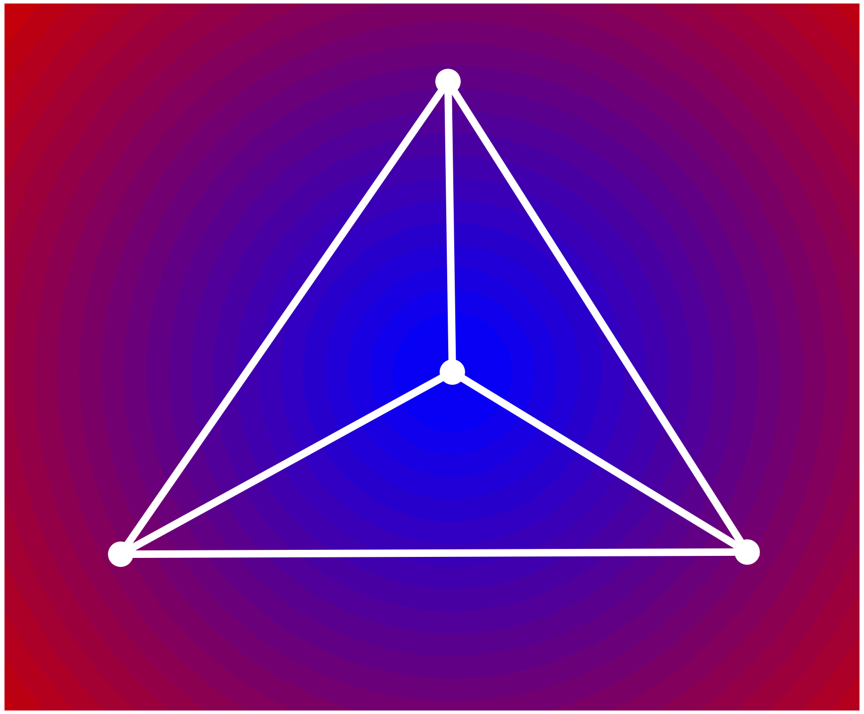




A polyhedron for pseudotriangulations

Wanted:
A perturbation of the constraints "$\exp_{ij} \geq 0$" such that the vertices are in 1-1 correspondence with pseudotriangulations.

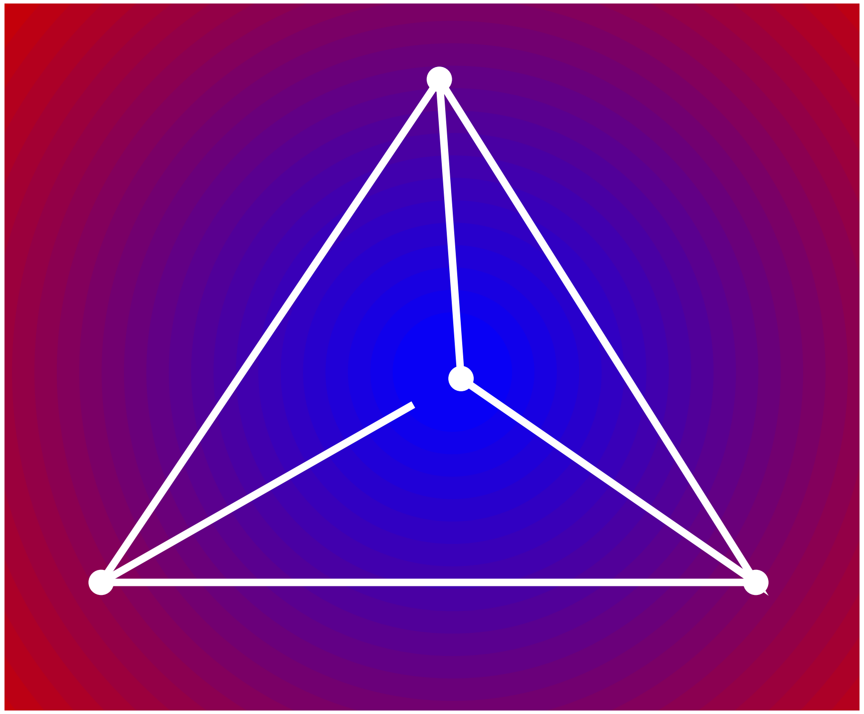
Heating up the bars



$$\Delta T = |x|^2$$

$$\text{Length increase } \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

Heating up the bars



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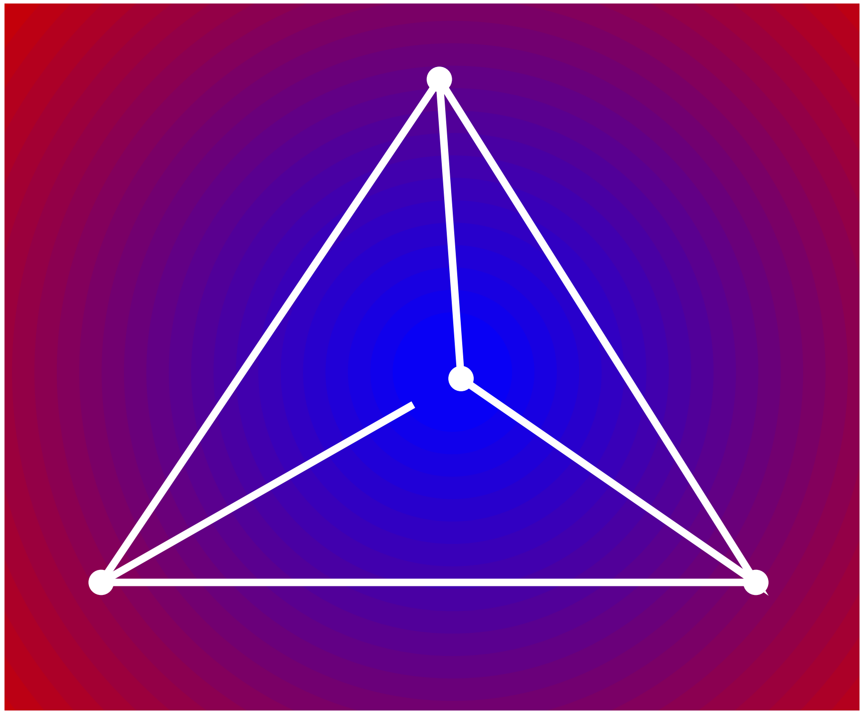
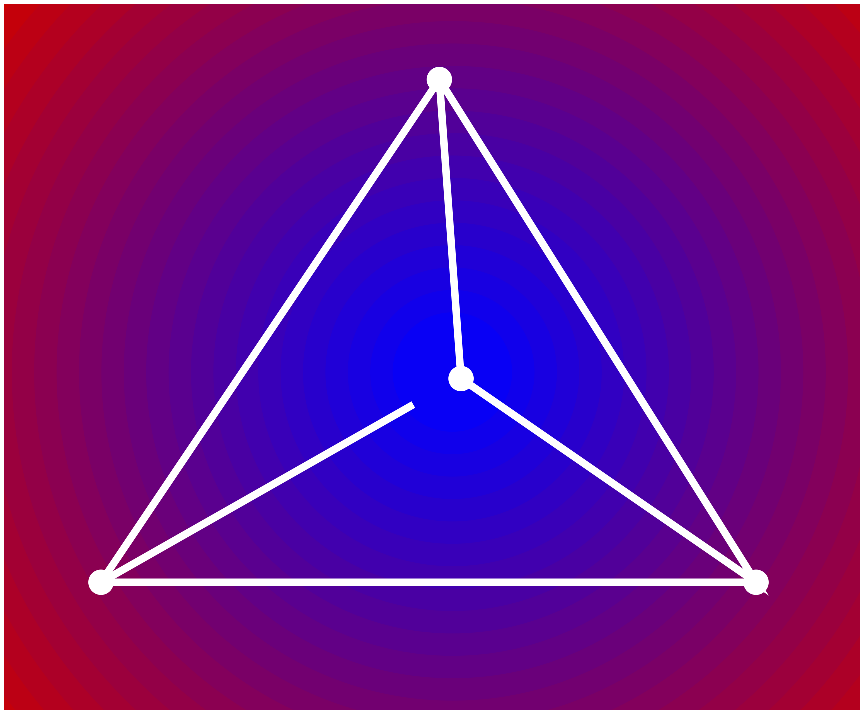


$$\Delta T = |x|^2$$

Length increase $\geq \displaystyle\int_{x \in p_i p_j} |x|^2 \, ds$

$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

Heating up the bars



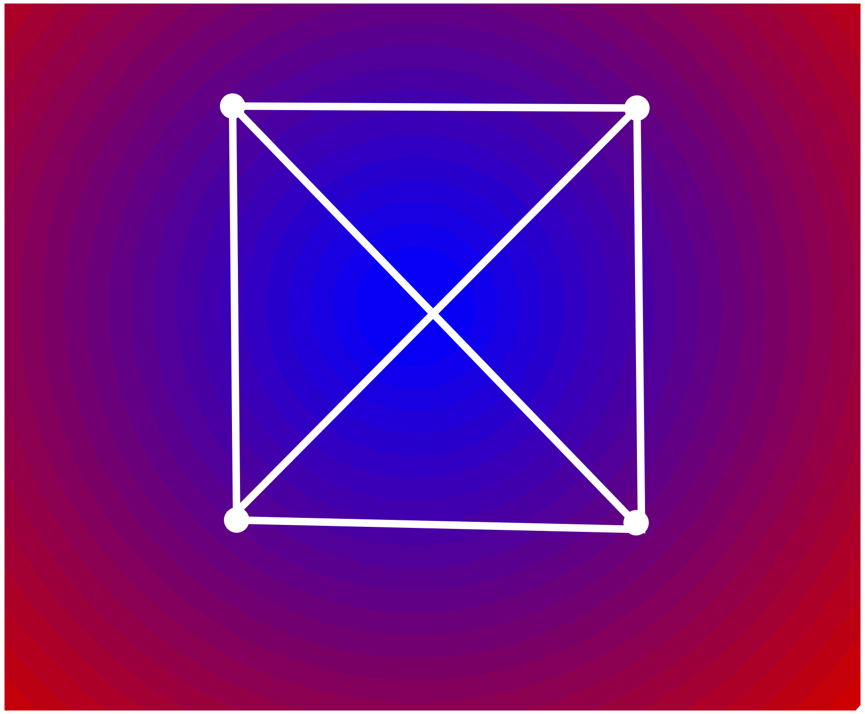
$$\Delta T = |x|^2$$

Length increase $\geq \int_{x \in p_i p_j} |x|^2 \, ds$

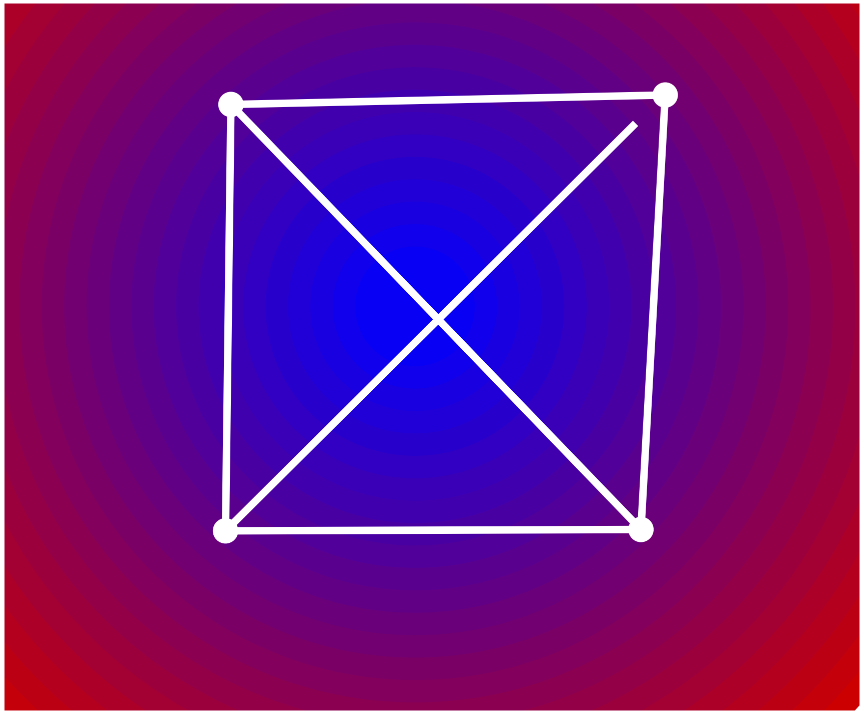
$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\exp_{ij} \geq |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \cdot \frac{1}{3}$$

Heating up the bars — points in convex position



$\Rightarrow$



The perturbed expansion cone = PPT polyhedron

$$\bar{X}_f = \{\, (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \,\}$$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

  $[x, y, z] =$ signed area of the triangle xyz
  a, b: two arbitrary points.

Tight edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$,

$$E(v) := \{ \, ij \mid \exp_{ij} = f_{ij} \, \}$$

is the *set of tight edges* at v.

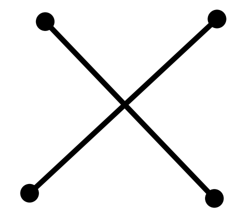
Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$.

What are good values of f_{ij}?

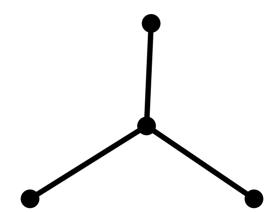
Which configurations of edges can occur in a set of tight edges?

We want:

- no crossing edges

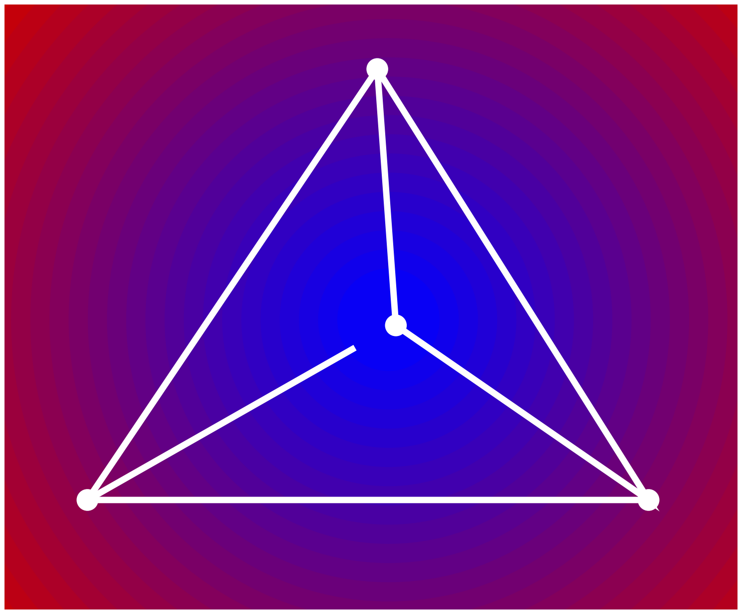


- no 3-star with all angles $\leq 180^\circ$



It is sufficient to look at 4-point subsets.

Good values f_{ij} for 4 points



f_{ij} is given on six edges.

Any five values $\exp_{ij}$ determine the last one.

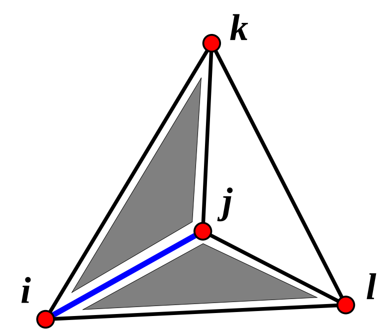
Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$)

$\longrightarrow$ checking the sign of an expression.

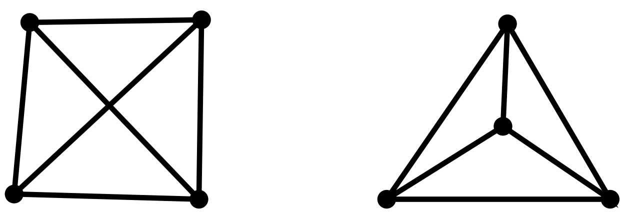
Good Values f_{ij} for 4 points

A 4-tuple p_1, p_2, p_3, p_4 has a unique self-stress (up to a scalar factor).

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \text{ for all } 1 \le i < j \le 4$$



$\omega_{ij} > 0$ for boundary edges.
$\omega_{ij} < 0$ for interior edges.



Why the stress?

If the *equation*

$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} = 0$$

holds, then f_{ij} are the expansion values $\exp_{ij}$ of a motion (v_1, v_2, v_3, v_4).

Actually, “if and only if”.

Why the stress?

If the *equation*

$$\sum_{1\le i<j\le 4} \omega_{ij} f_{ij} = 0$$

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Actually, “if and only if”.

[$M^{\mathrm{T}}\omega = 0$, $f = \exp = Mv$]

Good perturbations

We need

$$\sum_{1\leq i<j\leq 4} \omega_{ij} f_{ij} > 0$$

for all 4-tuples of points.

$\longrightarrow$ For every vertex v, $E(v)$ is non-crossing and pointed.

$\longrightarrow$ $\bar{X}_f$ is a simple polyhedron.

The PPT-polyhedron

Every vertex is incident to $2n - 3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

Removing an interior segment leads to an adjacent pseudotriangulation (flip).

Removing a hull segment is an extreme ray. $\square$

Proof of

$$\omega_{12}f_{12}+\omega_{13}f_{13}+\omega_{14}f_{14}+\omega_{23}f_{23}+\omega_{24}f_{24}+\omega_{34}f_{34} > 0$$

$$R(a,b) := \sum_{1\le i<j\le 4} \omega_{ij} \cdot [a,p_i,p_j][b,p_i,p_j]$$

$$R \equiv 1!$$

R is linear in a and linear in b. $R(p_i,p_j) = 1$ is sufficient.

$R(p_1,p_2)$: all $f_{ij} = 0$ except f_{34}

$$R(p_1,p_2) = \omega_{34}f_{34} = \frac{\det(p_1,p_3,p_4)\det(p_2,p_3,p_4)}{\det(p_3,p_4,p_1)\det(p_3,p_4,p_2)} = 1. \quad \square$$

The PPT polytope

Cut out all rays:
Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The PPT polytope

Cut out all rays:
Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The Expansion Cone $\bar{X}_0$:

collapse parallel rays into one ray. $\longrightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.

The PT polytope

Vertices correspond to *all* pseudotriangulations, pointed or not.

Change inequalities $\exp_{ij} \geq f_{ij}$ to

$$\exp_{ij} + (s_i + s_j)\|p_j - p_i\| \geq f_{ij}$$

with a "slack variable" s_i for every vertex.
$s_i = 0$ indicates that vertex i is pointed.

Faces are in one-to-one correspondence with all non-crossing graphs.

[Orden, Santos 2002]

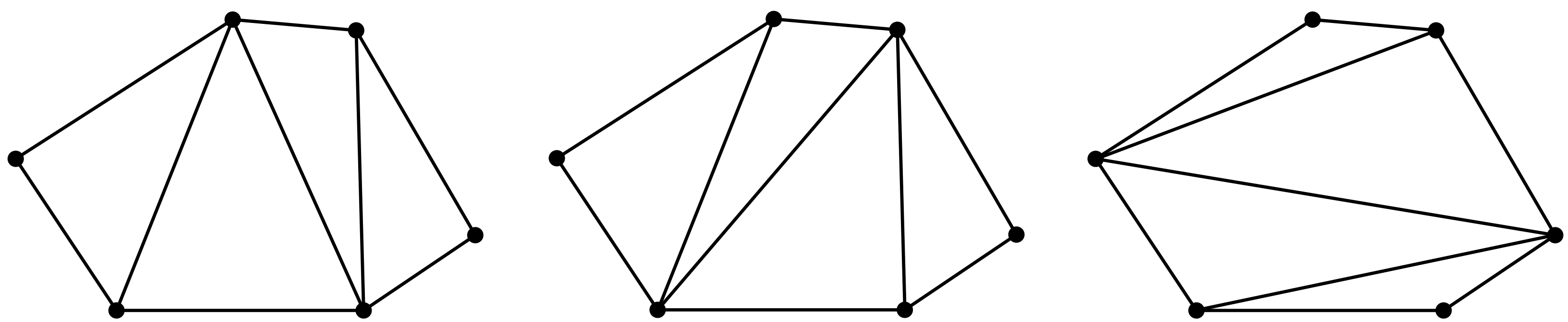
Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\longrightarrow$ expansive mechanism $\square$

Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:



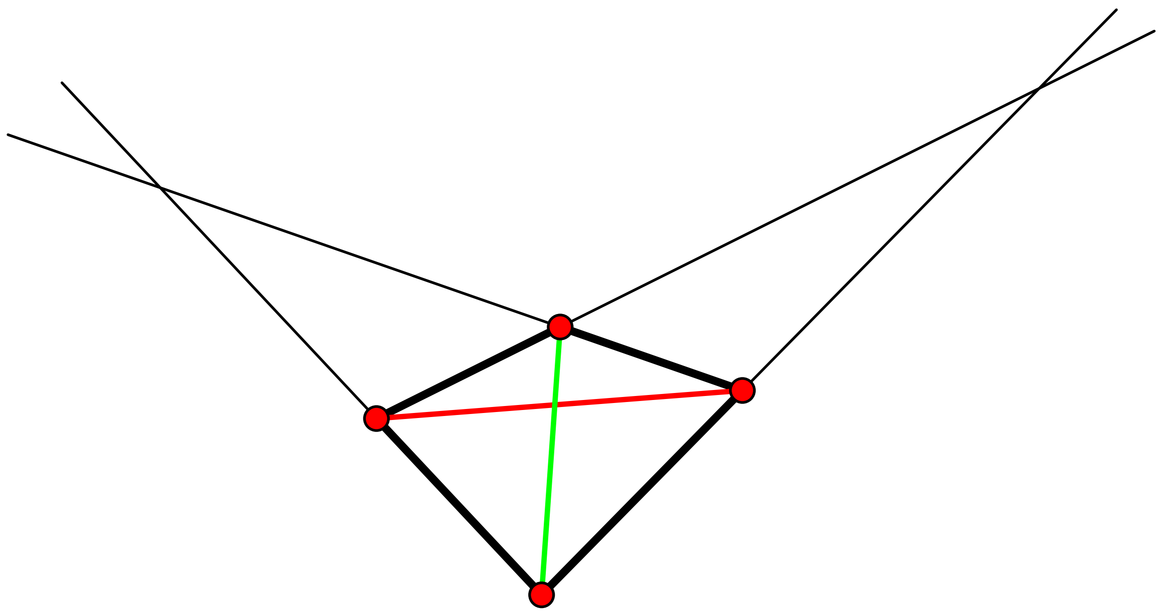
(a) (b) (c)

Delaunay triangulation

Max/Min $\sum p_i \cdot v_i$
(not affinely invariant)

(Can be constructed as the lower/upper convex hull of lifted points.) [André Schulz]

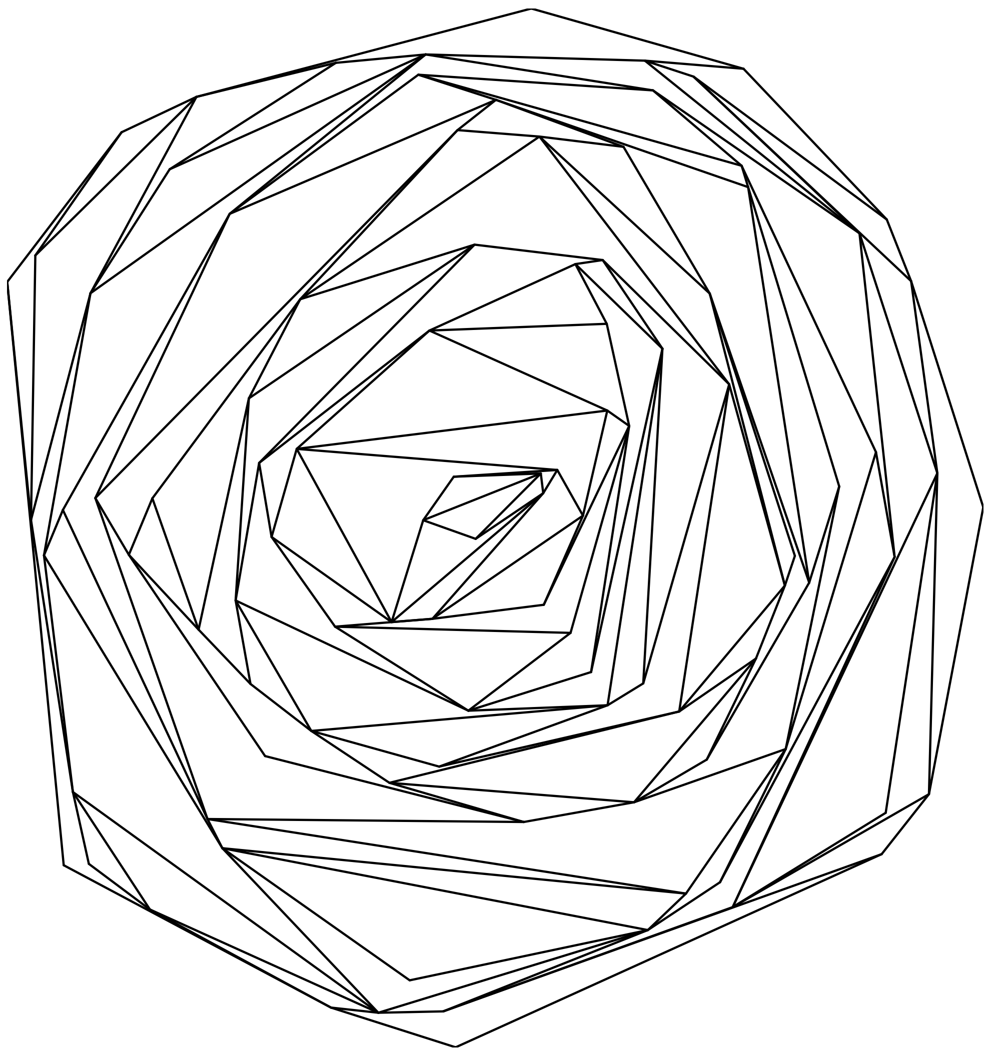
Edge flipping criterion for canonical pseudotriangulations of 4 points in convex position



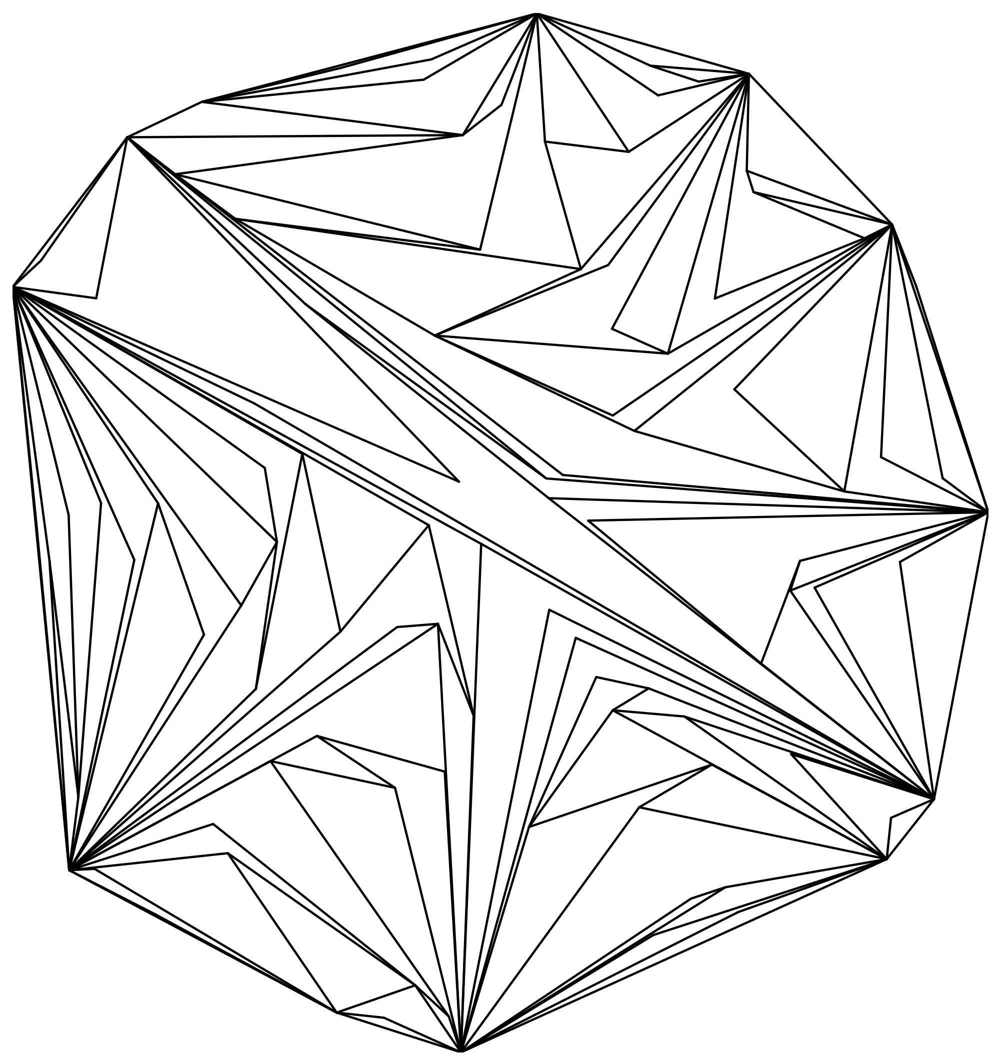
Maximize/minimize the product of the areas.

Invariant under affine transformations.

The “Delone pseudotriangulation” for 100 random points



The "Anti-Delone pseudotriangulation" for 100 random points



Which f_{ij} to choose?

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

Go to the space of the (exp_{ij}) variables instead of the (v_i) variables.

$$\mathrm{exp} = Mv$$

Characterization of the space $(\exp_{ij})_{i,j}$

A set of values $(\exp_{ij})_{1\leq i<j\leq n}$ forms the expansion values of a motion $(v_1, \ldots, v_n)$ if and only if the equation

$$\sum_{1\leq i<j\leq 4} \omega_{ij} \exp_{ij} = 0$$

holds for all 4-tuples.

SKIP

A canonical representation

$$\sum_{1 \le i < j \le 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \ge f_{ij}, \text{ for all pairs } i, j$$

A canonical representation

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \geq f_{ij}, \text{ for all pairs } i, j$$

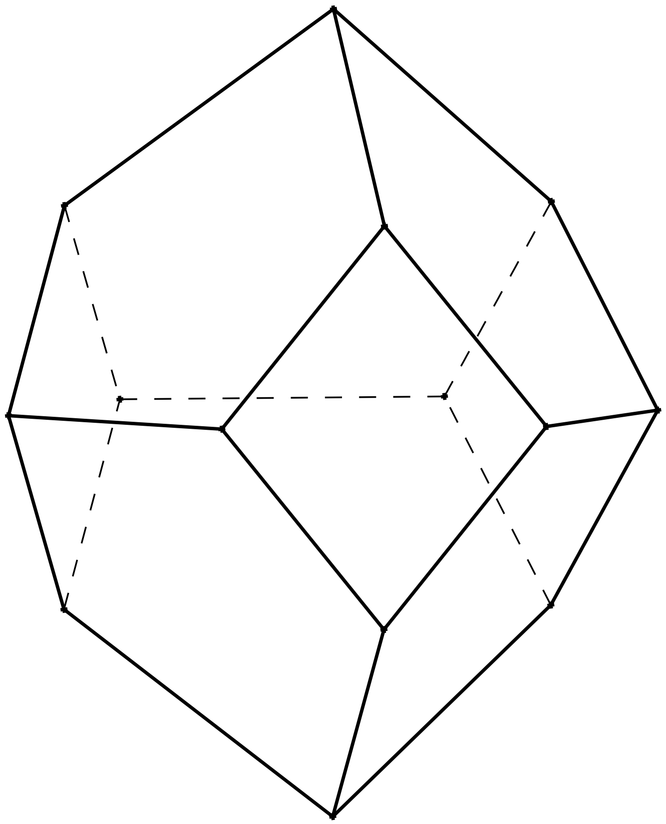
$$\sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 1, \text{ for all 4-tuples}$$

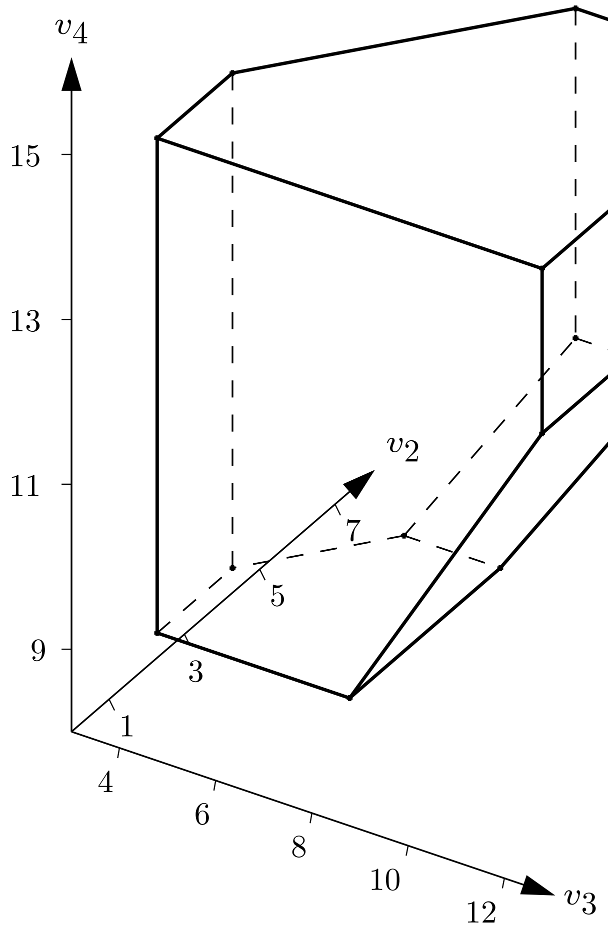
Substitute $d_{ij} := \exp_{ij} - f_{ij}$:

$$\textstyle\sum_{1 \leq i < j \leq 4} d_{ij} \exp_{ij} = -1, \text{ for all 4-tuples} \qquad (1)$$

$$d_{ij} \geq 0, \text{ for all } i, j \qquad (2)$$

The associahedron





Catalan structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$

Catalan structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$
- non-crossing alternating trees
- . . . .

The secondary polytope

Triangulation $T \mapsto (x_1, \ldots, x_n)$.

$$x_i := \text{total area of all triangles incident to } p_i$$

vertices $\equiv$ regular triangulations of $(p_1, \ldots, p_n)$

$(p_1, \ldots, p_n)$ in convex position:
pseudotriangulations $\equiv$ triangulations $\equiv$ regular triangulations.

$\longrightarrow$ two realizations of the associahedron.

These two associahedra are affinely equivalent.

Expansive motions in one dimension

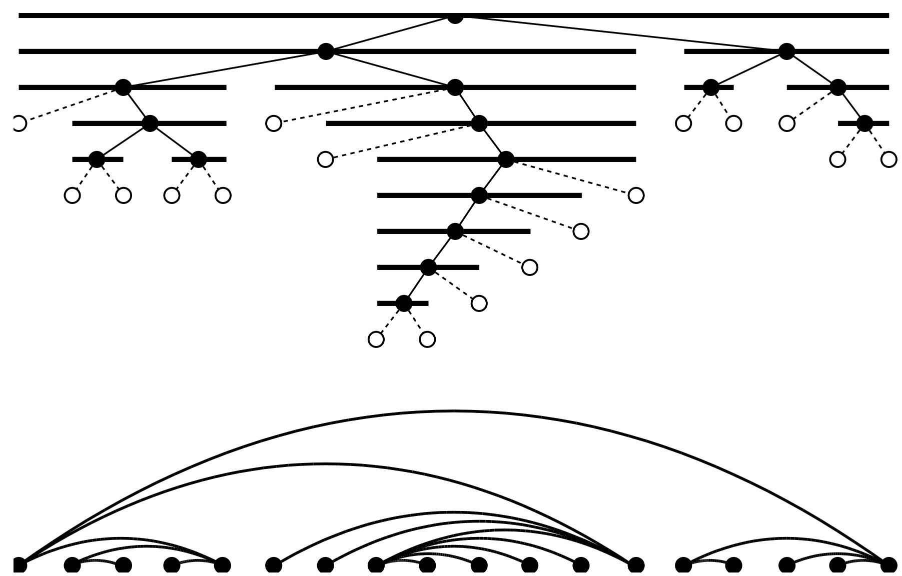
$$\{\,(v_i) \in \mathbb{R}^n \mid v_j - v_i \geq f_{ij} \text{ for } 1 \leq i < j \leq n\,\}$$

$$f_{il} + f_{jk} > f_{ik} + f_{jl}, \text{ for all } i < j < k < l.$$
$$f_{il} > f_{ik} + f_{kl}, \text{ for all } i < k < l.$$

For example, $f_{ij} := (i - j)^2$

related to the Monge Property.

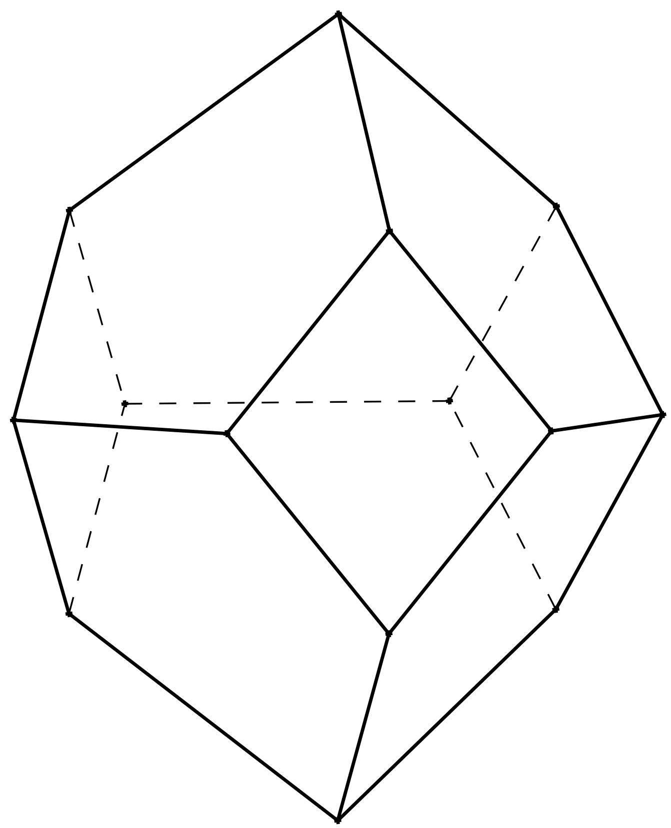
Non-crossing alternating trees

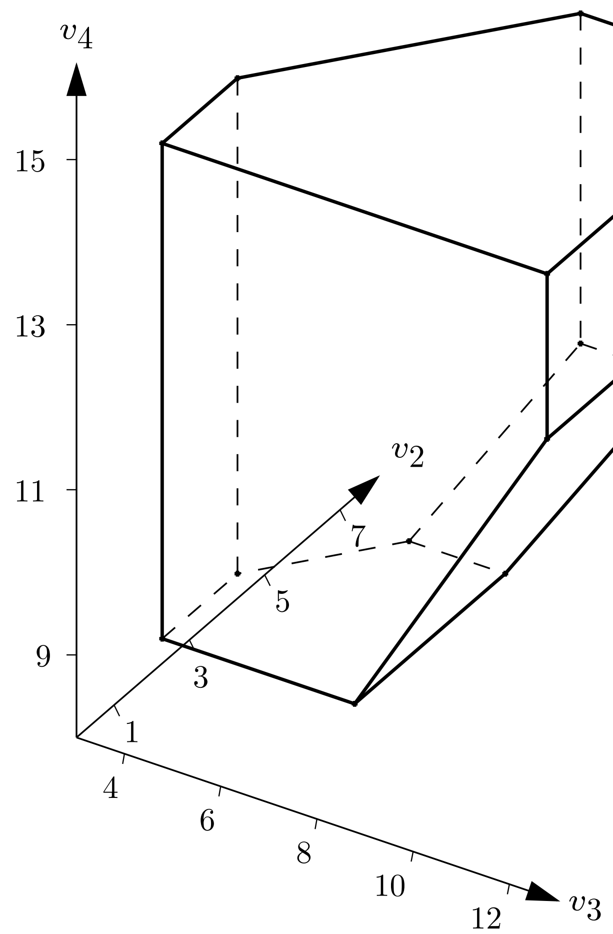


non-crossing: no two edges ik, jl with $i < j < k < l$.
alternating: no two edges ij, jk with $i < j < k$.

[Gelfand, Graev, and Postnikov 1997], in a dual setting.
[Postnikov 1997], [Zelevinsky ?]

The associahedron

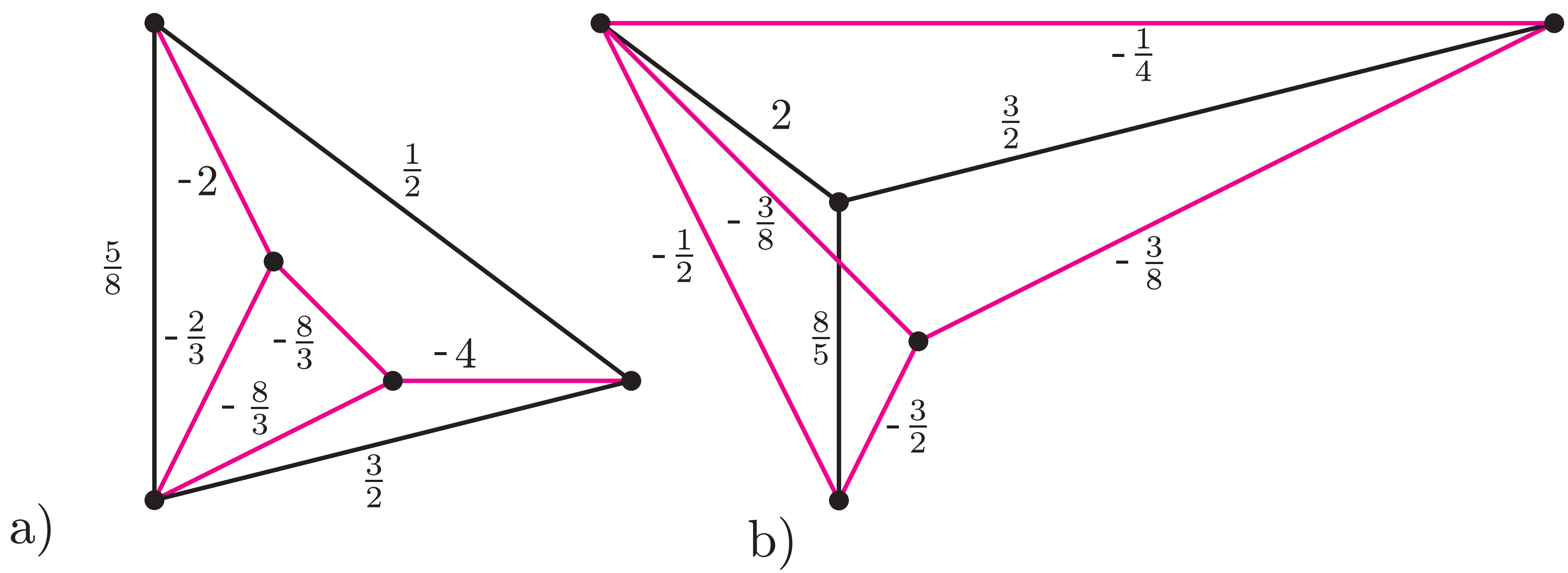




5. Reciprocal diagrams and stresses

Given: A plane graph G and its planar dual G^*.

A framework (G, p) is *reciprocal* to (G^*, p^*) if corresponding edges are parallel.

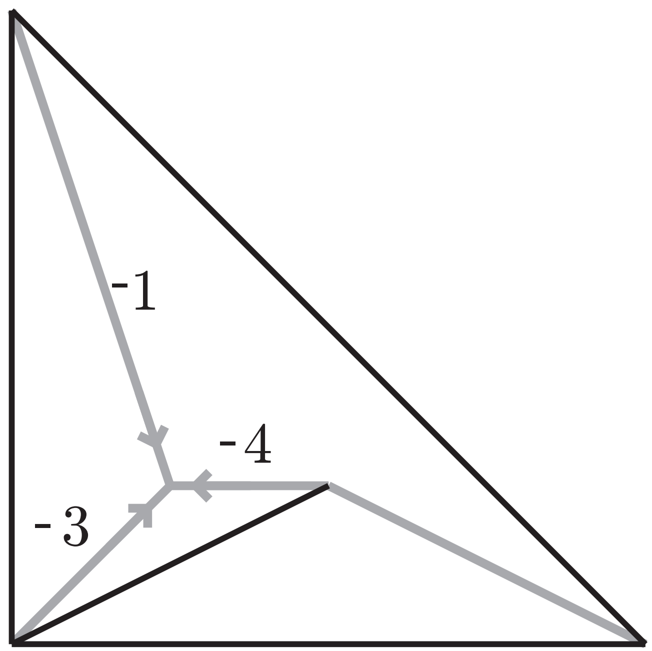


Variation: Maxwell uses *perpendicular* instead of parallel.

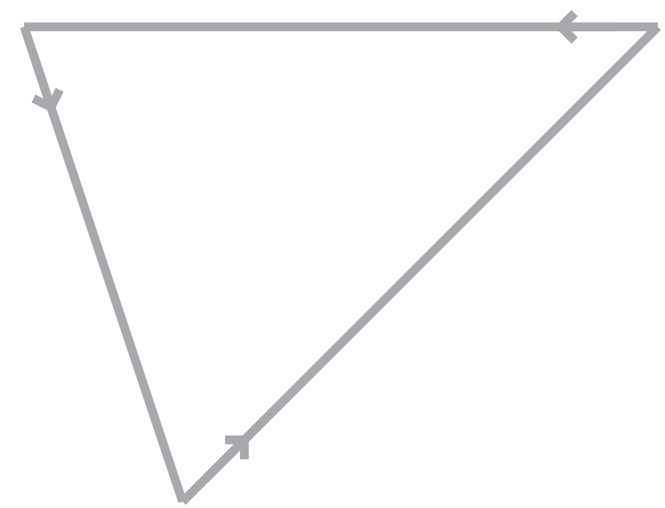
→ dynamic animation of reciprocal diagrams with *Cinderella* dynamic geometry software

Self-stresses and reciprocal frameworks

An equilibrium at a vertex gives rise to a polygon of forces:



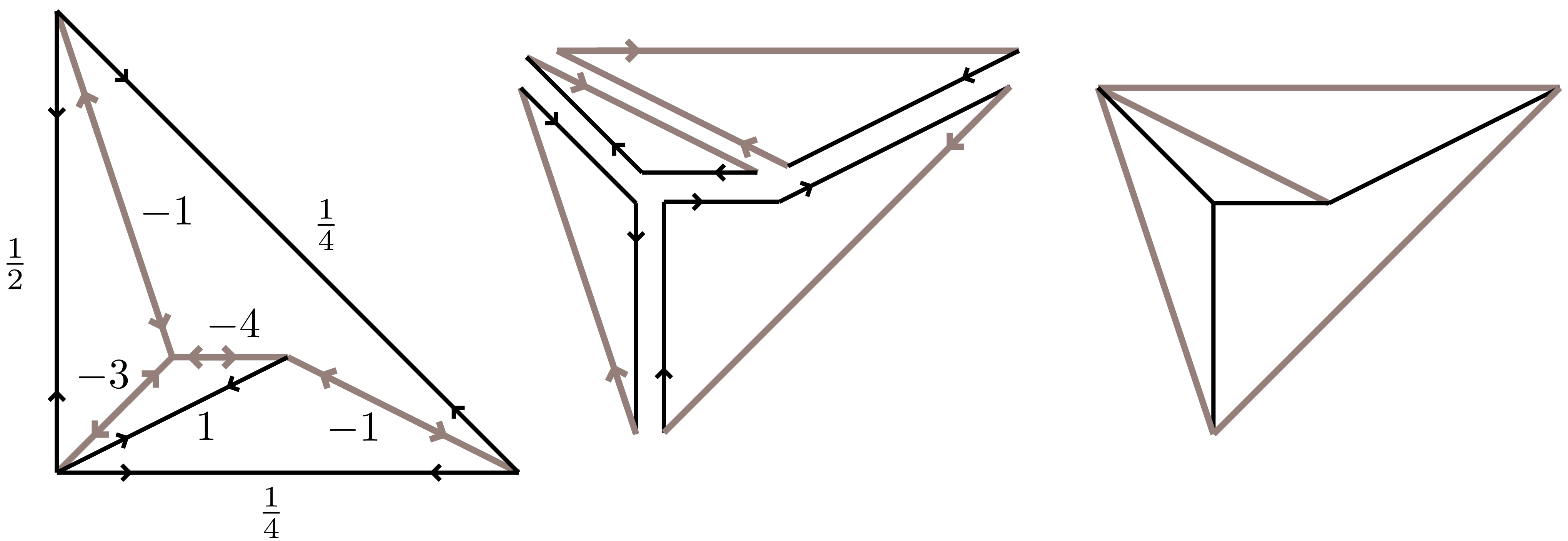
a)



b)

These polygons can be assembled to the reciprocal diagram.

Assembling the reciprocal framework



$\omega_{ij}^* := 1/\omega_{ij}$ defines a self-stress on the reciprocal.

The Maxwell-Cremona Correspondence [1864/1872]

self-stresses on a
planar framework

$\Updownarrow$ one-to-one correspondence

reciprocal diagram

The Maxwell-Cremona Correspondence [1864/1872]

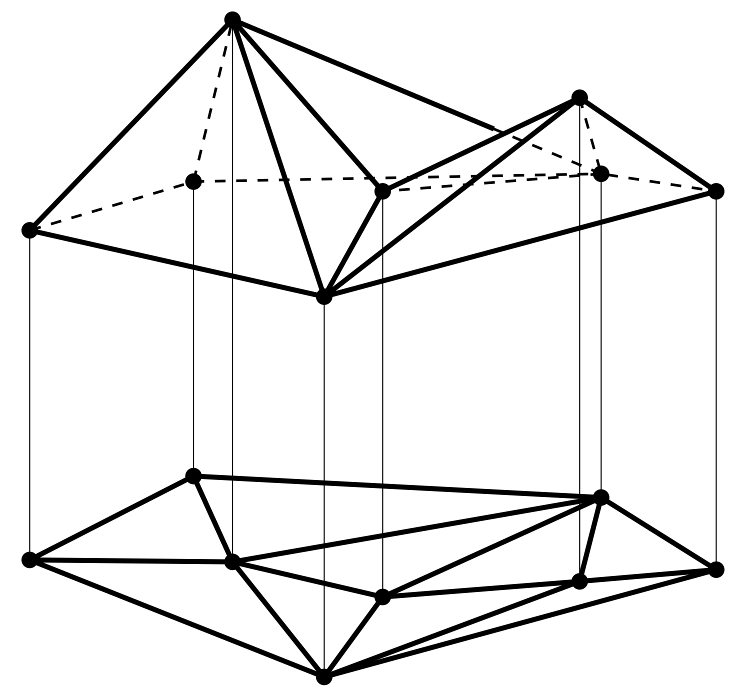
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

reciprocal diagram

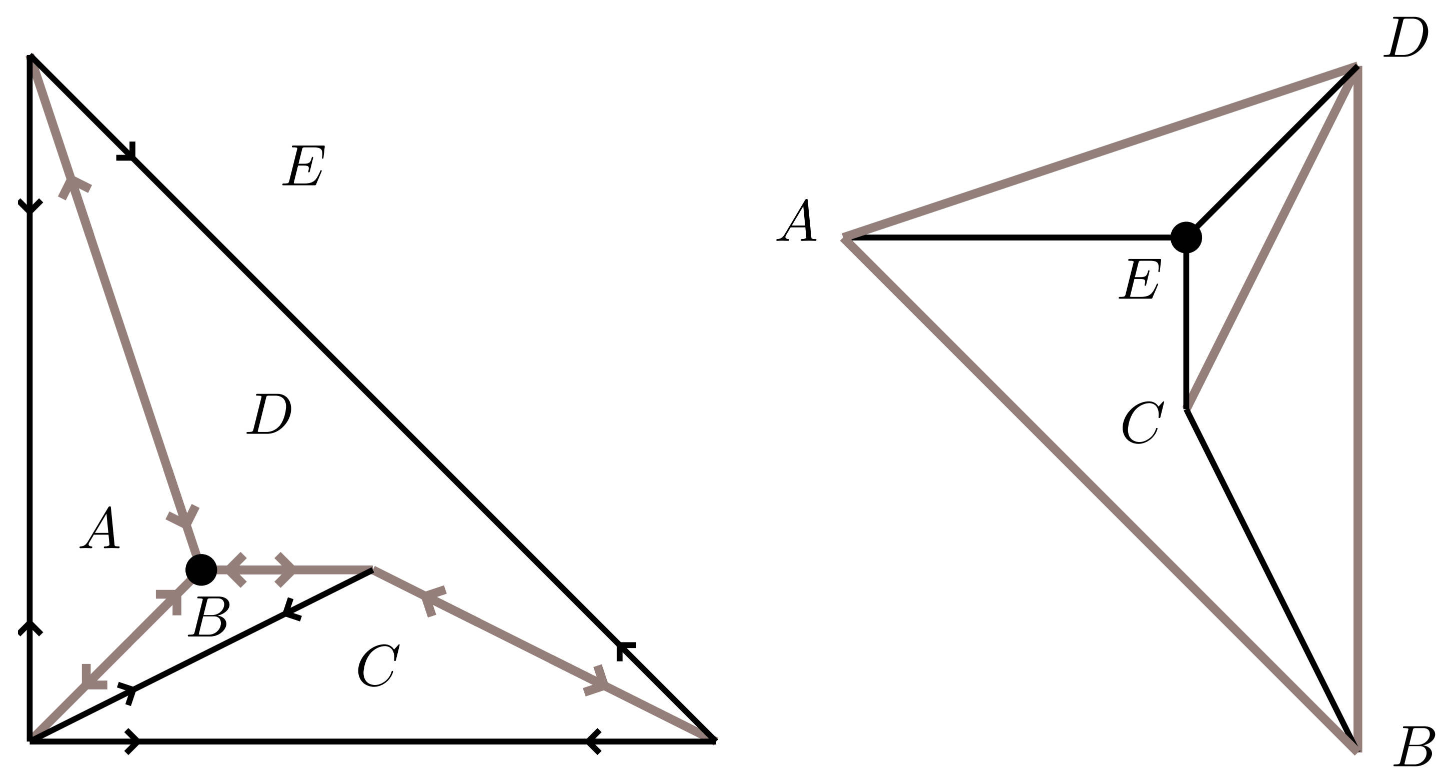
$\Updownarrow$ one-to-one correspondence

3-d lifting (polyhedral terrain)



The Maxwell reciprocal

In the *Maxwell reciprocal*, corresponding edges of the two frameworks (G, p) and (G^*, p^*) are *perpendicular*.



Interpret vertices (vectors) of (G^*, p^*) as *gradients* of faces in the lifted framework (G, p) (and vice versa).

The Maxwell reciprocal

Face f:

$$z = \langle f^*, \begin{pmatrix} x \\ y \end{pmatrix} \rangle + c_f$$

Need to determine scalars c_f (vertical shifts) so that lifted faces share common edges.

Lifted faces f and g in G with gradients f^* and g^*
$\rightarrow$ the intersection of the planes f and g (the lifted edge) is perpendicular to the dual edge f^*g^*.

$f\colon z = \langle f^*, \binom{x}{y} \rangle + c_f$
$g\colon z = \langle g^*, \binom{x}{y} \rangle + c_g$
$f \cup g\colon \langle f^* - g^*, \binom{x}{y} \rangle = c_g - c_f$

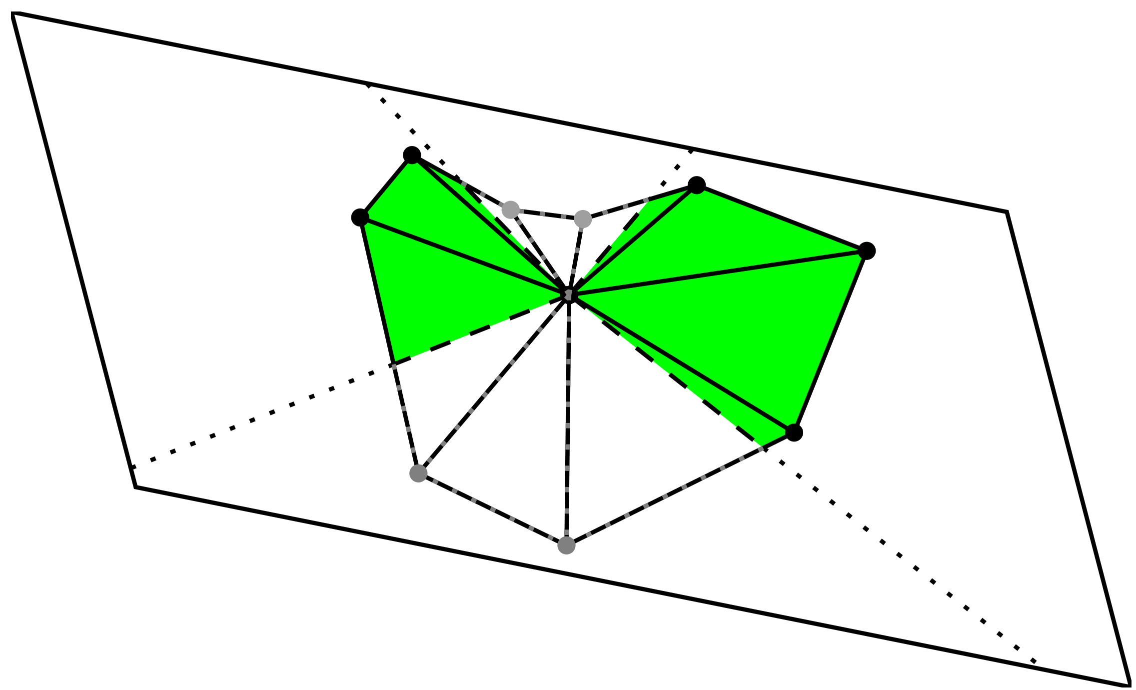
Planar frameworks with planar reciprocals

Theorem. *Let G be a pseudotriangulation with $2n-2$ edges (and hence with a single nonpointed vertex). Then G has a unique self-stress, and the reciprocal G^* is non-crossing. Moreover, if the stress on G is nonzero on all edges, G^* is also a pseudotriangulation with $2n-2$ edges.*

[Orden, Rote, Santos, B. Servatius, H. Servatius, Whiteley 2003]

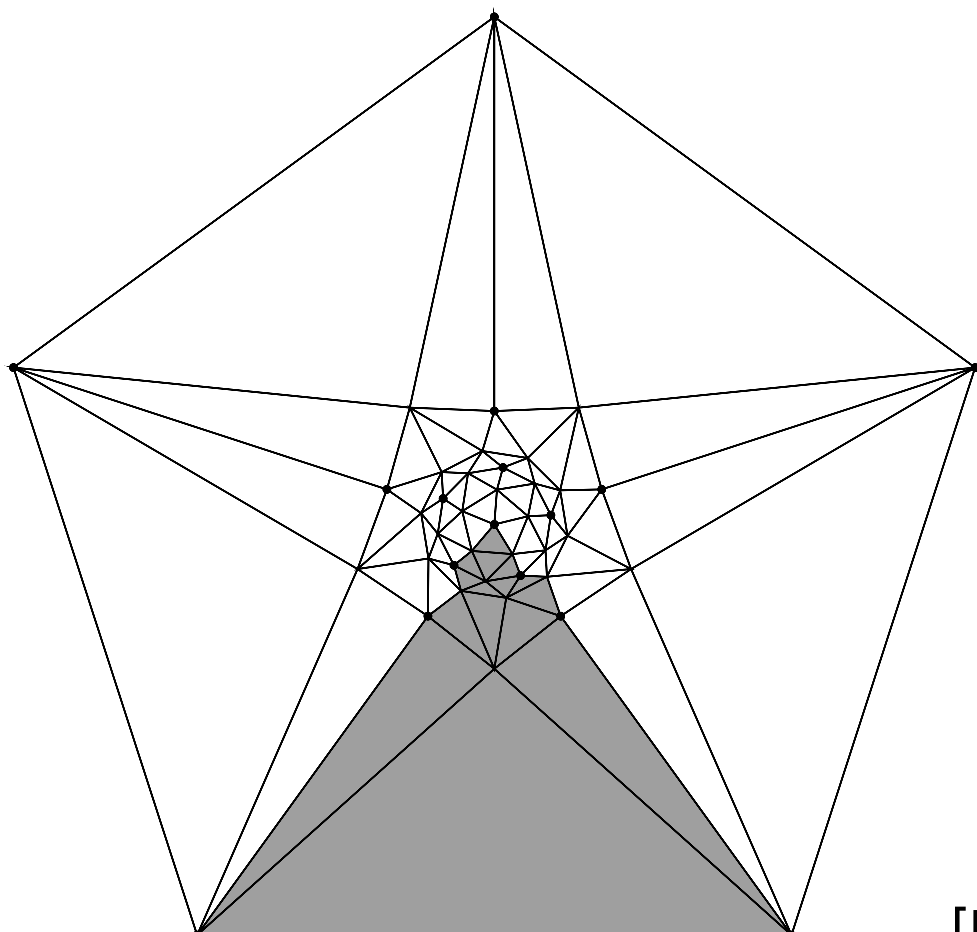
Liftings of non-crossing reciprocals

Theorem. *If G and G^* are non-crossing reciprocals, the lifting has a unique maximum. There are no other critical points. Every other point p is a "twisted saddle": Its neighborhood is cut into four pieces by some plane through v (but not more).*



Minimal pseudotriangulations

Minimal pseudotriangulations (w.r.t. $\subseteq$) are not necessarily minimum-cardinality pseudotriangulations.



A minimal pseudotriangulation has at most $3n - 8$ edges, and this is tight for infinitely many values of n.

(see Exercise 7)

[Rote, C. A. Wang, L. Wang, Xu 2003]

Pseudotriangulations in 3-space?

Rigid graphs are not well-understood in 3-space.

INPUT-A NO INPUT