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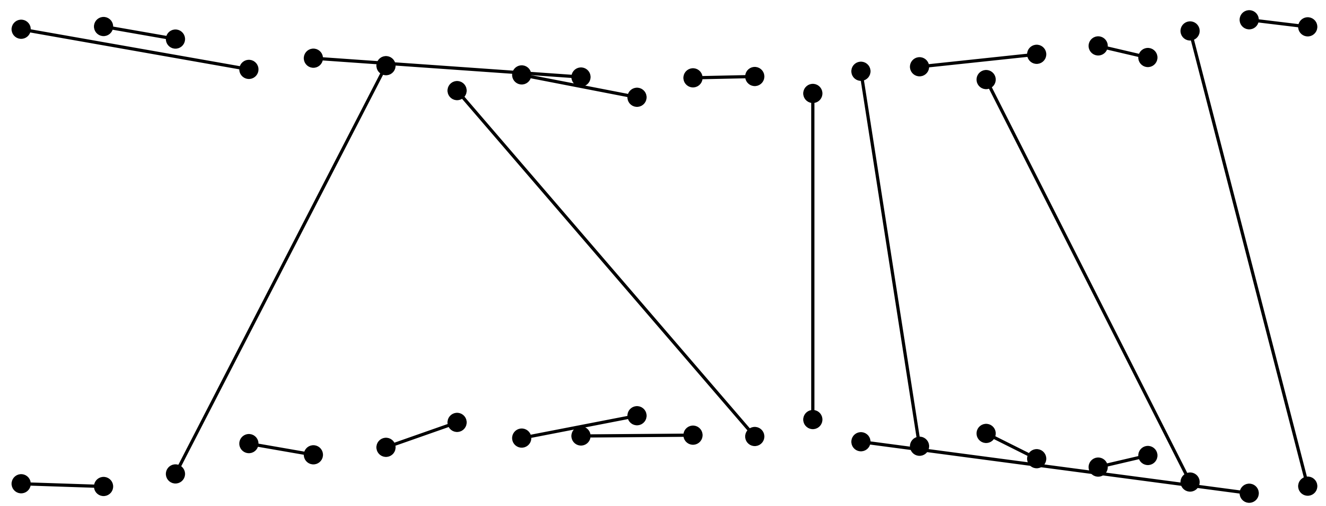
Lattice Paths with States, and Counting Geometric Objects via Production Matrices

Günter Rote
Freie Universität Berlin

ongoing joint work with Andrei Asinowski and Alexander Pilz



a non-crossing perfect matching

Extremal Combinatorial Geometry

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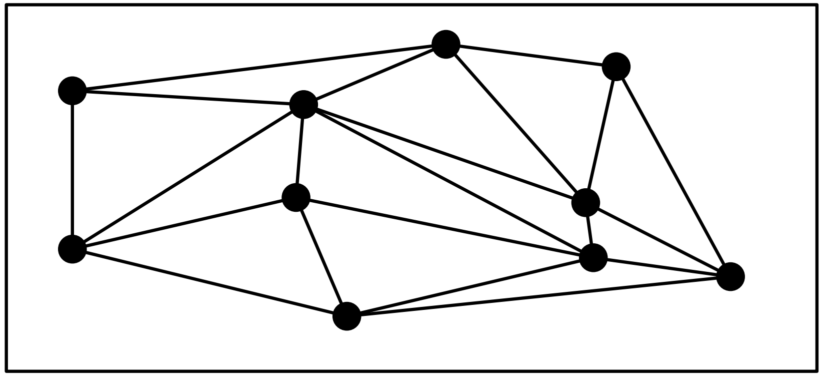
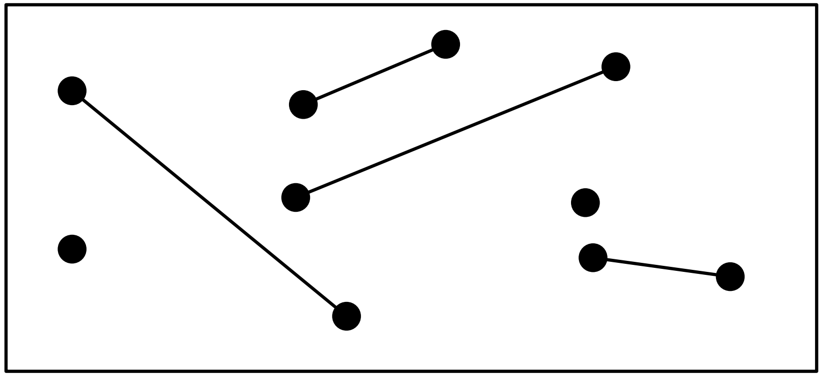
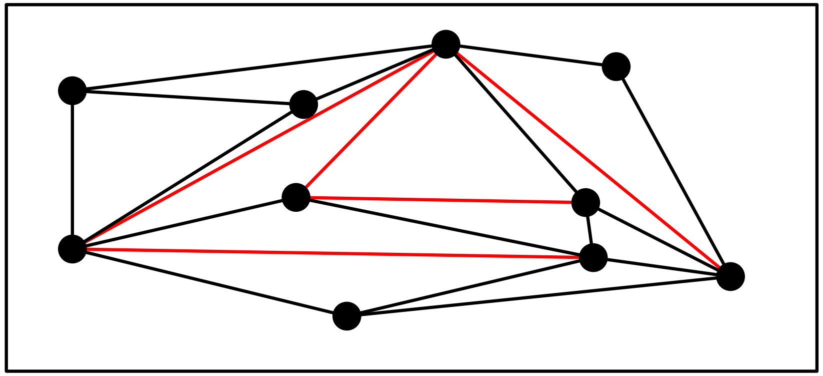
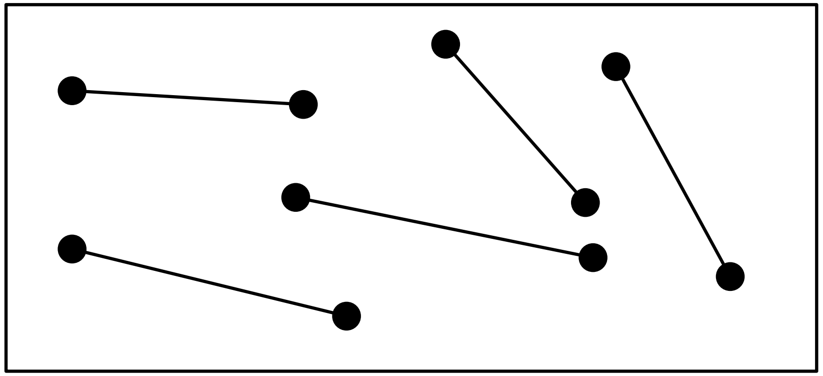
How many $\left\{ \begin{array}{l} \text{triangulations} \\ \text{non-crossing matchings} \\ \text{non-crossing } X \end{array} \right\}$ can a set of n points have $\left\{ \begin{array}{l} \text{at most?} \\ \text{at least?} \end{array} \right.$





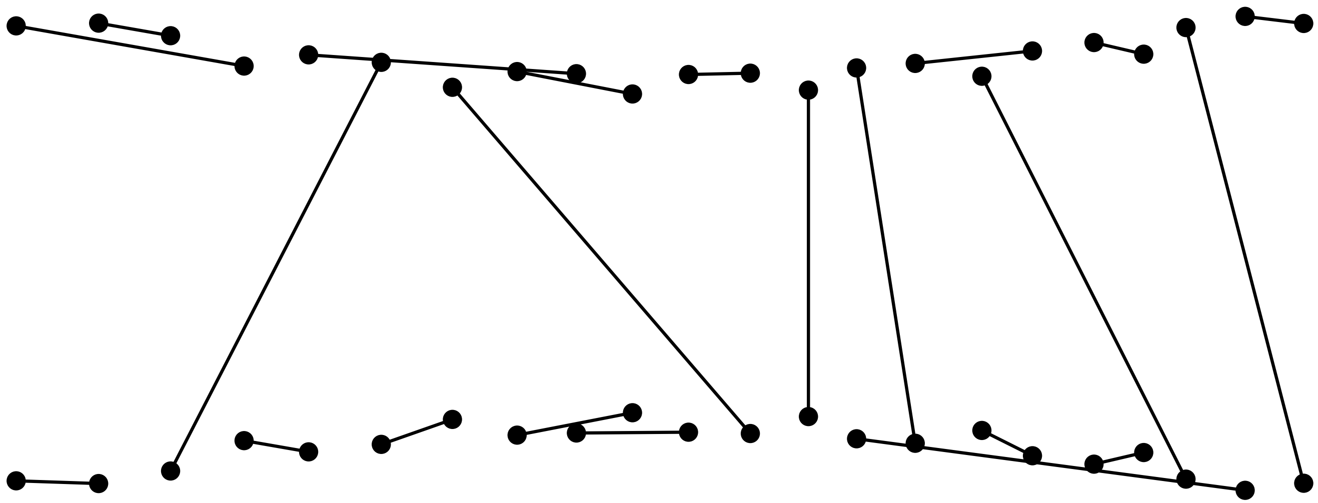




https://adamsheffer.wordpress.com/numbers-of-plane-graphs/

Lower Bound: Explicit Construction

- Think of some type of regular construction
- Find a formula for the number of non-crossing X



a non-crossing
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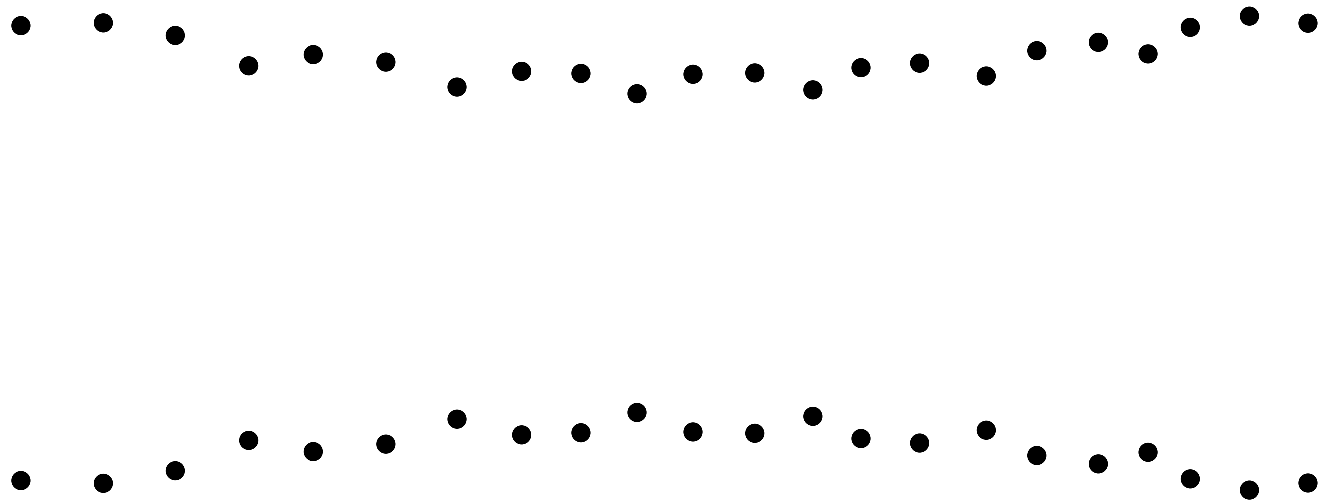

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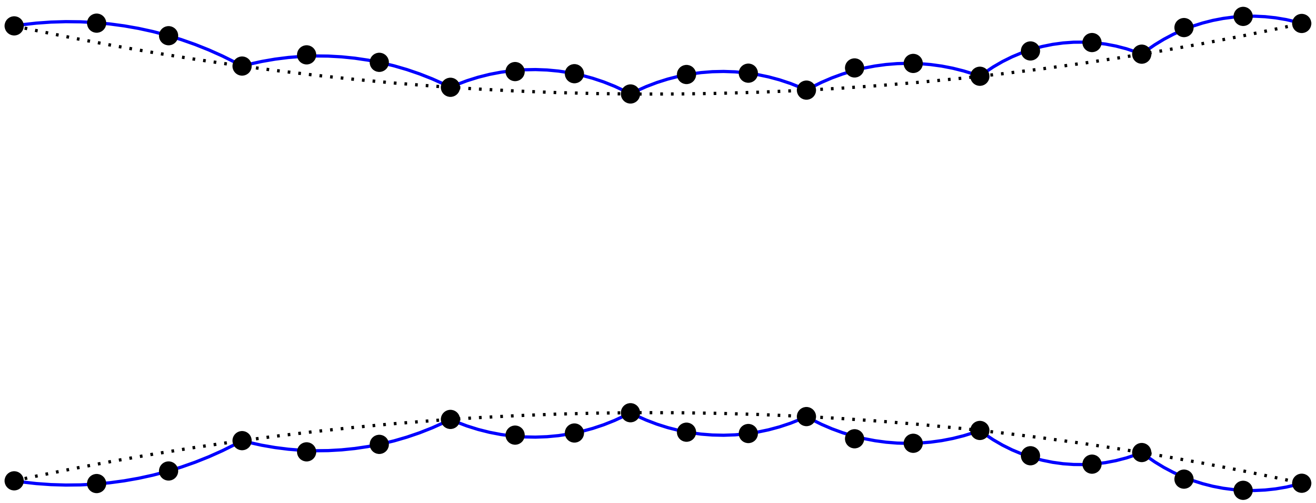
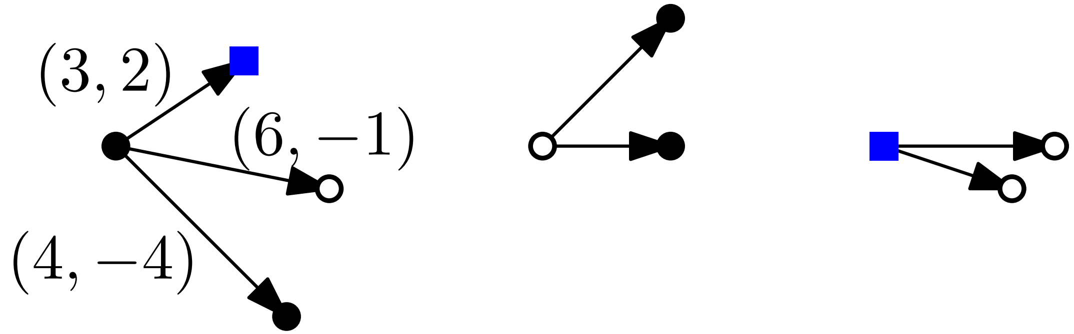


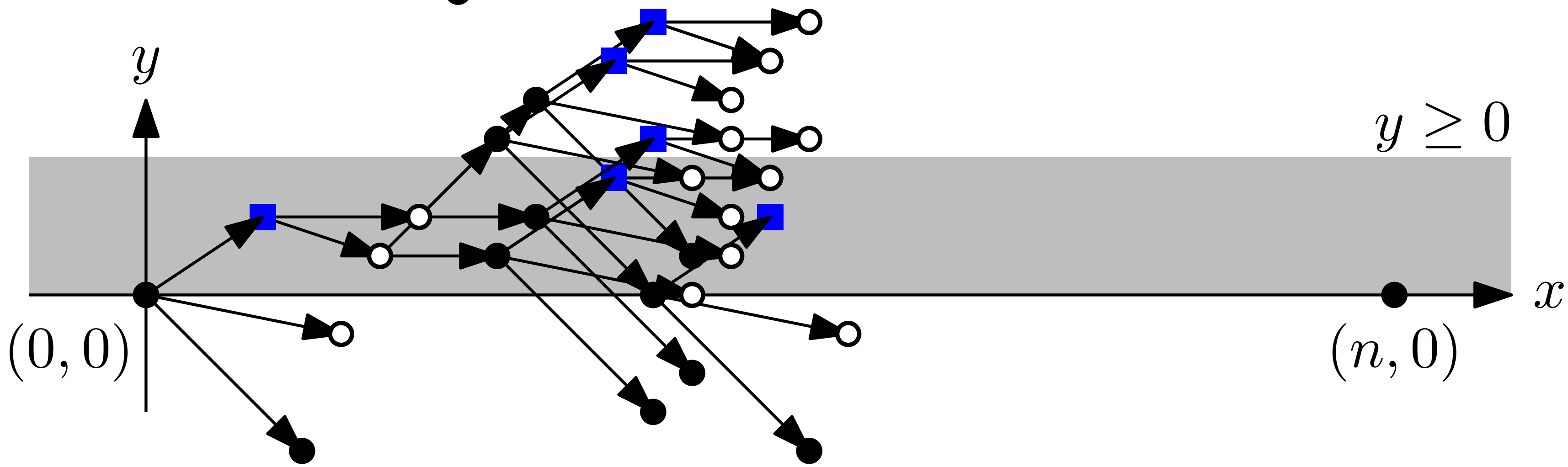
the generalized
double zigzag chain

Lattice Paths with States

- Finite set of *states* $Q = \{\bullet, \circ, \blacksquare, \square, \triangle, \ldots\}$
- For each $q \in Q$, a set S_q of permissible *steps* $((i,j), q')$:

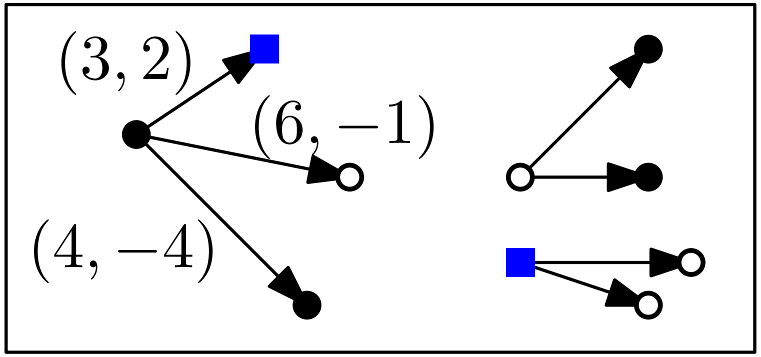
  From point (x,y) in state q, can go to $(x+i, y+j)$ in state q'.





Wanted: The number of paths from $(0,0)$ in state q_0 to $(n,0)$ in state q_1 that don't go below the x-axis.

Formula for Lattice Paths with States



$$(i, j) \mapsto t^i u^j$$

$A(t, u) =$ *characteristic matrix*

$$\left(\begin{array}{c|ccc} & \bullet & \circ & \blacksquare \\ \hline \bullet & t^4u^{-4} & t^6u^{-1} & t^3u^2 \\ \circ & t^3 + t^3u^3 & 0 & 0 \\ \blacksquare & 0 & t^4 + t^3u^{-1} & 0 \end{array}\right)$$

Conjecture: The number of paths from $(0, 0)$ in state q_0 to $(n, 0)$ in state q_1 that don't go below the x-axis is

$$\sim \text{const} \cdot (1/t^*)^n \cdot n^{-3/2},$$

where

(1) $A(t^*, u^*)$ has largest (Perron-Frobenius) eigenvalue 1.
$[\implies \det(A(t, u) - I) = 0\ ]$

(2) $u^* > 0$ is chosen such that the value $t^* > 0$ that fulfills (1) is as large as possible. $[\implies \frac{\partial}{\partial u} \det(A(t, u) - I) = 0\ ]$

Formula for Lattice Paths with States

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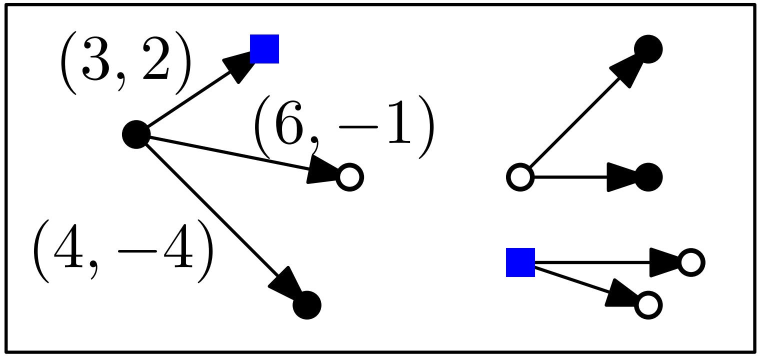


$$(i, j) \mapsto t^i u^j$$

$A(t, u) =$ *characteristic matrix*

	●	○	■
●	t^4u^{-4}	t^6u^{-1}	t^3u^2
○	$t^3 + t^3u^3$	0	0
■	0	$t^4 + t^3u^{-1}$	0

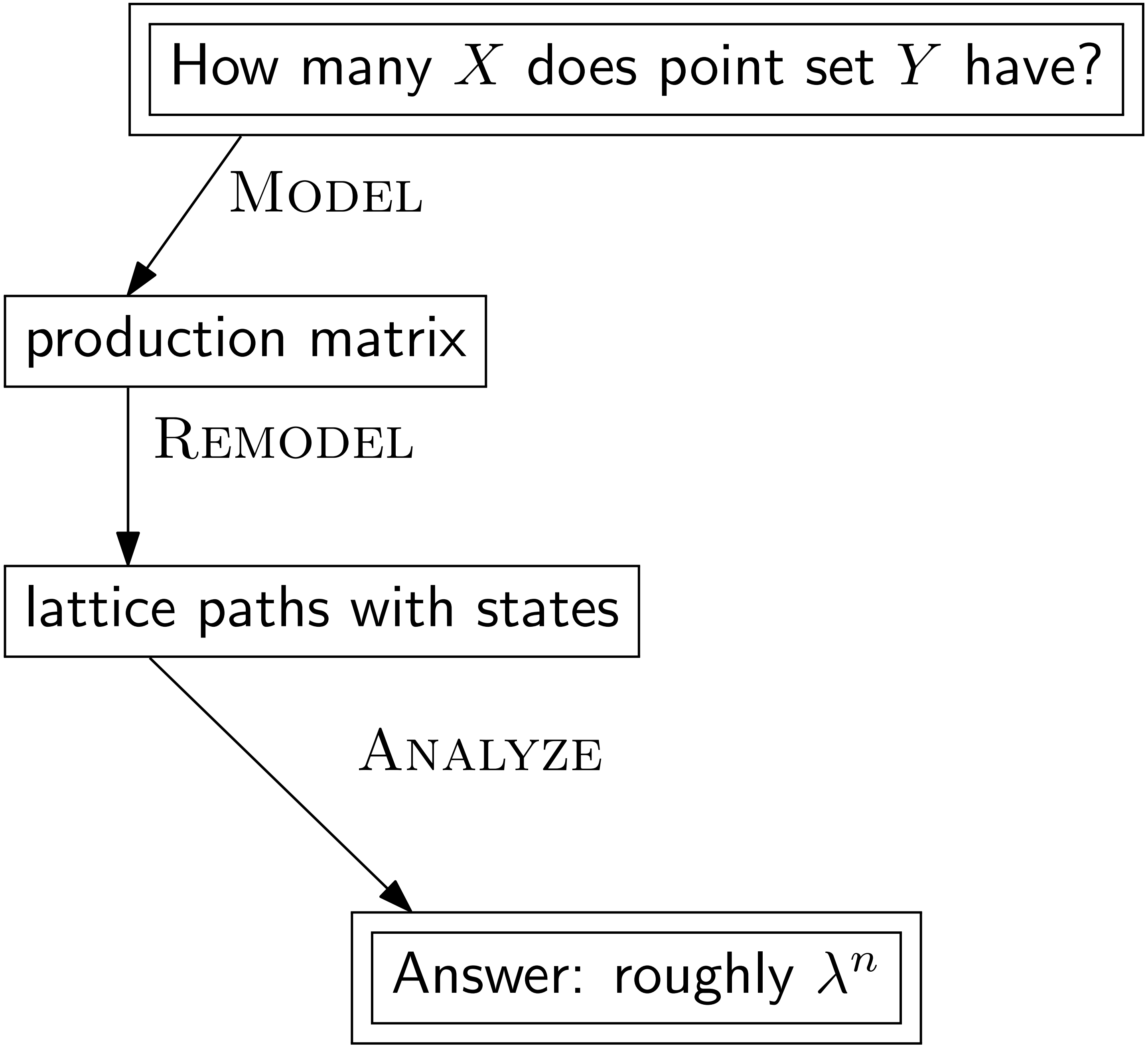
Conjecture: The number of paths from $(0, 0)$ in state q_0 to $(n, 0)$ in state q_1 that don't go below the x-axis is

$$\sim \text{const} \cdot (1/t^*)^n \cdot n^{-3/2},$$

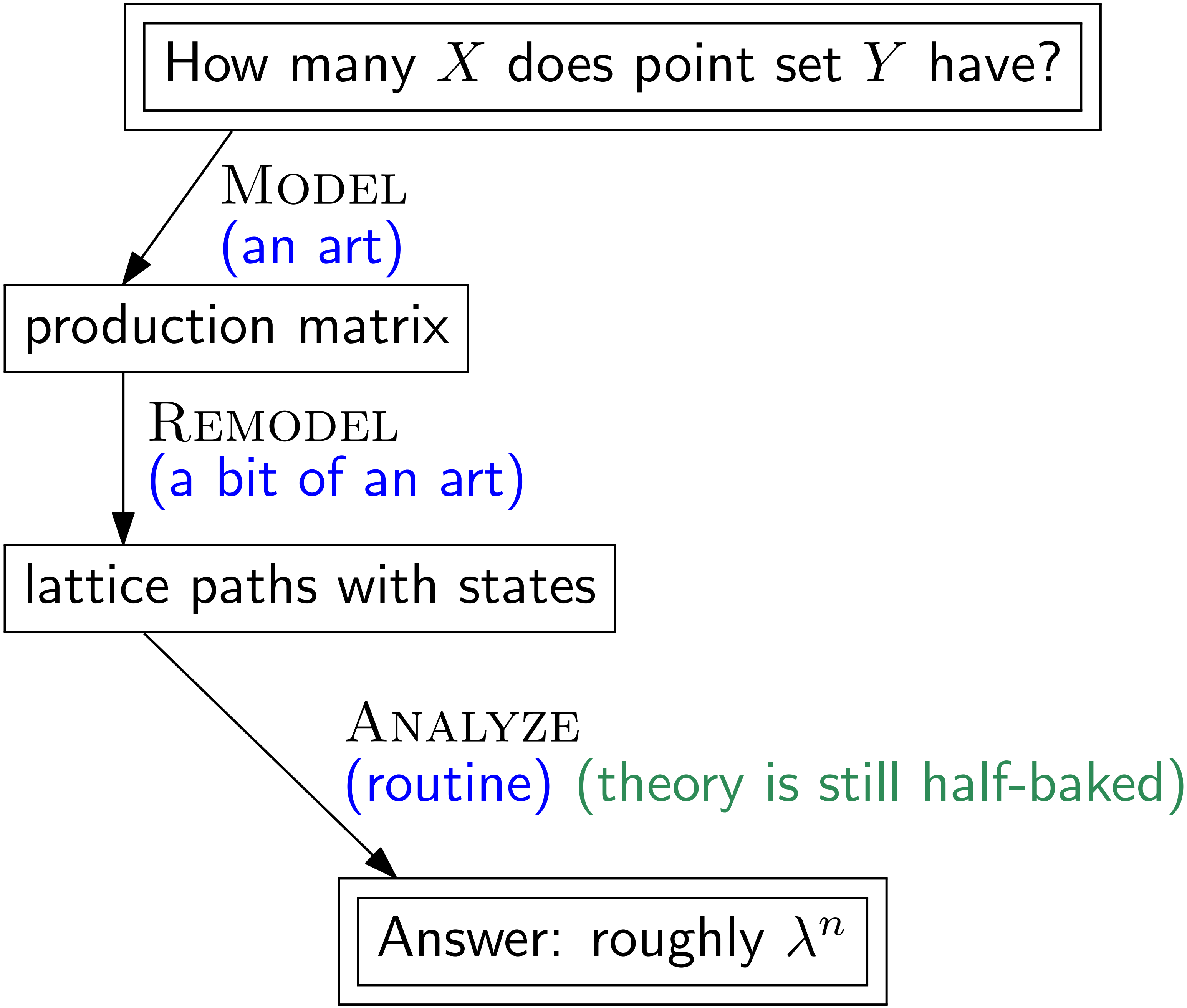
under some obvious *technical conditions*:

- state graph is strongly connected
- no cycles in the lattice paths
- aperiodic
- . . .

Method Pipeline



Method Pipeline



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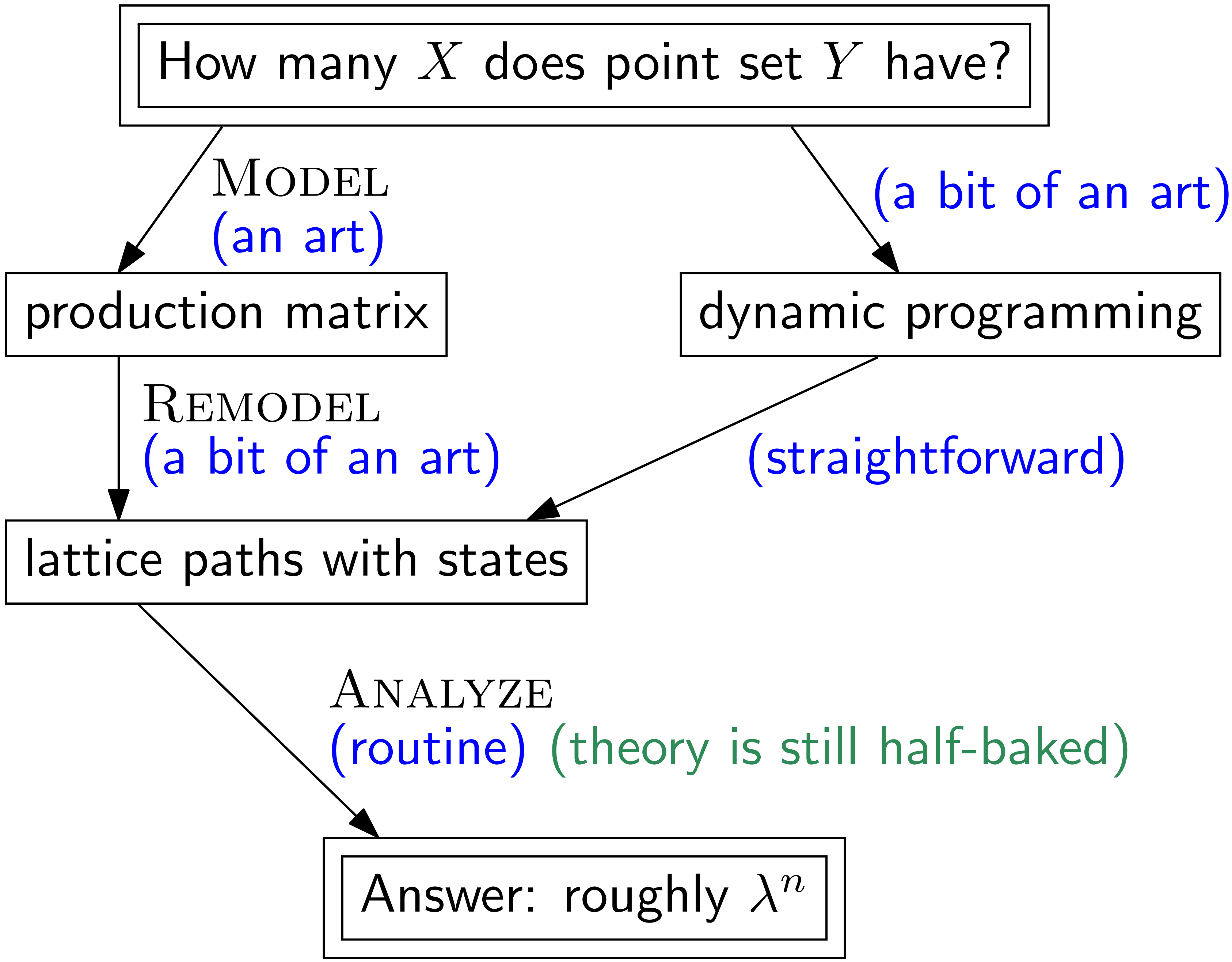

Overview

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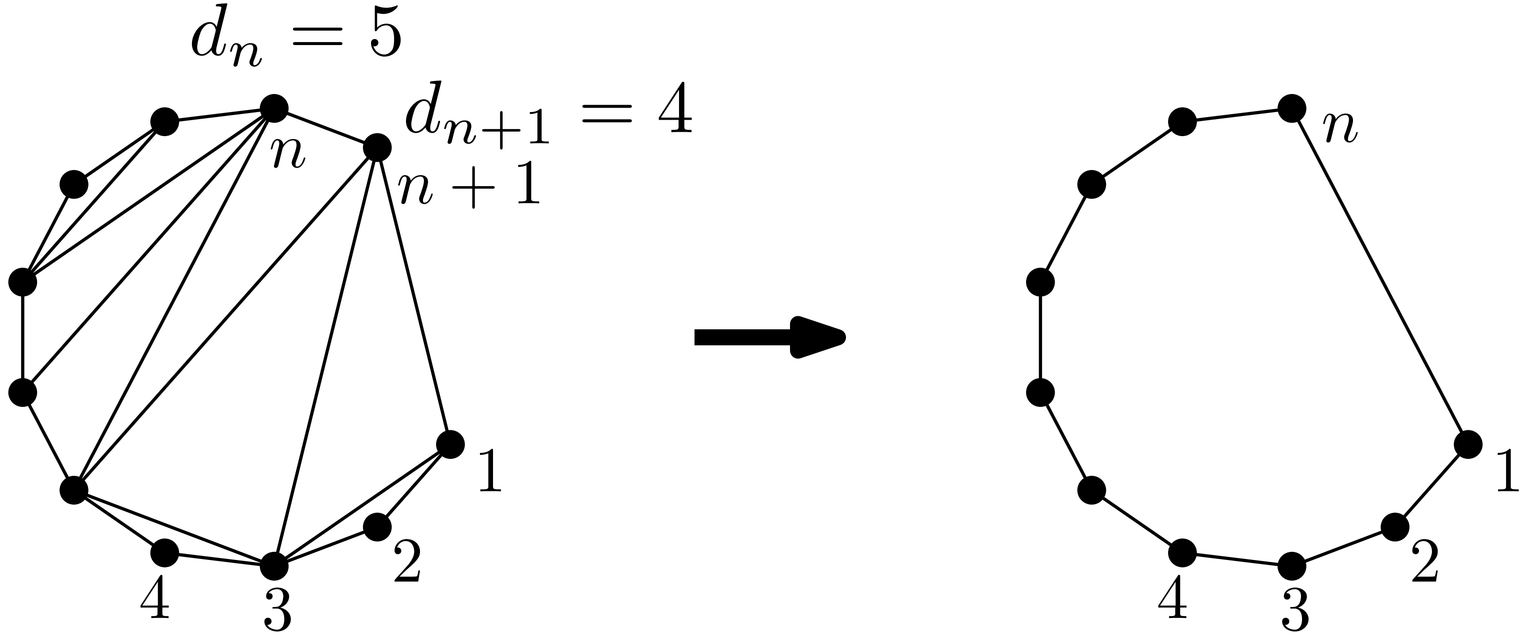
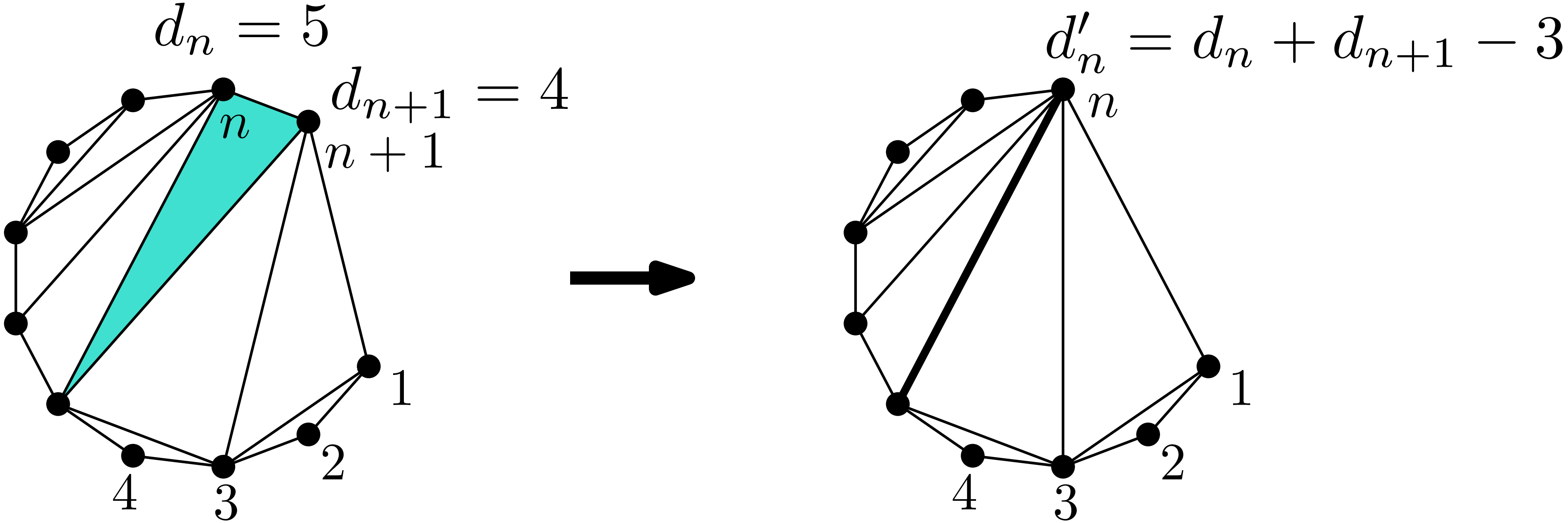
- Introduction. Point sets with many noncrossing X
- The lattice path formula with states (preview)
- Method pipeline
- Overview
- Example 1: Triangulations of a convex n-gon
- Production matrices
- Example 2: Noncrossing forests in a convex n-gon
- Example 3: The generalized double zigzag chain.

- Proof idea 1. Analytic combinatorics
- Proof idea 2. Random walk

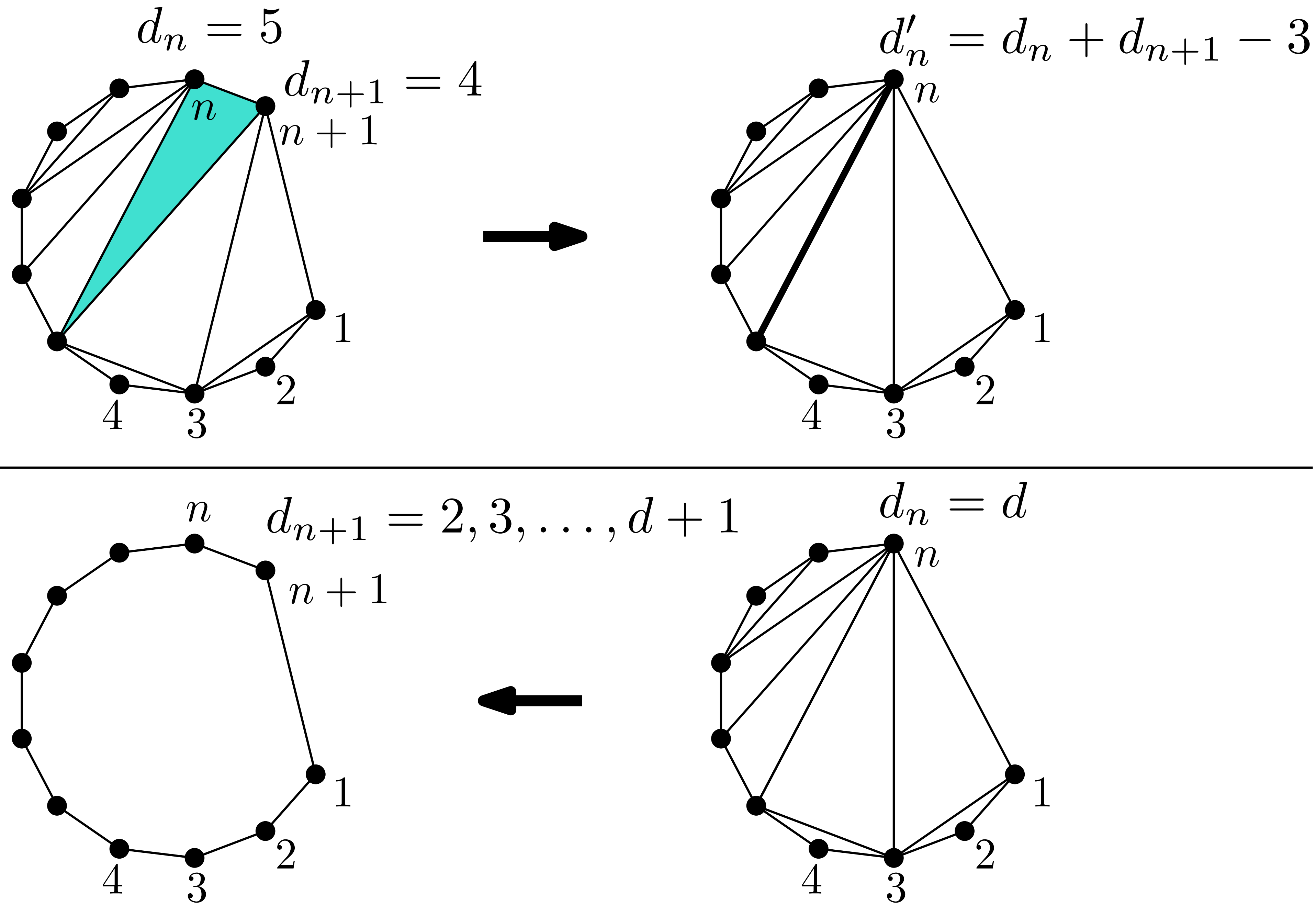
Triangulations of a convex n-gon



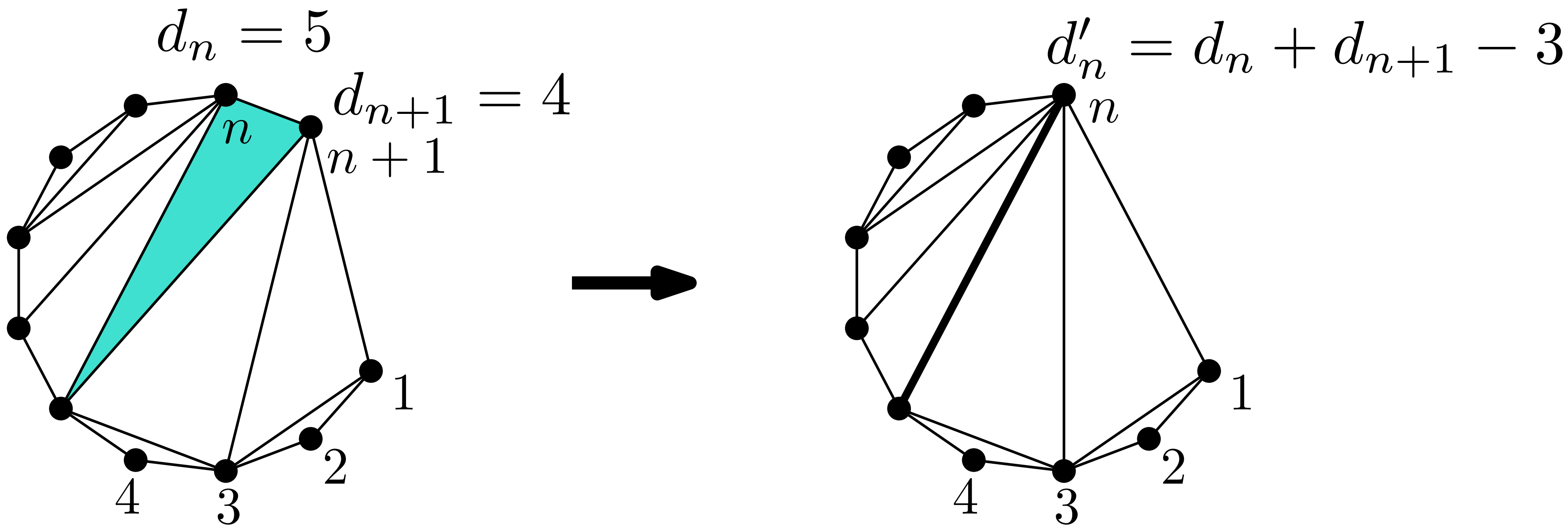
Triangulations of a convex n-gon

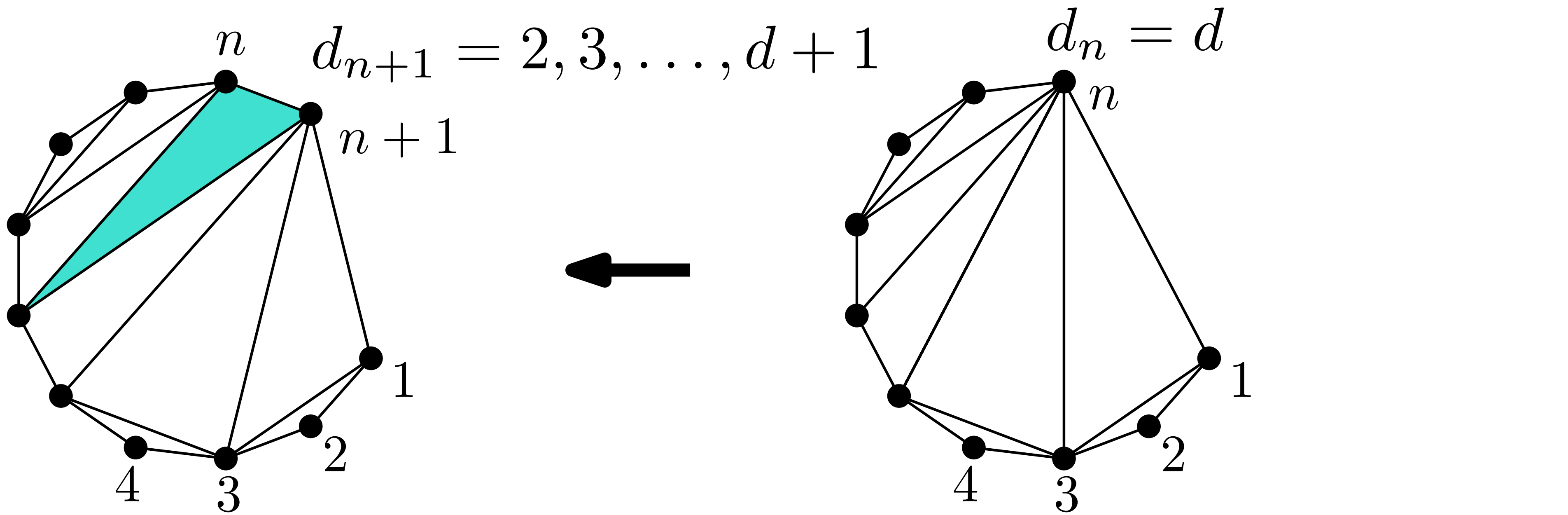


Triangulations of a convex n-gon



Triangulations of a convex n-gon





Triangulations of a convex n-gon

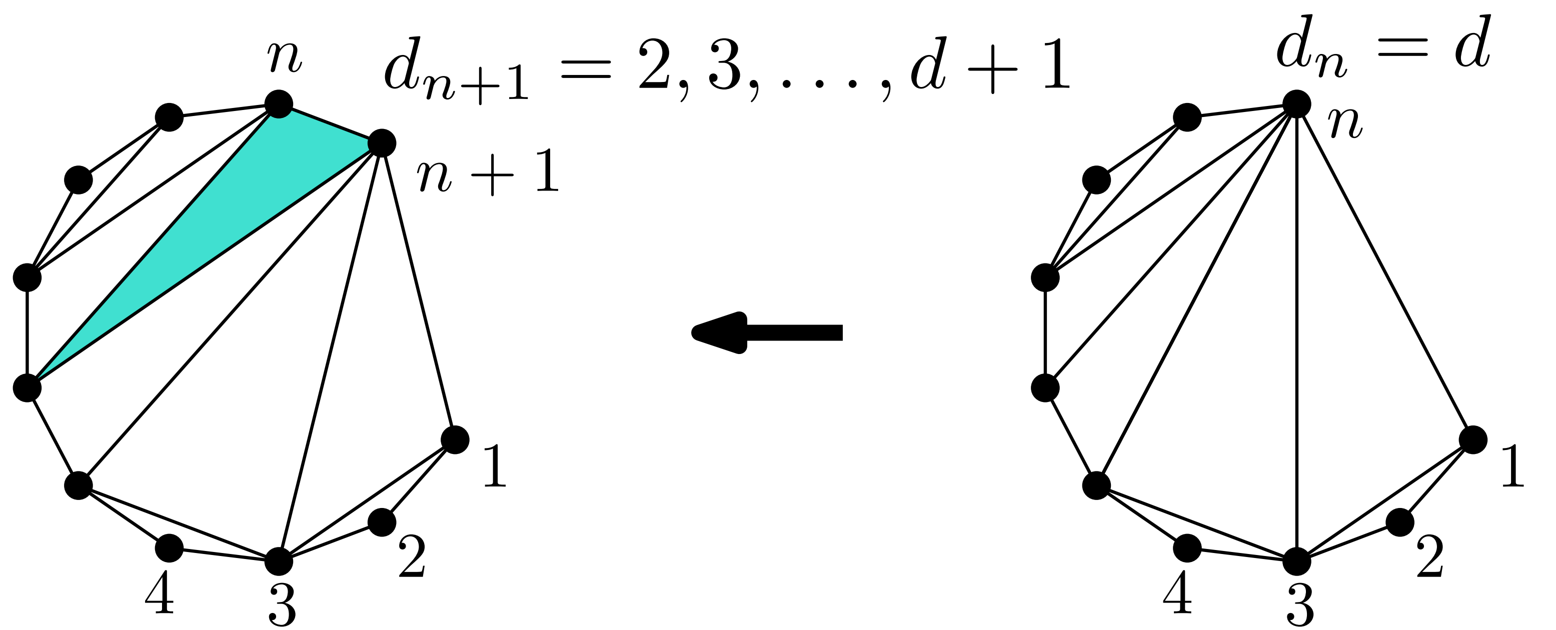
Triangulation of n-gon with last vertex of degree $d_n = d$

$\rightarrow$

Triangulation of $(n+1)$-gon with last vertex of degree

$$d_{n+1} = 2 \text{ or } 3 \text{ or } 4 \text{ or } \ldots \text{ or } d, \text{ or } d+1$$

[Hurtado & Noy 1999]
"tree of triangulations"



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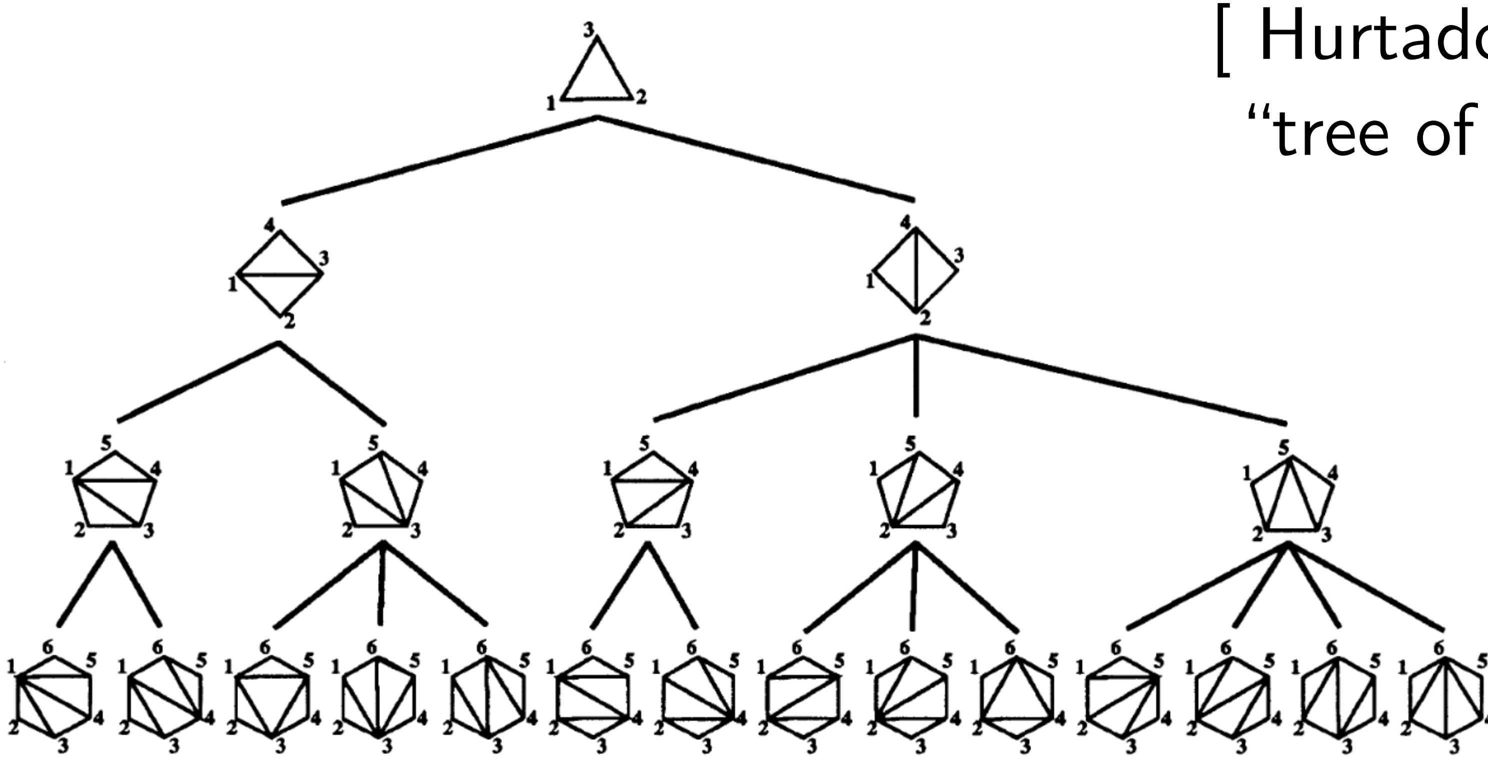

Fig. 4. Levels three to six of the tree of triangulations.

Triangulations of a convex n-gon

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Triangulation of n-gon with last vertex of degree $d_n = d$

$\rightarrow$

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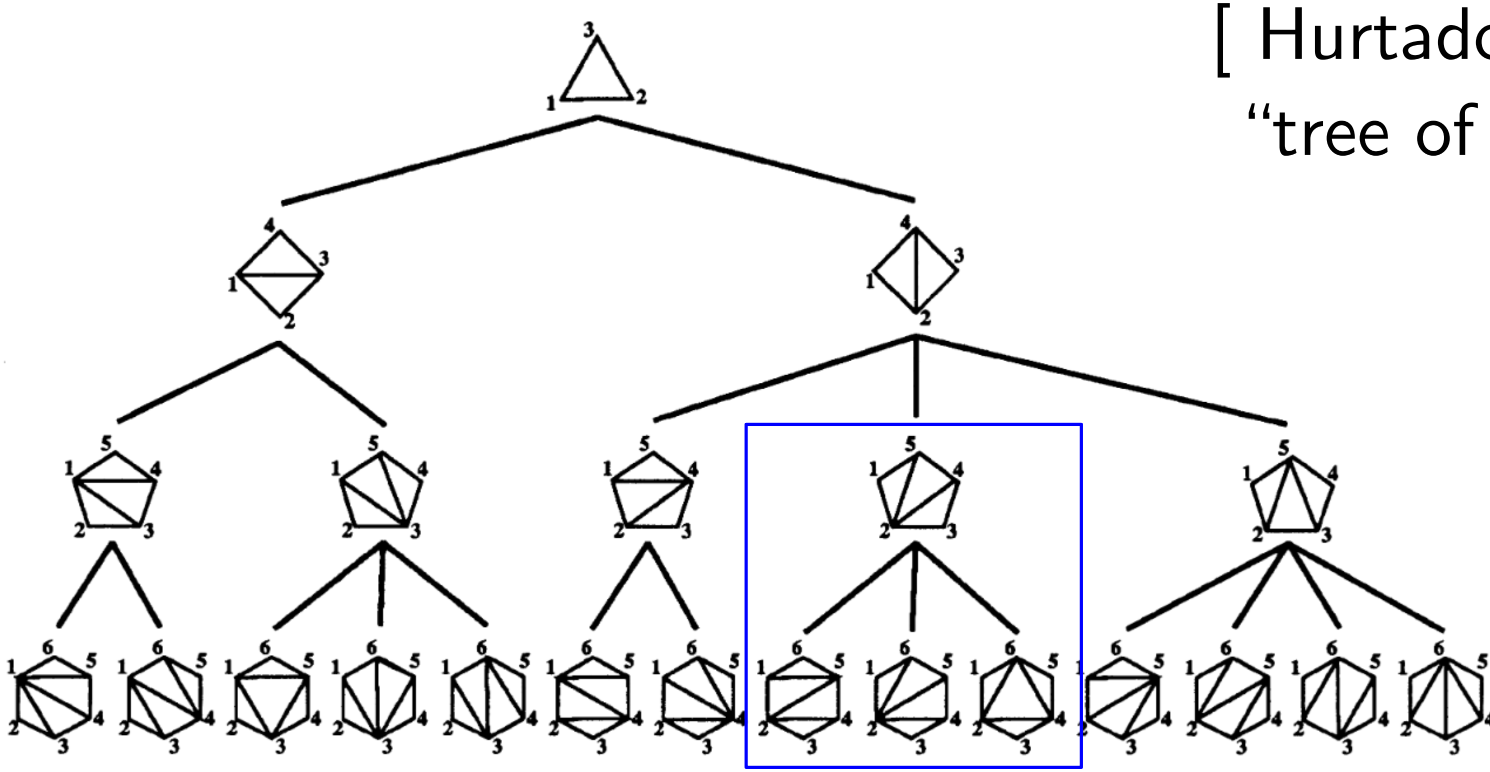

Fig. 4. Levels three to six of the tree of triangulations.

Triangulations of a convex n-gon

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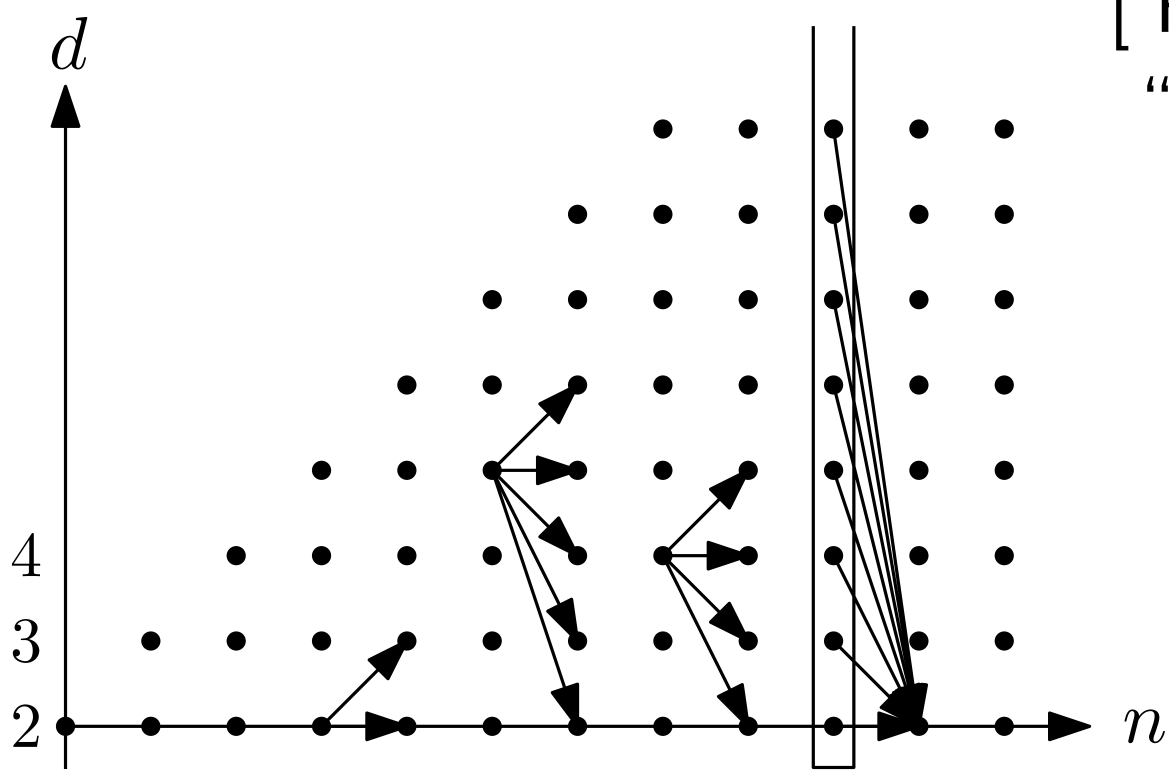
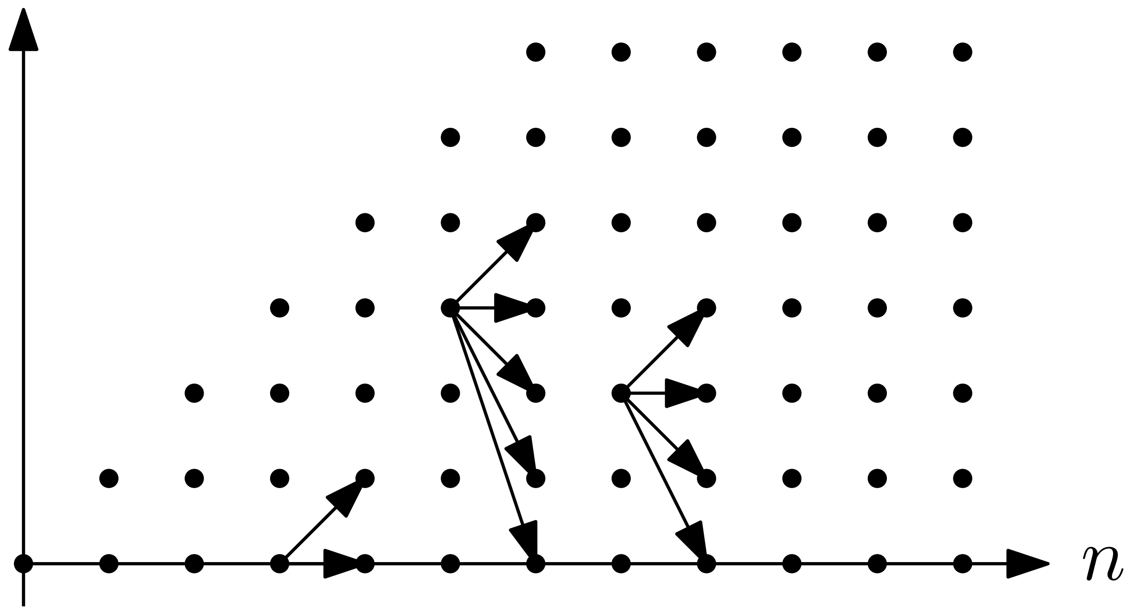


triangulation
$\updownarrow$
lattice path

Production matrices



count paths in a layered graph

The answer is

$$\begin{pmatrix}1 & 0 & 0 & \dots\end{pmatrix}\underbrace{\begin{pmatrix}1 & 1 & 1 & 1 & 1 & \dots\\ 1 & 1 & 1 & 1 & 1 & \dots\\ 0 & 1 & 1 & 1 & 1 & \dots\\ 0 & 0 & 1 & 1 & 1 & \dots\\ 0 & 0 & 0 & 1 & 1 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}^n}_{\text{the “production matrix” } P}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix}$$

Production matrices for enumeration

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were introduced by Emeric Deutsch, Luca Ferrari, and Simone Rinaldi (2005).

were used for counting noncrossing graphs for points in convex position: Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 5 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 1 & 2 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 4 & 5 & \dots \\ 0 & 1 & 3 & 4 & \dots \\ 0 & 0 & 1 & 3 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

matchings — spanning trees — forests

Making the degree finite

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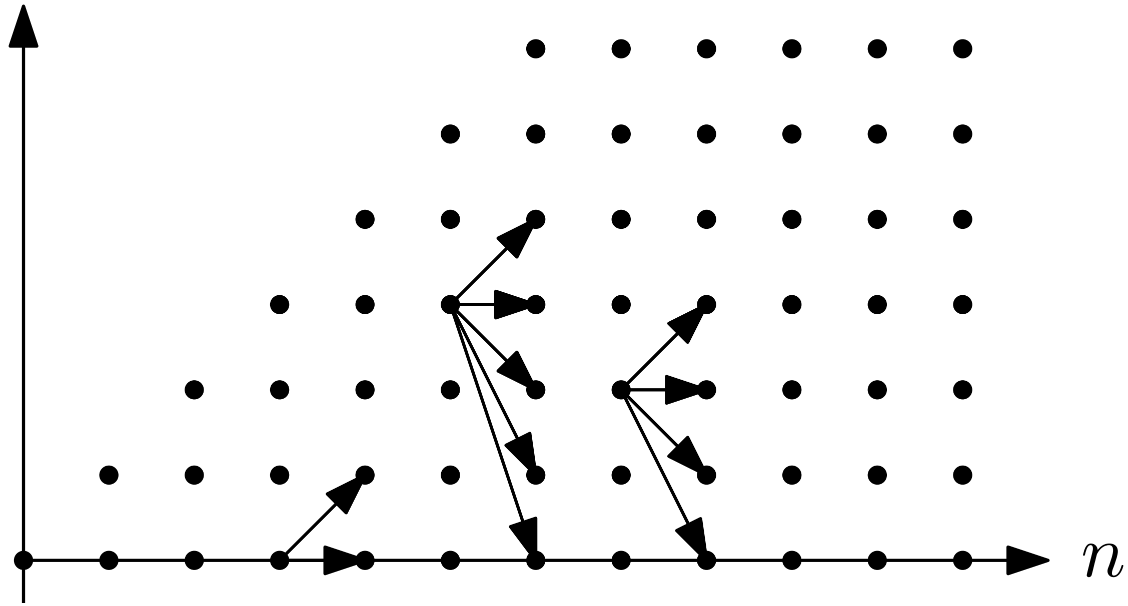
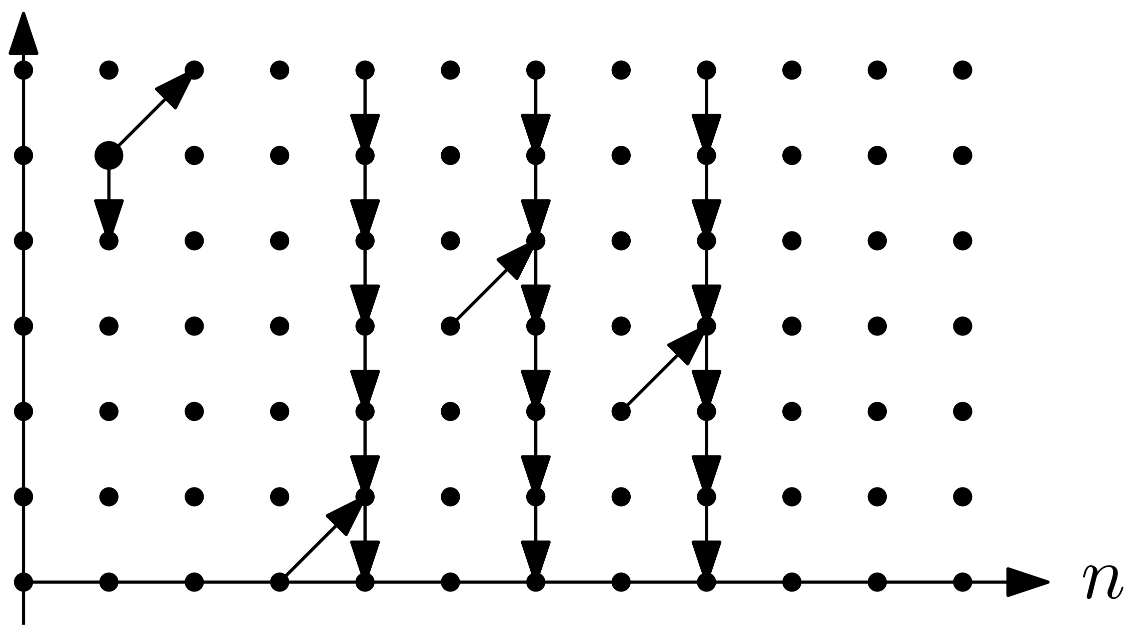
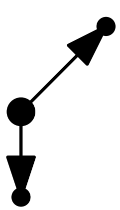


METHOD I:
vertical edges for partial summation

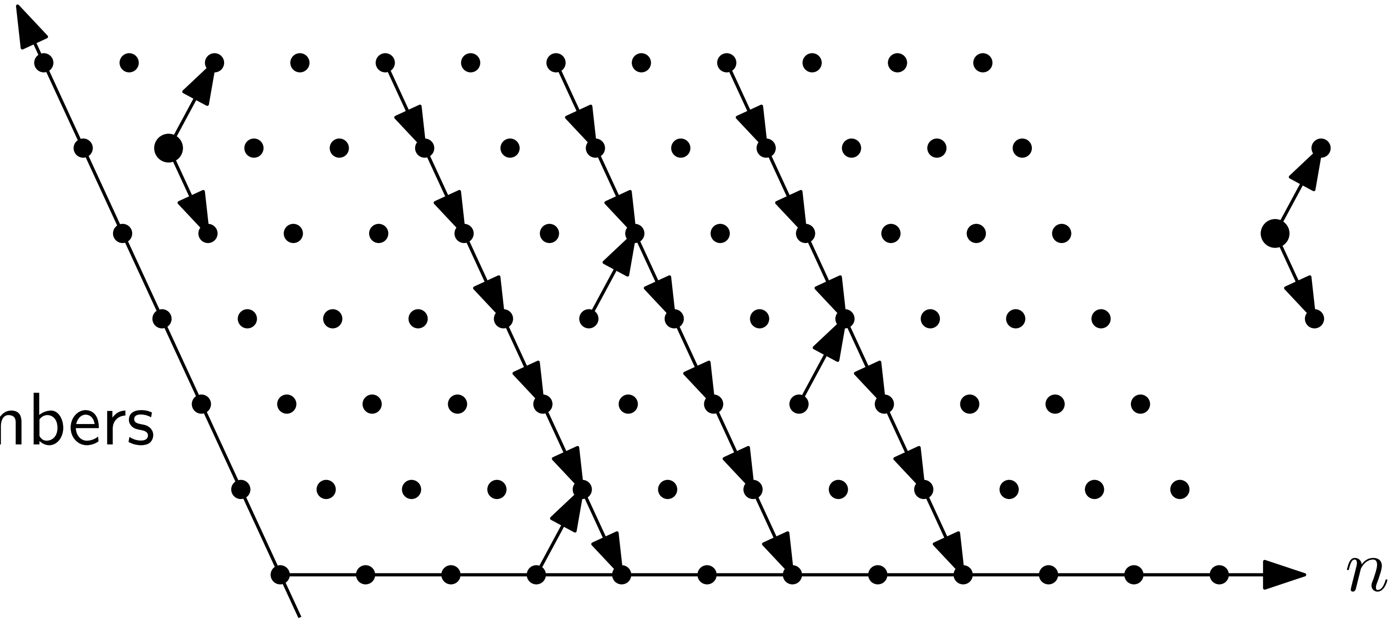


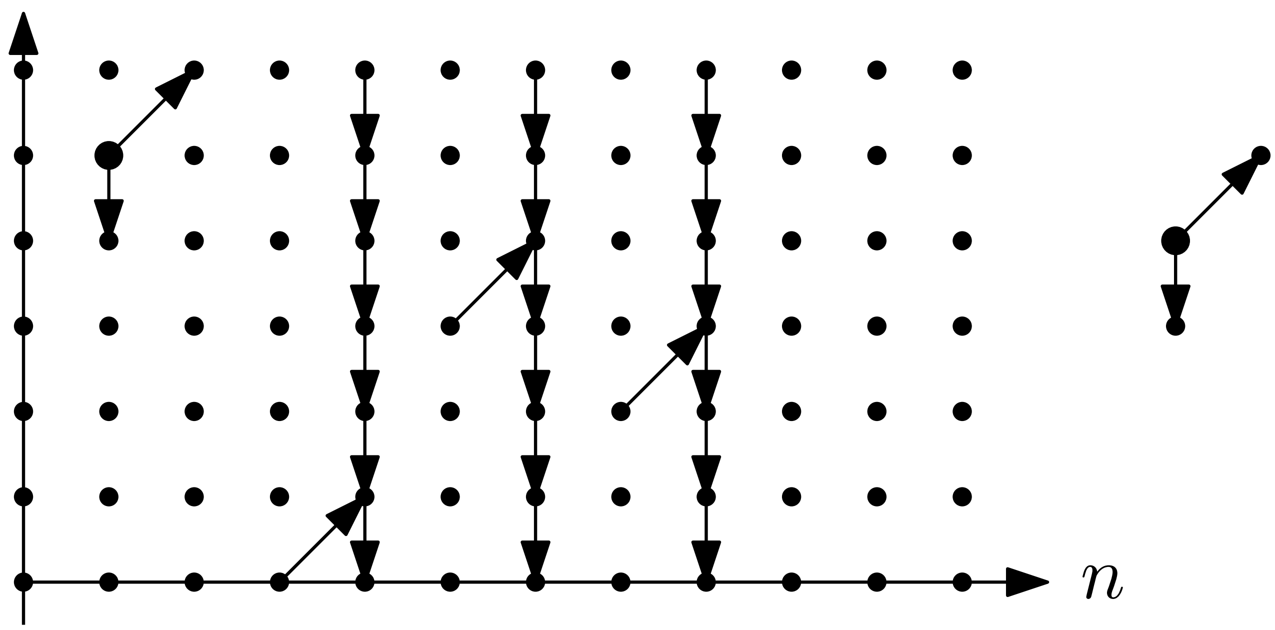


Number of paths is preserved:
1 forward step
+ any number of vertical steps

Making the degree finite

Shearing
→ Dyck paths
→ Catalan numbers





Number of paths is preserved:

1 forward step + any number of vertical steps

Example 2: Forests

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 4 & 5 & 6 & \dots \\ 0 & 1 & 3 & 4 & 5 & \dots \\ 0 & 0 & 1 & 3 & 4 & \dots \\ 0 & 0 & 0 & 1 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Irregularities at the boundary can be ignored.

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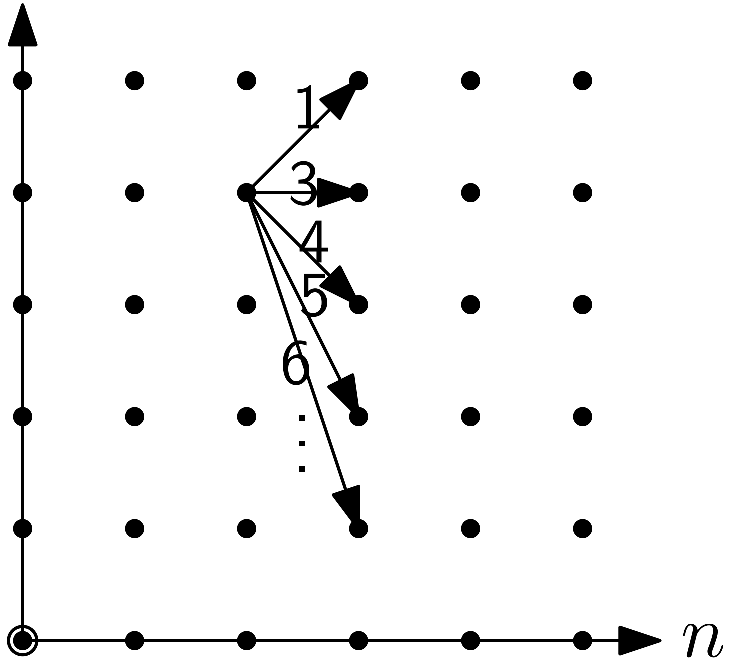

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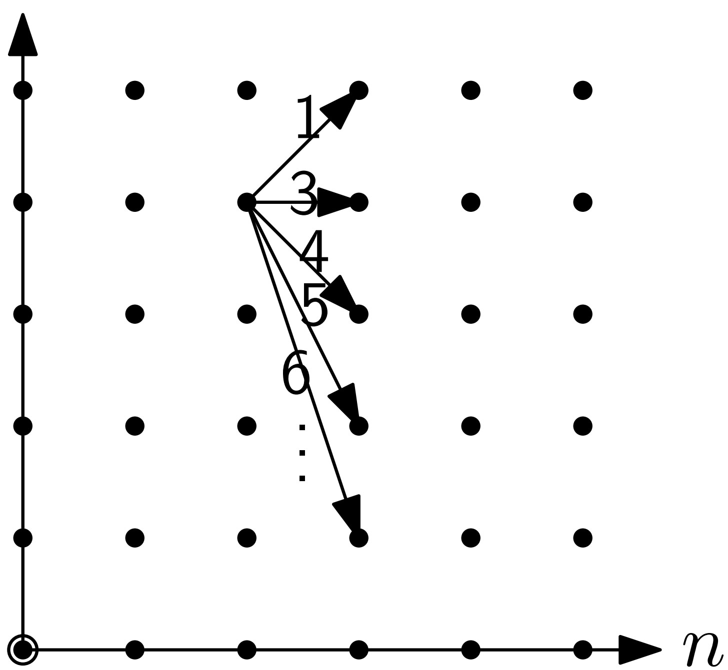
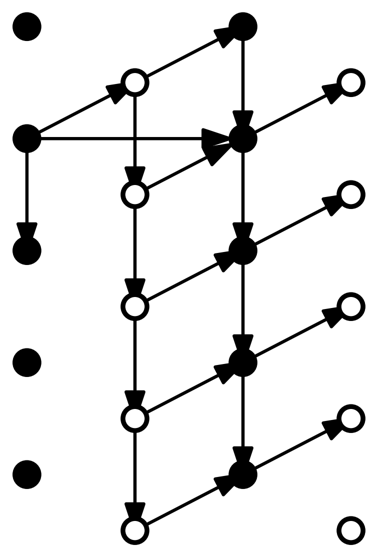
METHOD II: intermediate layers





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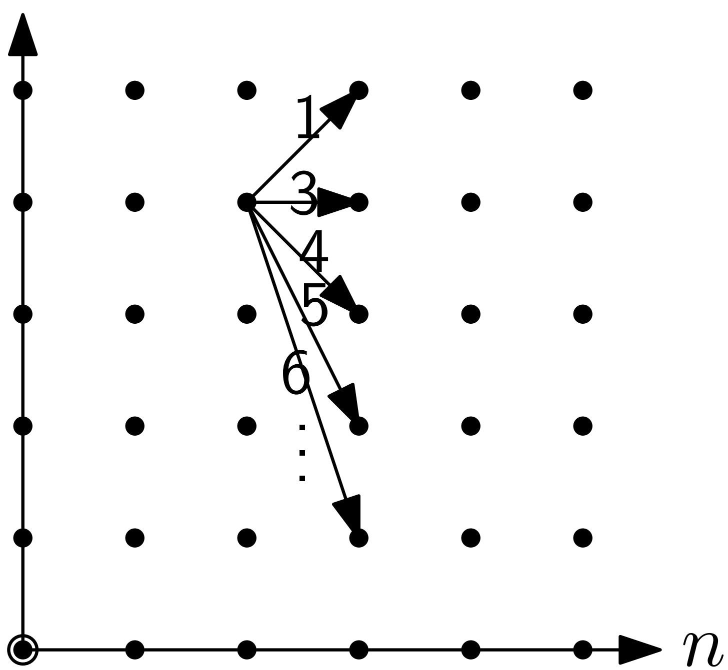

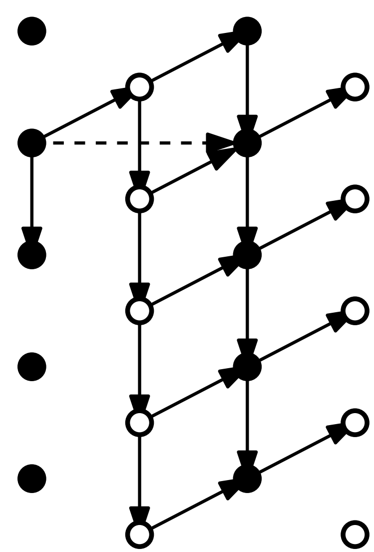

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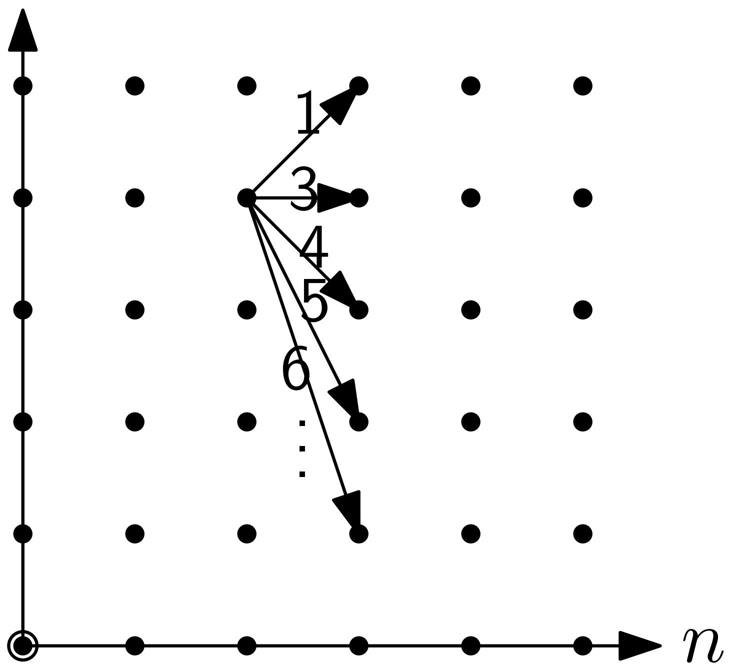
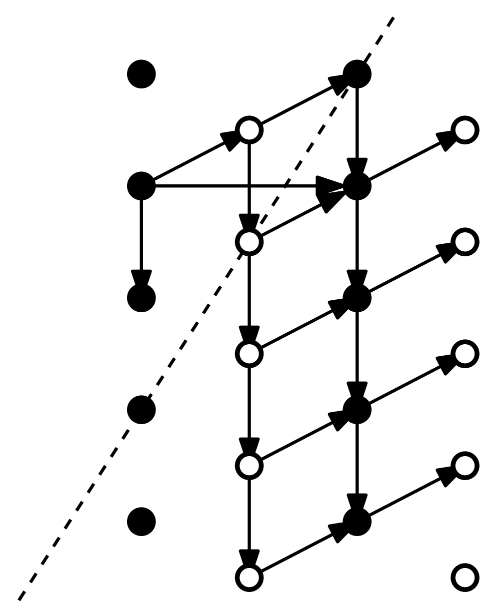
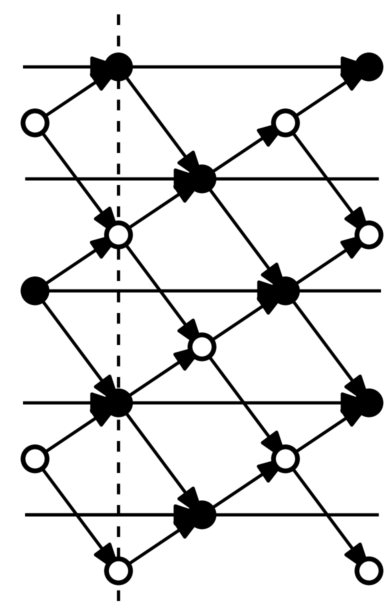
Irregularities at the boundary can be ignored.

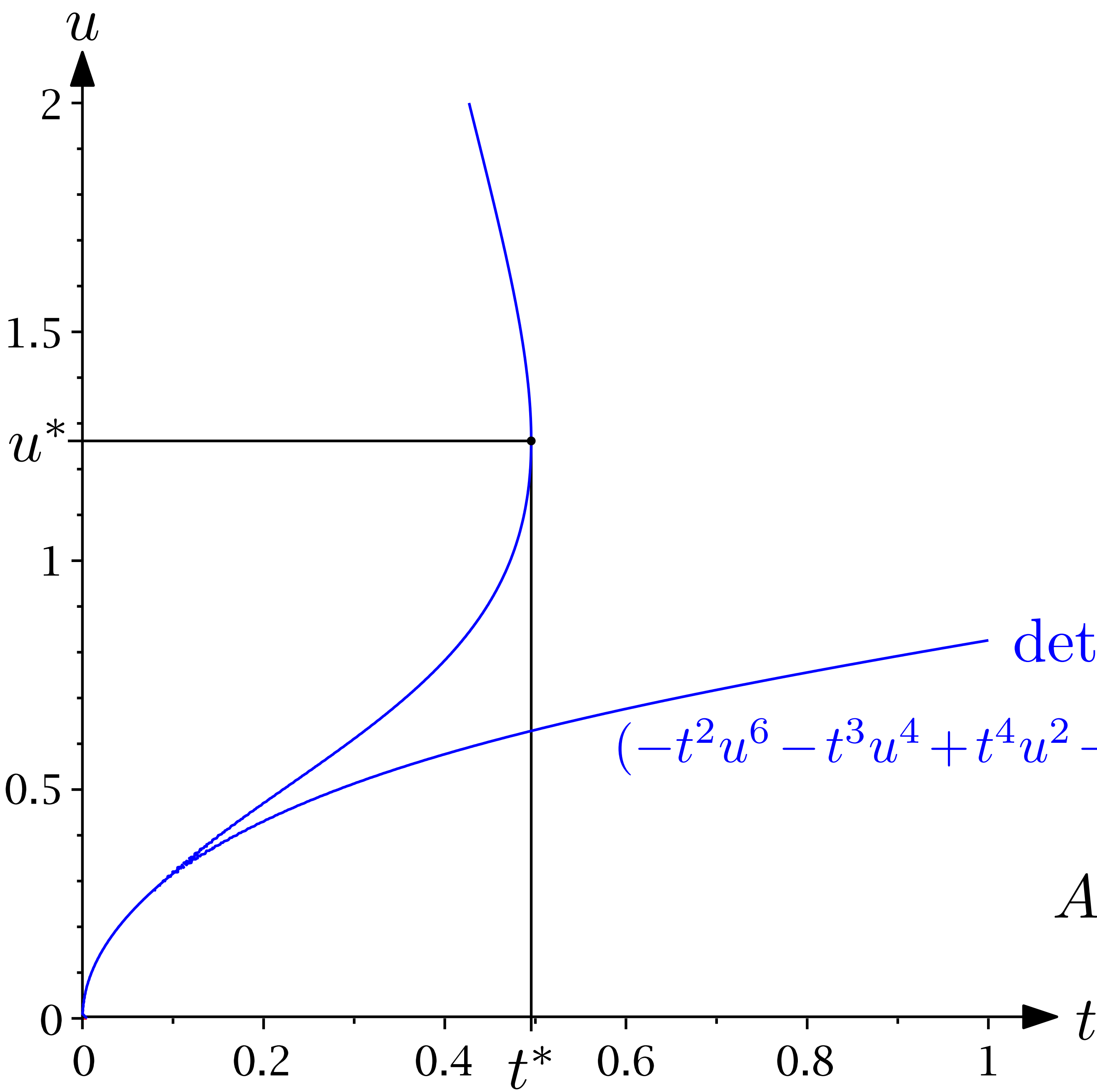
$$A = \left(\begin{array}{c|cc} & \bullet & \circ \\ \hline \bullet & t^3 + tu^{-2} & tu \\ \circ & tu & tu^{-2} \end{array}\right)$$







Solving for t^* and u^*

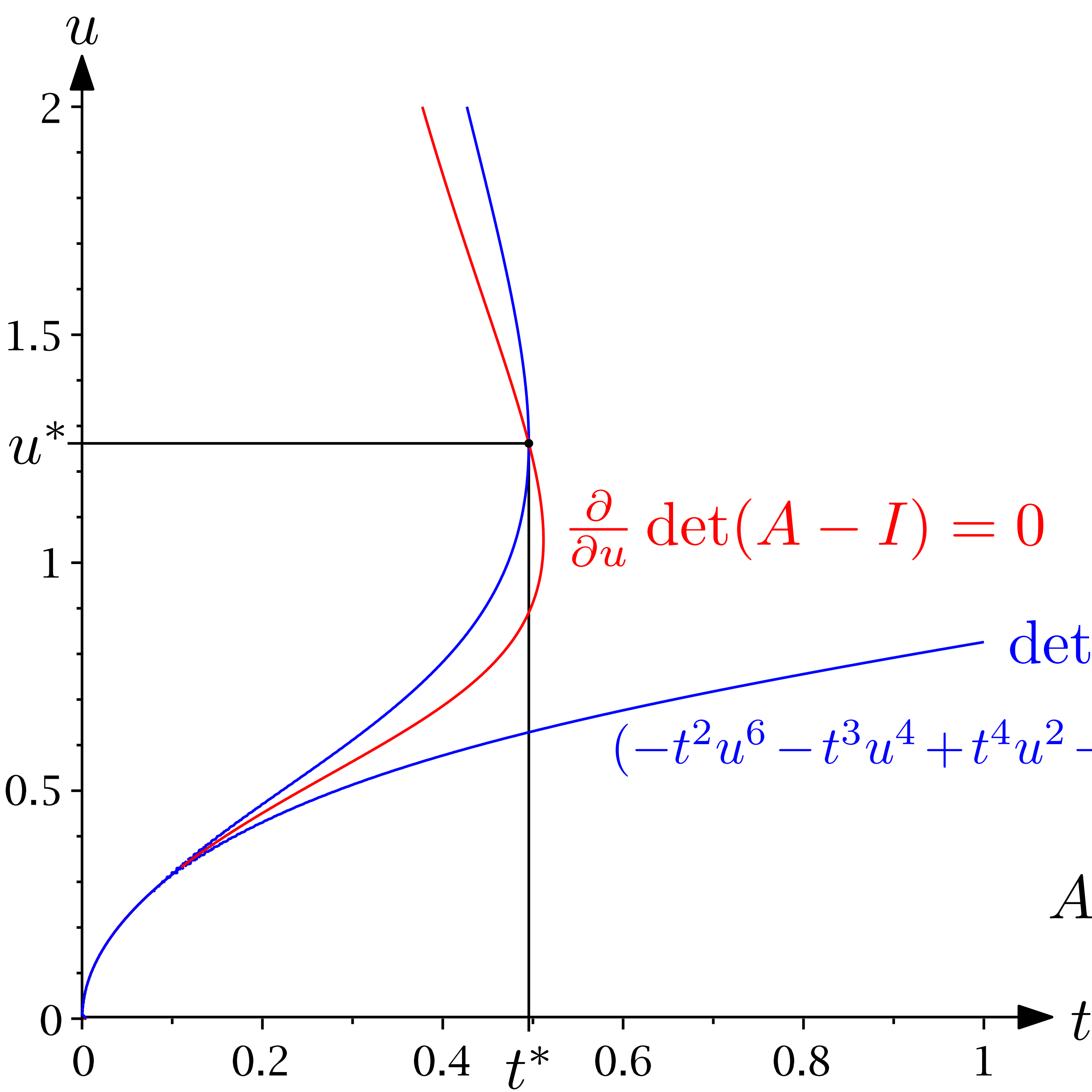


$$\det(A - I) = 0$$

$$(-t^2u^6 - t^3u^4 + t^4u^2 + u^4 - 2\,tu^2 + t^2)u^{-4} = 0$$

$$A = \begin{pmatrix} t^3 + tu^{-2} & tu \\ tu & tu^{-2} \end{pmatrix}$$

Solving for t^* and u^*



$$\frac{\partial}{\partial u}\det(A-I)=0$$

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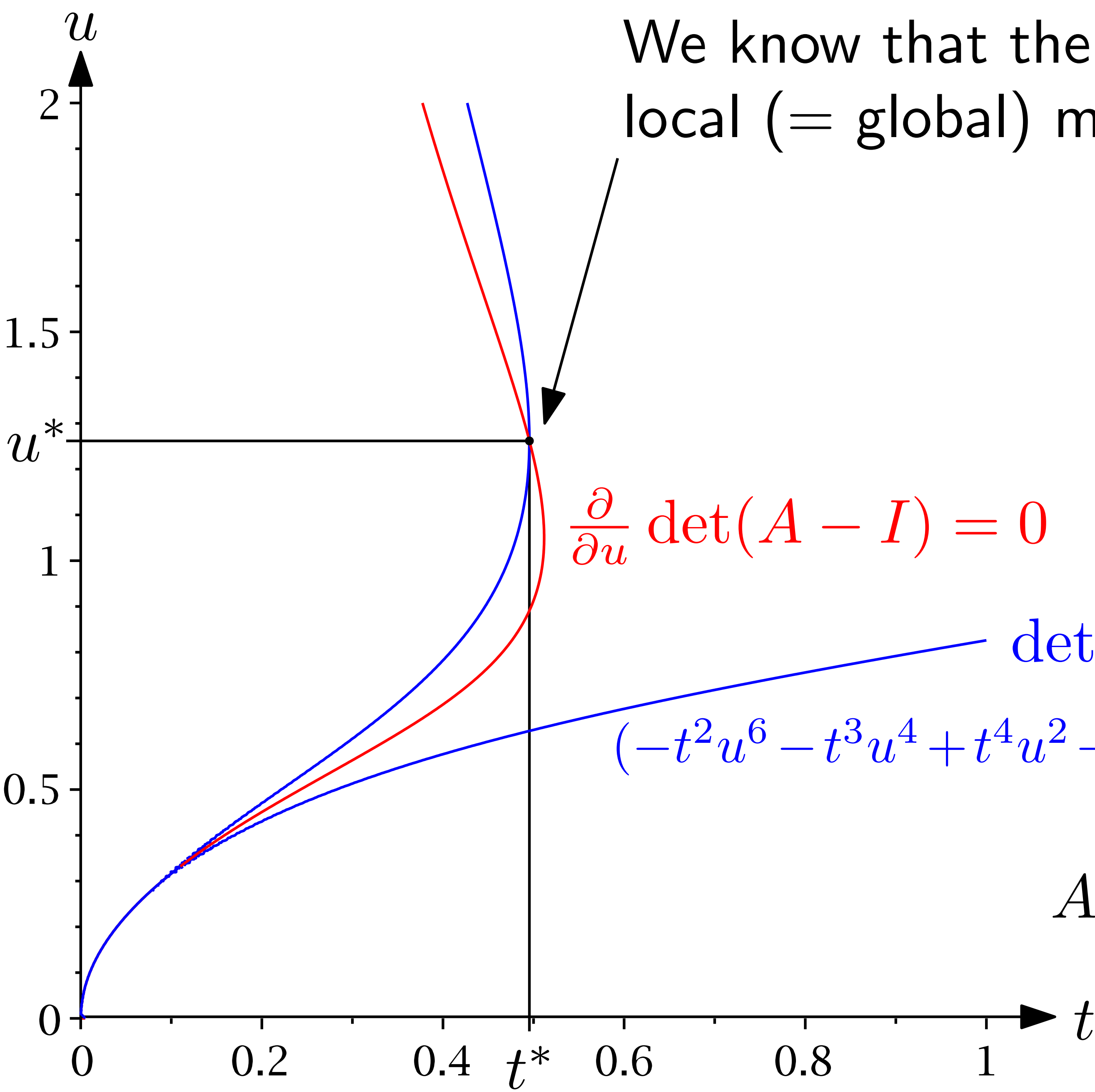
Solving for t^* and u^*

We know that there is a unique local (= global) maximum u^*



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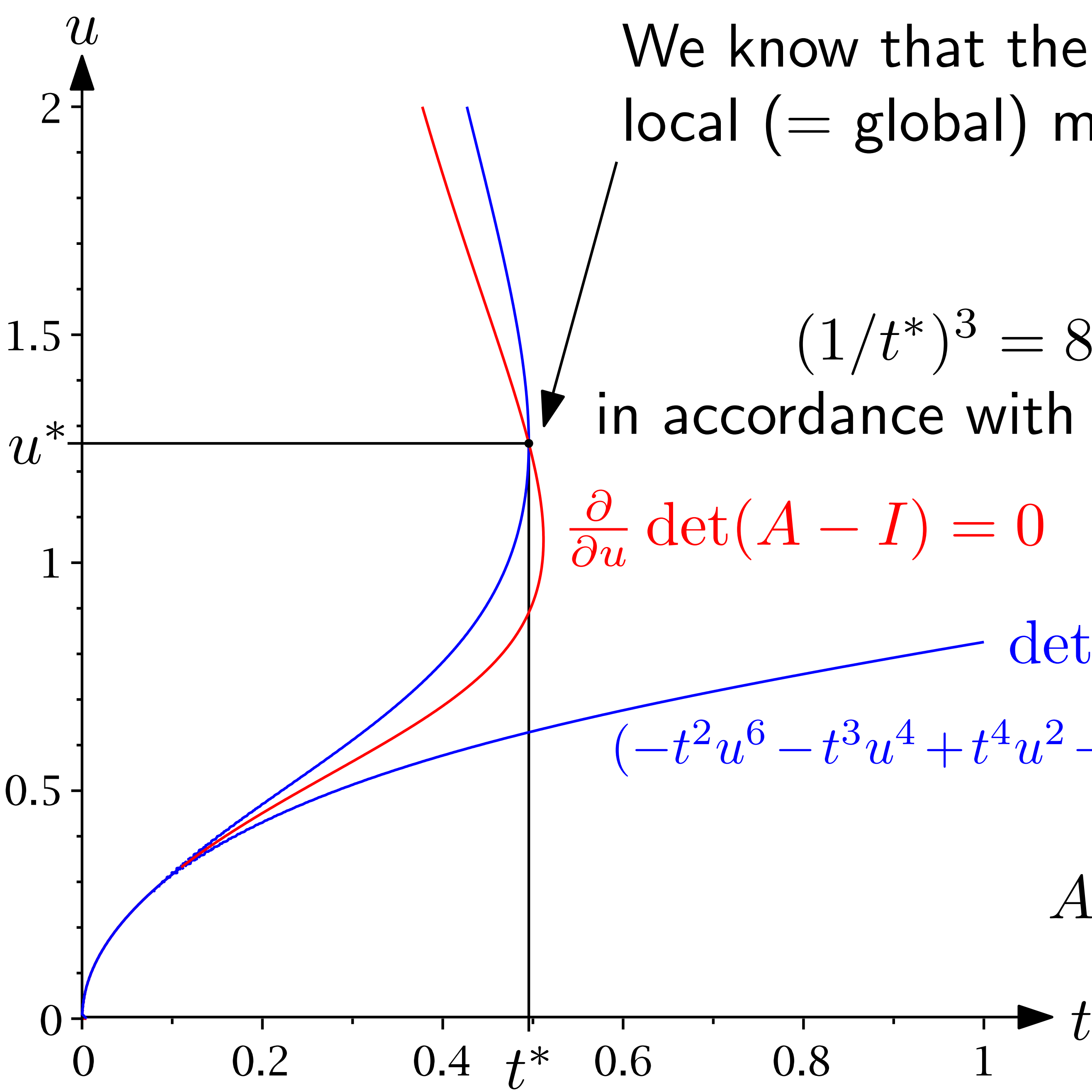
Solving for t^* and u^*



We know that there is a unique local (= global) maximum u^*

$$(1/t^*)^3 = 8.22469154098$$

in accordance with Flajolet & Noy (1999)

$$\frac{\partial}{\partial u}\det(A-I)=0$$

$$\det(A-I)=0$$

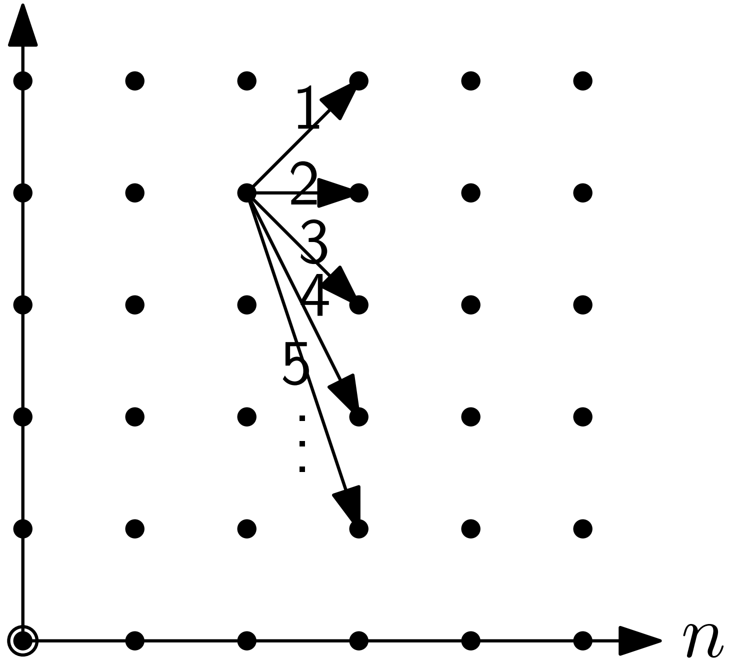
$$(-t^2u^6 - t^3u^4 + t^4u^2 + u^4 - 2\,tu^2 + t^2)u^{-4} = 0$$

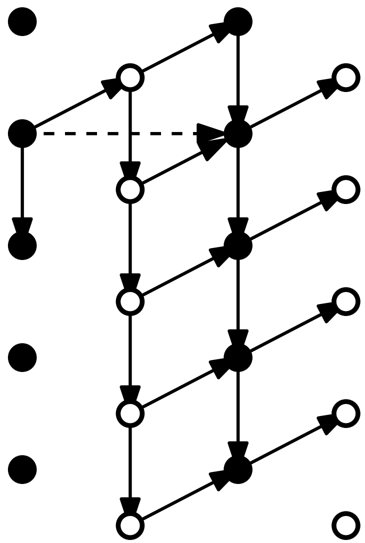
$$A = \begin{pmatrix} t^3 + tu^{-2} & tu \\ tu & tu^{-2} \end{pmatrix}$$

Example 2a: Trees and Serendipity

$$P = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots \\ 0 & 1 & 2 & 3 & 4 & \dots \\ 0 & 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The extra edge is not needed.

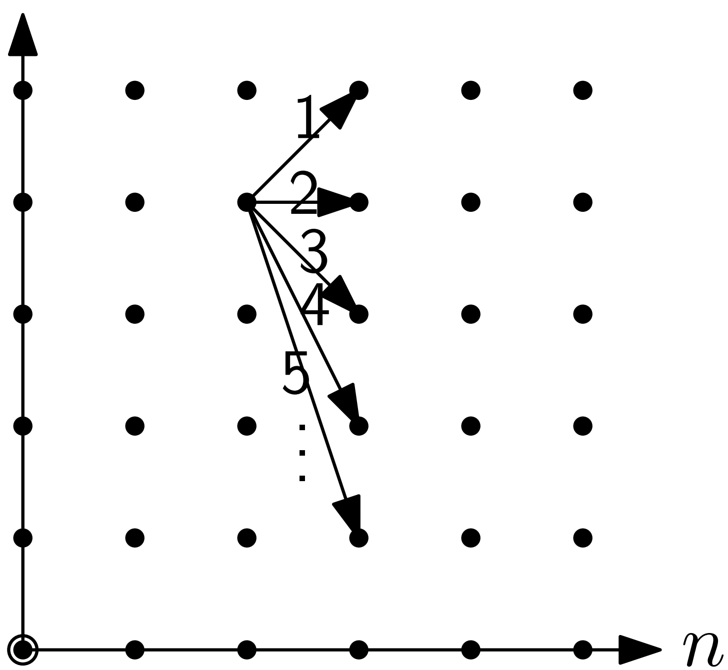


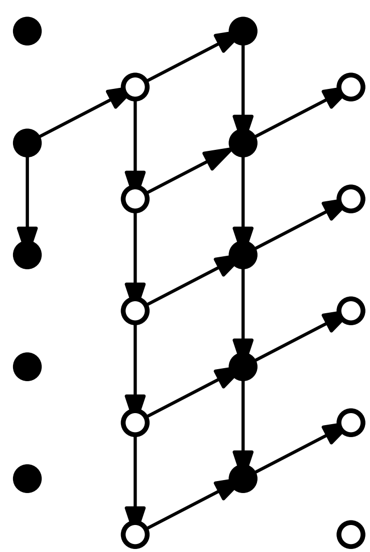


Example 2a: Trees and Serendipity

$$P = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 2 & 3 & 4 & 5 & \dots \\ 0 & 1 & 2 & 3 & 4 & \dots \\ 0 & 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 0 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The extra edge is not needed.





Only one type of node!
One state is sufficient.

Example 2b: Graphs, and 2c: Paths

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 3 & 3 & 3 & 3 & \dots \\ 0 & 2 & 4 & 4 & 4 & 4 & \dots \\ 0 & 0 & 2 & 4 & 4 & 4 & \dots \\ 0 & 0 & 0 & 2 & 4 & 4 & \dots \\ 0 & 0 & 0 & 0 & 2 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

geometric graphs paths

Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

Example 2b: Graphs, and 2c: Paths

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 3 & 3 & 3 & 3 & \dots \\ 0 & 2 & 4 & 4 & 4 & 4 & \dots \\ 0 & 0 & 2 & 4 & 4 & 4 & \dots \\ 0 & 0 & 0 & 2 & 4 & 4 & \dots \\ 0 & 0 & 0 & 0 & 2 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

geometric graphs

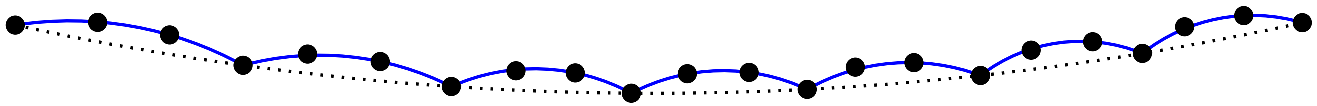
$$\left(\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}\right)$$

paths

two states

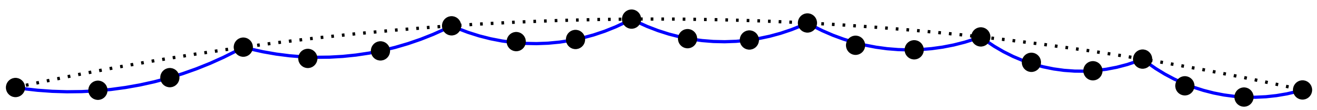
Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

Example 3: Geometric graphs



the generalized double zigzag chain
[Huemer, Pilz, and Silveira 2018]

[Asinowski and Rote 2018] for *matchings*

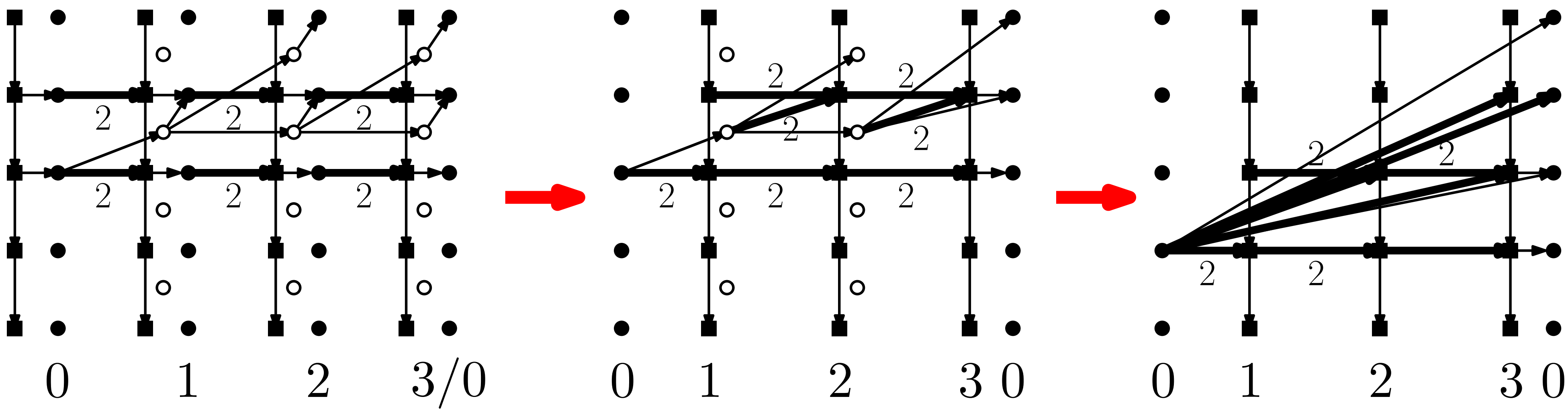


$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 2 & 2 & 2 & 2 & \dots \\ 0 & 0 & 2 & 2 & 2 & \dots \\ 0 & 0 & 0 & 2 & 2 & \dots \\ 0 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$P = R^3 + SR^2 + S(I+S)R + S(I+S)^2$$

Example 3: Geometric graphs

$$P = R^3 + SR^2 + S(I+S)R + S(I+S)^2$$



$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 2 & 2 & 2 & \dots \\ 0 & 0 & 2 & 2 & \dots \\ 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 3: Geometric graphs

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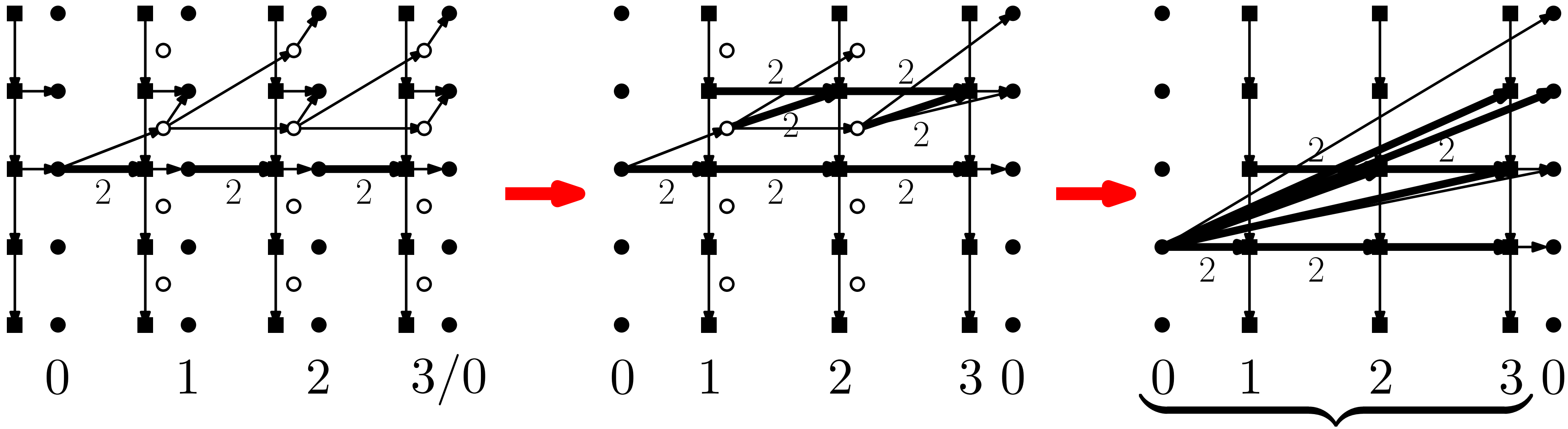

$$A = \left(\begin{array}{c|cccc} & \bullet & \blacksquare_1 & \blacksquare_2 & \blacksquare_3 \\ \hline \bullet & t(u+2u^2+u^3) & 2 & 2u & 2u+2u^2 \\ \blacksquare_1 & 0 & u^{-1} & 2 & 0 \\ \blacksquare_2 & 0 & 0 & u^{-1} & 2 \\ \blacksquare_3 & t & 0 & 0 & u^{-1} \end{array}\right)$$

$$1/t^* = 44\sqrt{2} + 62 \approx 124.225$$

Possible Proofs

Freie Universität Berlin

Conjecture: The number of paths from $(0,0)$ in state q_0 to $(n,0)$ in state q_1 that don't go below the x-axis is

$$\sim \text{const} \cdot (1/t^*)^n \cdot n^{-3/2},$$

where

(1) $A(t^*, u^*)$ has largest (Perron-Frobenius) eigenvalue 1.

$$[\implies \det(A(t,u) - I) = 0\,]$$

(2) u^* is chosen such that the value t^* that fulfills (1) is as large as possible. $[\implies \frac{\partial}{\partial u} \det(A(t,u) - I) = 0\,]$

APPROACHES:

A) Analytic Combinatorics, "square-root-type" singularity

B) Probabilistic interpretation, random walk

C) Pedestrian, induction

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APPROACHES:

A) Analytic Combinatorics, "square-root-type" singularity

Special case 1: One state. All steps of the form $(1,j)$. $\to t^1 u^j$
[Banderier and Flajolet, 2002]

$[\det(A(t,u) - I) = t \cdot Q(u) - 1 = 0, \ \ Q'(u) = 0]$

Analytic Combinatorics

Freie Universität Berlin

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Special case 2: Lattice paths with forbidden patterns

use the "vectorial kernel method"

[Asinowski, Bacher, Banderier, Gittenberger, 2019]

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Special case 2: Lattice paths with forbidden patterns

use the "vectorial kernel method"

[Asinowski, Bacher, Banderier, Gittenberger, 2019]

Use an unambiguous context-free grammar

E. g. $D \rightarrow \varepsilon \mid$ +D–D for Dyck paths

Chomsky–Schützenberger enumeration theorem from 1963
$\rightarrow$ generating function is algebraic.

Possible Proofs

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(3) Let $\vec{v}$ and $\vec{w}$ be left and right eigenvectors of $A(t^*, u^*)$ with eigenvalue 1. Then
$$\vec{v} \cdot \frac{\partial}{\partial u} A(t,u) \cdot \vec{w} = 0 \text{ at } (t^*, u^*).$$

---

(1)∧(2) ⇔ (1)∧(3). (linear algebra)

(1) ⇒ $N_{(x,y),q} \le v_q t^{-x} u^{-y}$ by induction, ⇒ $N_{(n,0)} = O(t^{-n})$

↑ *all* paths to (x,y), including those that go negative

Random walk

The effect of edge weights $t^i u^j$:

t: Path weights from $(0,0)$ to $(n,0)$ are multiplied by t^n.

u: Path weights from $(0,0)$ to $(n,0)$ are unaffected by u!

Use entries a_{qr} of $A = A(t,u)$ as "weights" for a random walk.

$$A = \begin{pmatrix} 0.71 & 0.25 & 0.05 \\ 0.31 & 0.00 & 0.02 \\ 3.15 & 0.66 & 0.12 \end{pmatrix}$$, eigenvalue 1, right eigenvector $\vec{w}$

Use right eigenvector $\vec{w}$ to rescale: $p_{qr} := a_{qr} \frac{w_r}{w_q}$

$\to$ stochastic matrix with transition probabilities p_{qr}

Path weights from $(0,0)$ to $(n,0)$ are multiplied by w_{q_1}/w_{q_0}.

$$\#\text{paths} = \text{const} \cdot (1/t)^n \cdot \Pr[\text{walk nonnegative \& reaches } (n,0)]$$

Random walk

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$$\#\text{paths} = \text{const} \cdot (1/t)^n \cdot \Pr[\text{walk nonnegative \& reaches } (n,0)]$$

The place where the walk hits the line $x = n$ is approximately Gaussian.

If the mean is not 0, then this is exponentially small.

Use u to make the walk *balanced*. In $t^i u^j$, Up-steps $(j > 0)$ are favored $(u > 1)$ or penalized $(u < 1)$ over down-steps.

$$\text{Average vertical drift} = \sum_q \pi_q \cdot \sum_{((i,j),r)\in S_q} j \cdot p_{q,(i,j),r} \stackrel{!}{=} 0$$

stationary distribution over the states

Make the random walk balanced

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Average vertical drift $= \sum_q \pi_q \cdot \sum_{((i,j),r)\in S_q} j \cdot p_{q,(i,j),r} \overset{!}{=} 0$

$\frac{\partial}{\partial u}$ brings out the factor j from $t^i u^j$:

Example: Step $(8,5) \to a_{qr} = t^8 u^5$

$$\frac{\partial}{\partial u} t^8 u^5 = 5t^8 u^4 \implies u \frac{\partial}{\partial u} t^8 u^5 = 5t^8 u^5 = 5a_{qr}$$

"No-drift" condition: $\vec{v} \cdot \left(u \cdot \frac{\partial}{\partial u} A(t,u) \right) \cdot \vec{w} = 0$

left eigenvector = stationary distribution

Local Limit Theorems

Still want to show, for a balanced walk:

$$\Pr[\text{walk nonnegative} \wedge \text{reaches } (n,0)] \sim \text{const} \cdot n^{-3/2}$$

Classical Local Limit Theorem:
Prob[sum of n i.i.d. random variables with mean 0 lies in some small region around 0] $\sim$ const $\cdot\, n^{-1/2}$ [Gnedenko, Stone]

Needs to be adapted to sign-restricted case $(y \geq 0)$ and several states.

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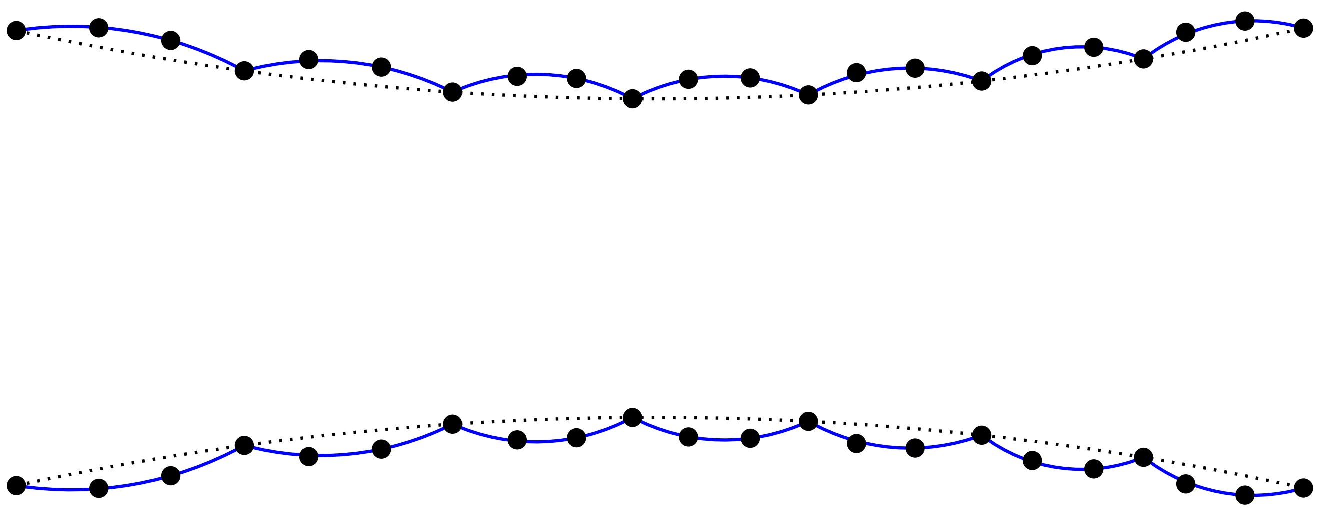
C) "Pedestrian" approach. Pioneered for a special case with two states in Asinowski and Rote (2018).

- $O((1/t^*)^n)$ by induction.
- $\Omega((1/t^* - \varepsilon)^n)$ for every $\varepsilon > 0$, by induction.

The necessity of two states

- Count non-crossing perfect matchings in the generalized double zigzag chain



the generalized
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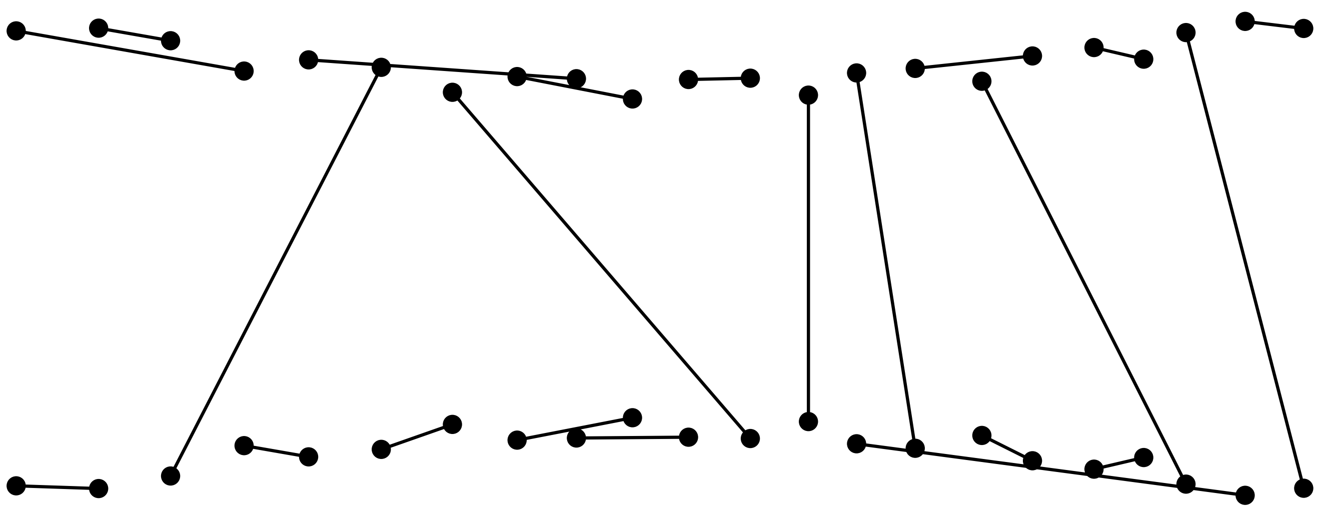

a non-crossing perfect matching

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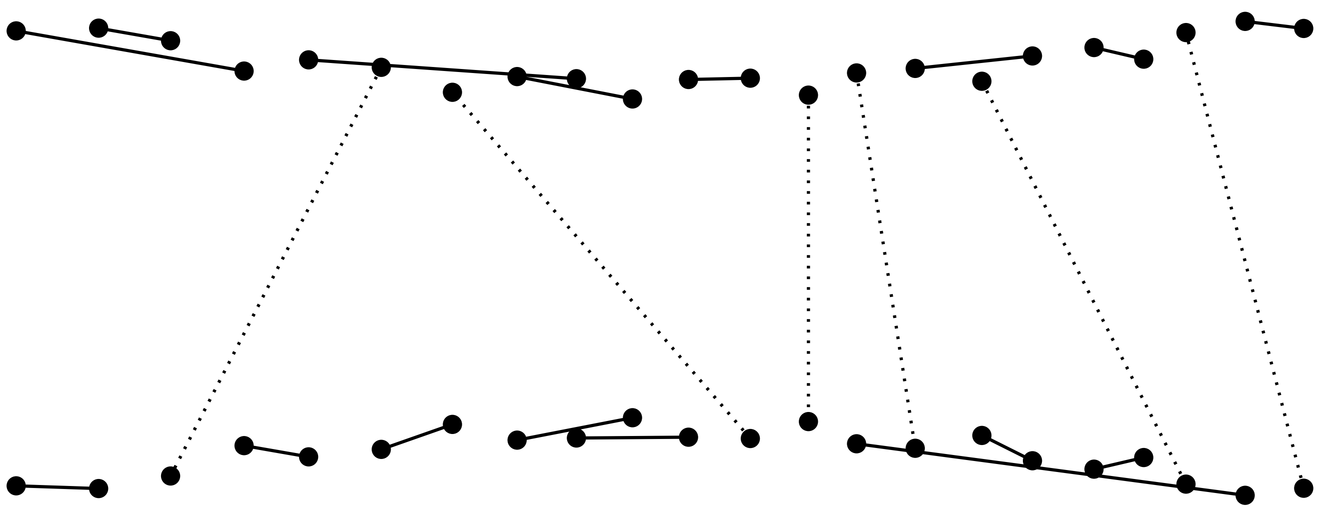

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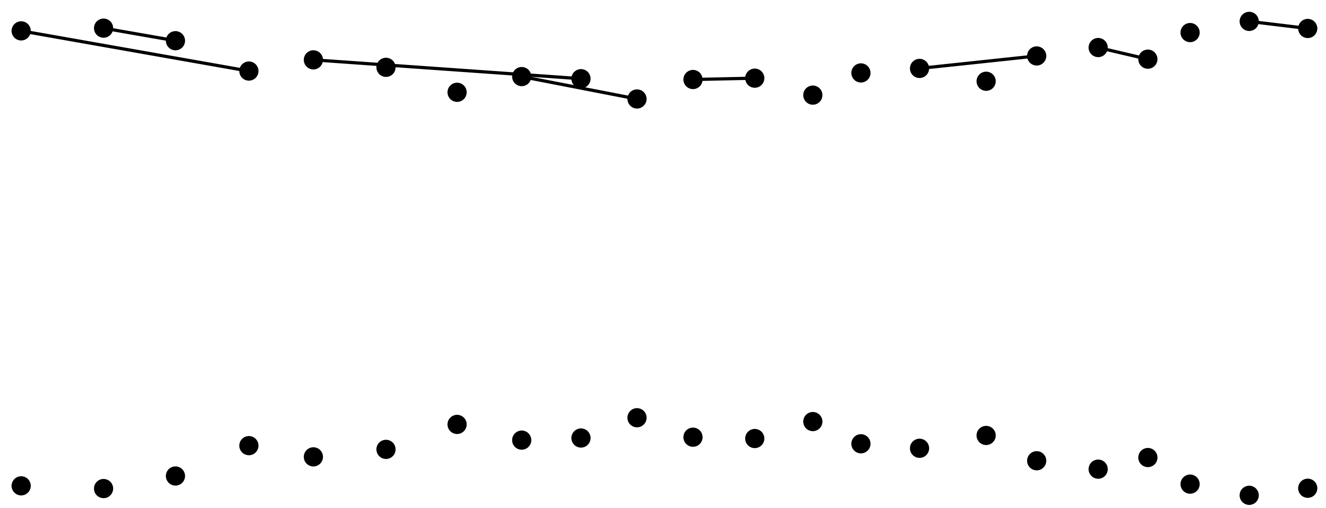
- Count non-crossing perfect matchings in the generalized double zigzag chain
- Count *down-free* matchings in a *single* zigzag chain



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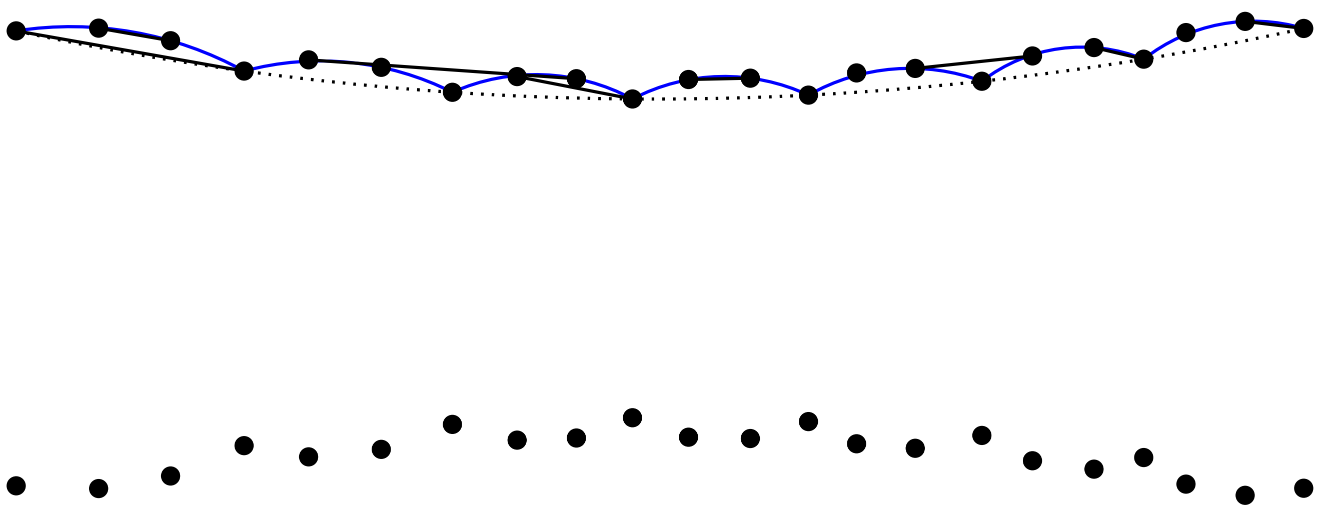

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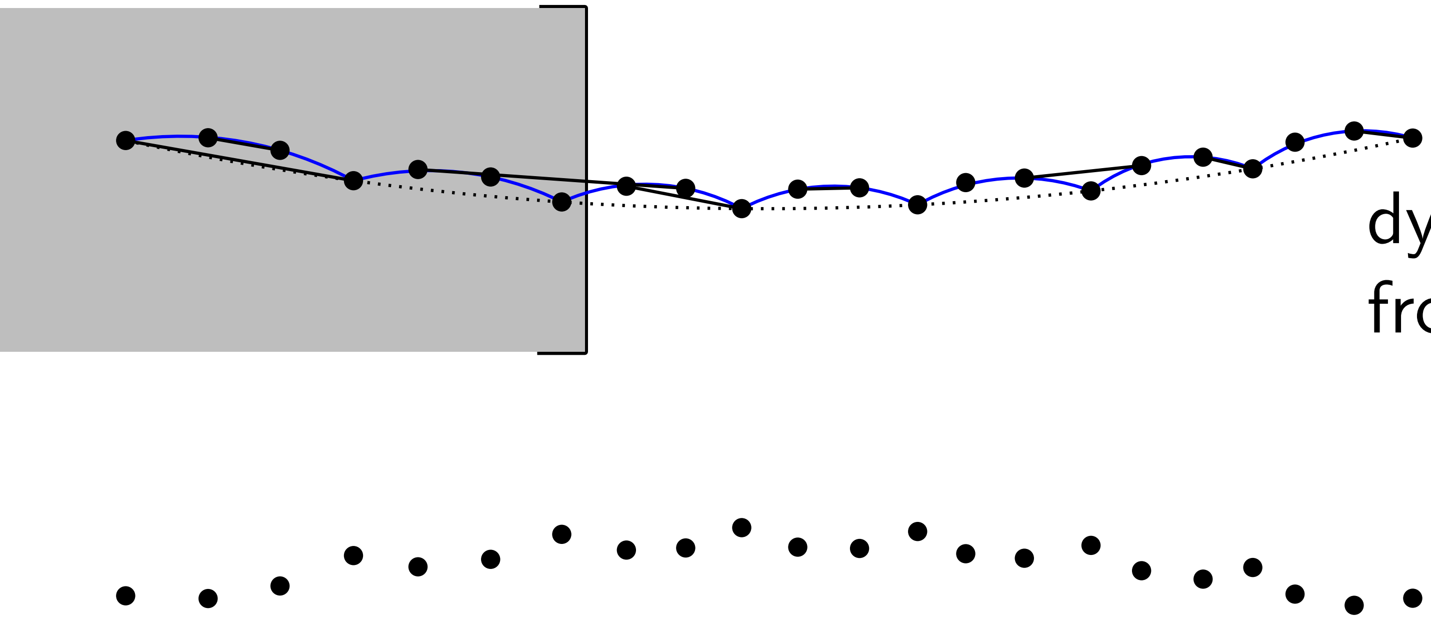

dynamic programming from left to right

The necessity of two states

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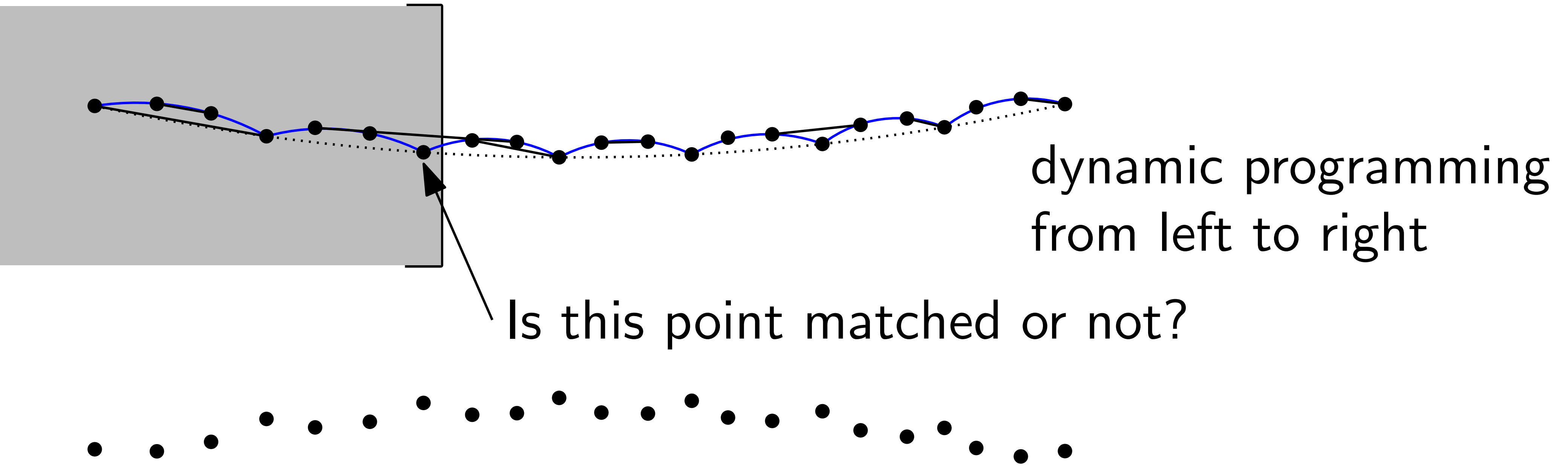
- Count *down-free* matchings in a *single* zigzag chain



Extensions and Questions

- higher dimensions: jumps (i, j, k)
- jumps $(i, j) \in \mathbb{R}^2$, not necessarily on the grid
- Prove that the local maximum u^* is a *strong* maximum
- real weights $c \geq 0$ ✔
  weights $c < 0$?
- other applications of production matrices or lattice paths with states