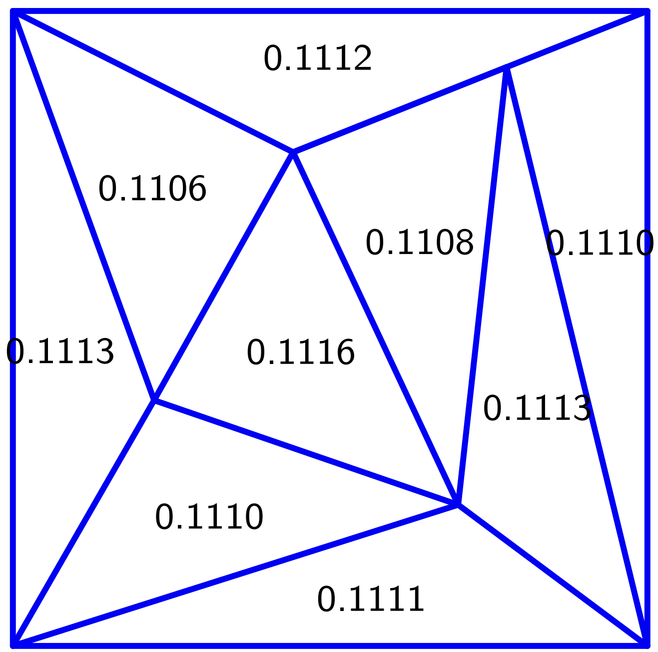
Dissecting a Square into an Odd Number of Triangles of Almost Equal Area

Jean-Philippe Labbé, Günter Rote, Günter M. Ziegler
Freie Universität Berlin



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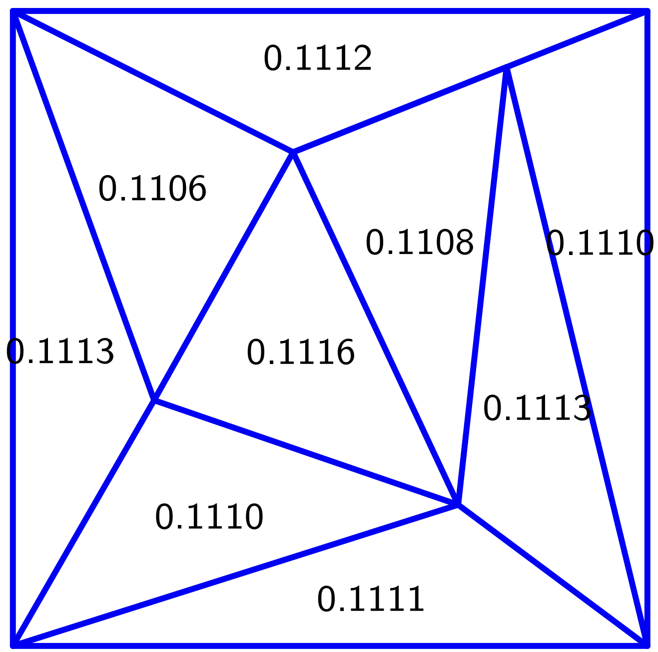

$n = 9$ triangles

root-mean-square (RMS) error:

$$0.0002737$$

The optimum among dissections with at most 8 nodes.

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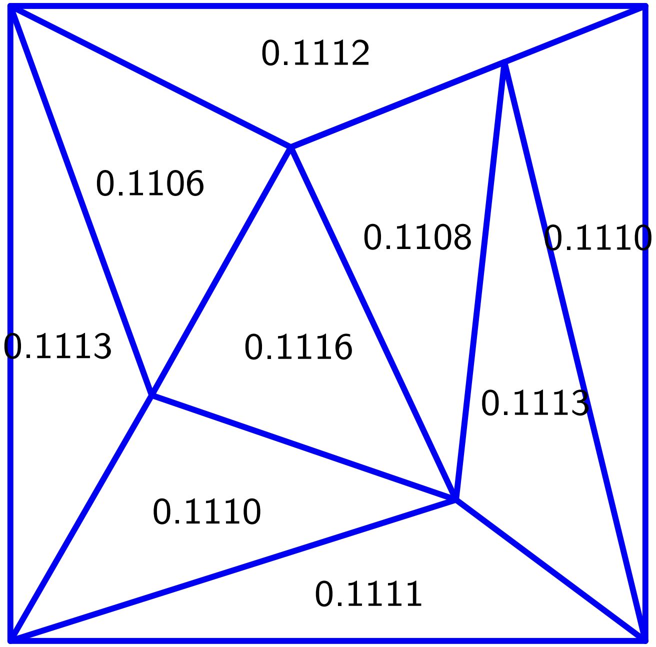

THEOREM (P. Monsky, 1970)

If n is odd, there is no dissection of the square into n triangles of equal area.

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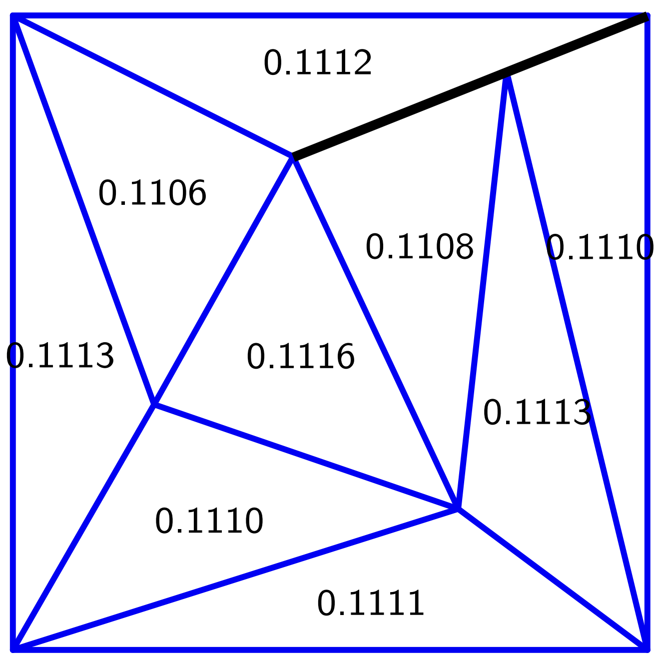

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dissection $\neq$ triangulation

Measuring Area Deviation

areas $a_1, \ldots, a_n$, target area $= \frac{1}{n}$

- Root-mean-square error (RMS, standard deviation):

$$\text{RMS} := \sqrt{\frac{1}{n} \cdot \sum_{i=1}^{n} (a_i - \tfrac{1}{n})^2}$$

- Range:

$$\text{range} := \max_{1 \le i \le n} a_i - \min_{1 \le i \le n} a_i$$

$$\boxed{\frac{\text{range}}{2\sqrt{n}} \le \text{RMS} \le \text{range}}$$

Lower and Upper Bounds

$$\text{range} \geq \frac{1}{2^{2^{O(n)}}} \text{ (doubly-exponential)}$$

Proof: Gap theorems from real algebraic geometry

A family of dissections for every n with

$$\text{range} \leq \frac{1}{n^{\log_2 n - 5}} = \frac{1}{2^{\Omega(\log^2 n)}} \text{ (superpolynomial)}$$

Previous results:

- Numerical experiments, exhaustive enumeration for small n (Katja Mansow, 2003)
- A family of triangulations with range $\leq 1/n^3$ (Bernd Schulze, 2011)

Overview

- Introduction: Problem statement and results
- Review of Monsky's proof (2-adic valuation, Sperner's lemma)
- Modeling the problem
- Lower bound via a gap theorem
- Numerical experiments
- Systematic construction (Thue-Morse sequence)
- More numerical experiments
- Speculations

Monsky: 3-coloring of the plane

2-adic valuation of $\mathbb{R}$

$\rightarrow$ coloring of $\mathbb{R} \times \mathbb{R}$ with three colors A, B, C

Crucial property:

A rainbow triangle cannot have area 0 or $\frac{1}{n}$ for odd n.

Monsky: 3-coloring of the plane

2-adic valuation of $\mathbb{R}$
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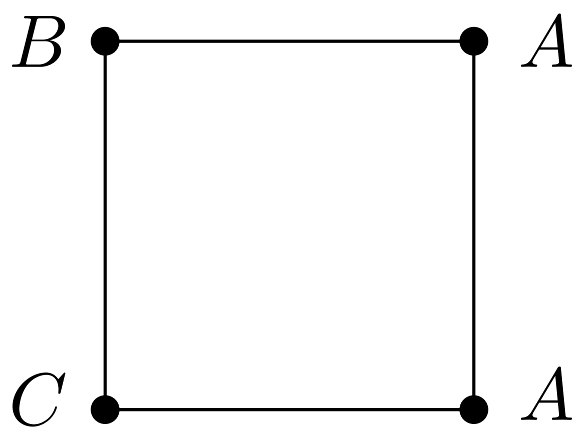
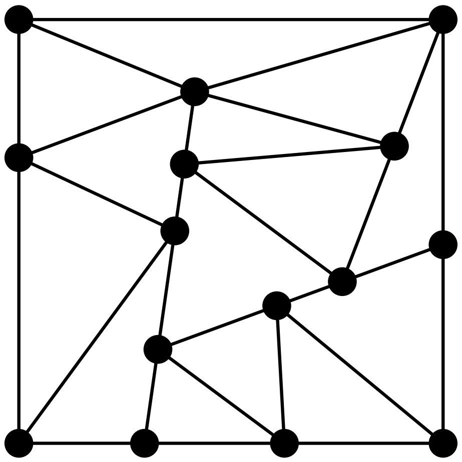
Crucial property:
A rainbow triangle cannot have area 0 or $\frac{1}{n}$ for odd n.

Parity argument like for Sperner's lemma:

If the boundary of a polygon has an odd number of AB-colored edges,
then every dissection has an odd number of rainbow triangles.

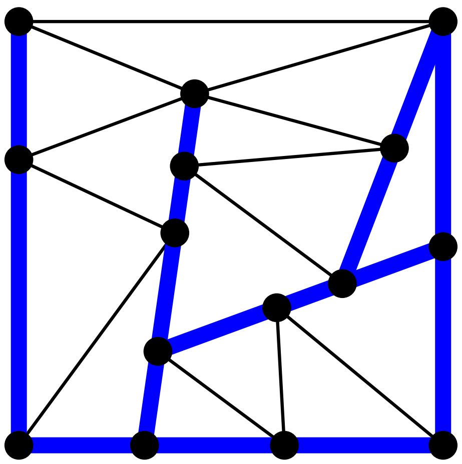


Lower Bound: Modeling Collinearity



- Lock at all maximal line segments

Lower Bound: Modeling Collinearity



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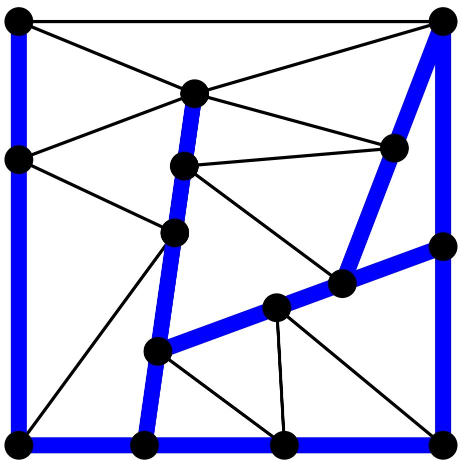
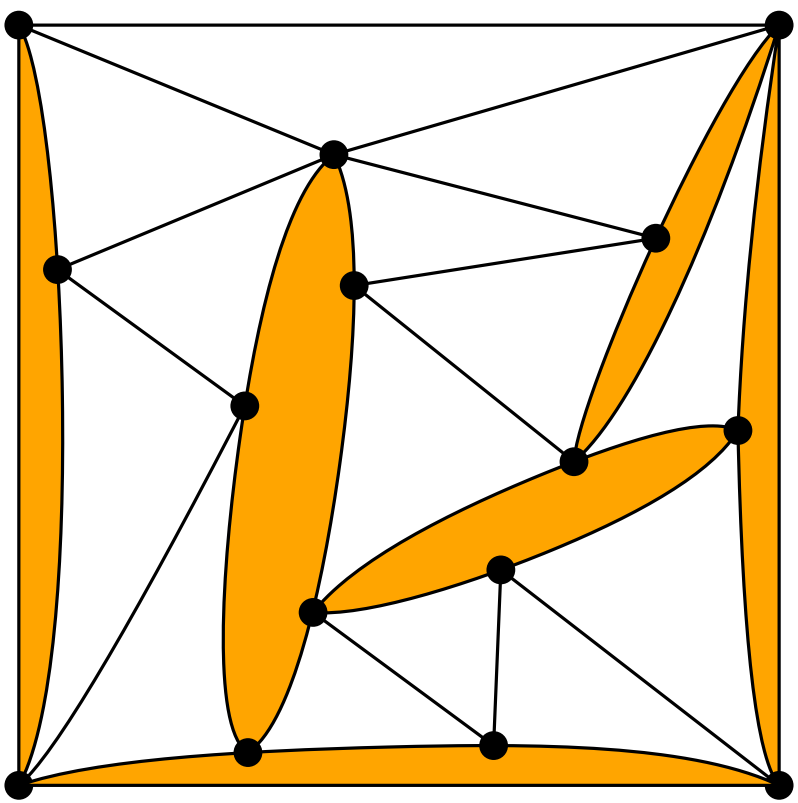


- Lock at all maximal line segments
- Open them up
- Triangulate them arbitrarily.
  → combinatorial triangulation of a 4-gon, with additional *zero-area triangles* Z



Lower Bound: Modeling Collinearity

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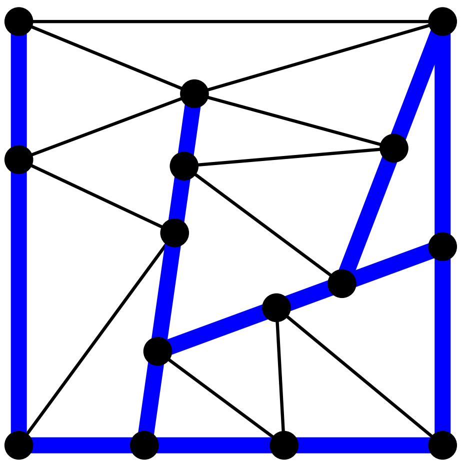

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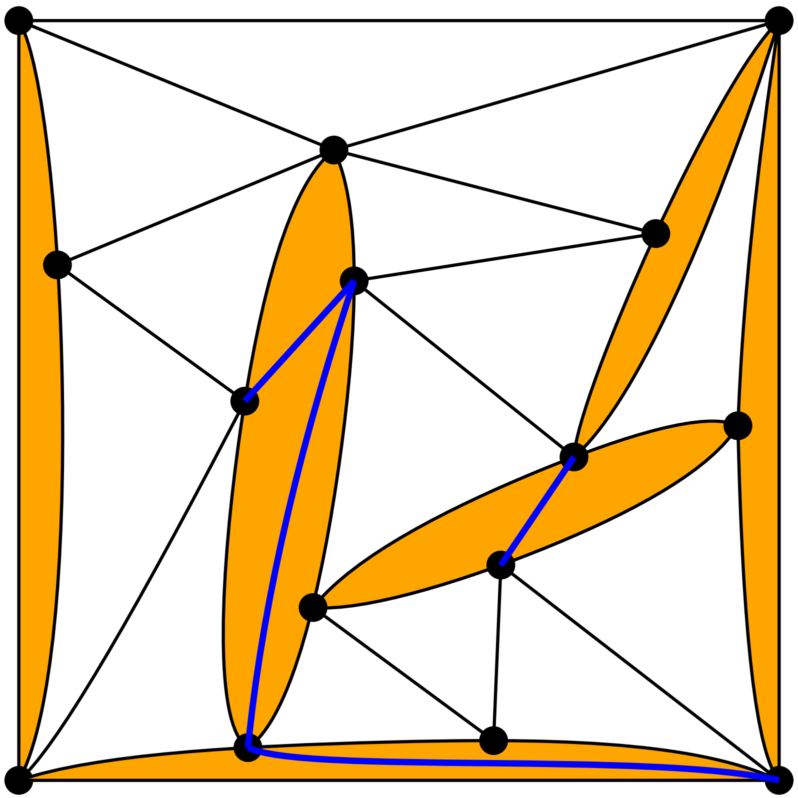

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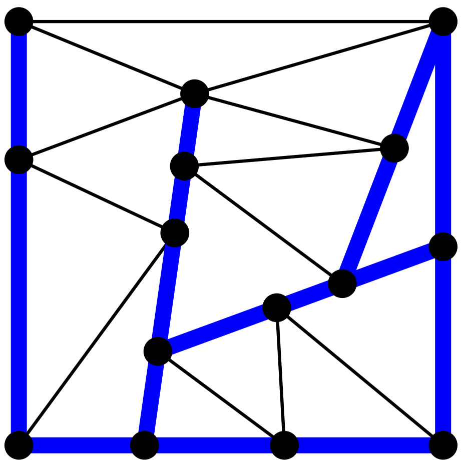

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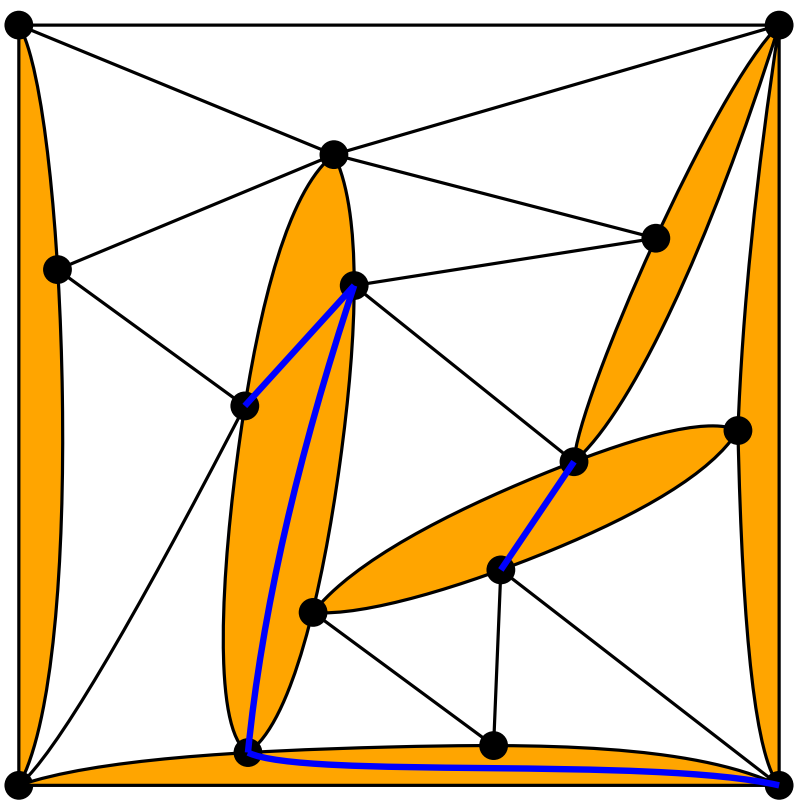
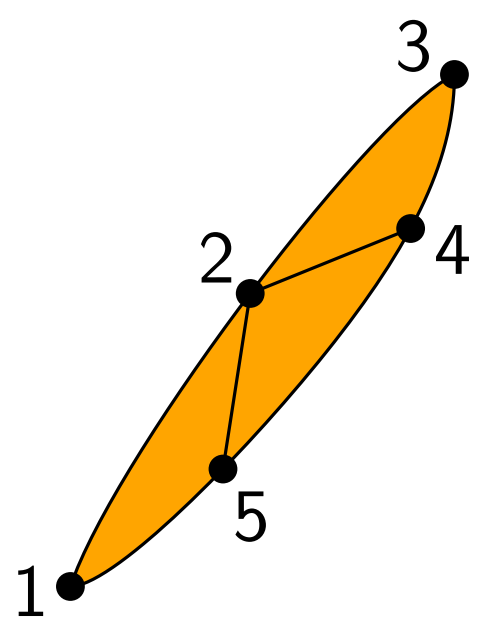
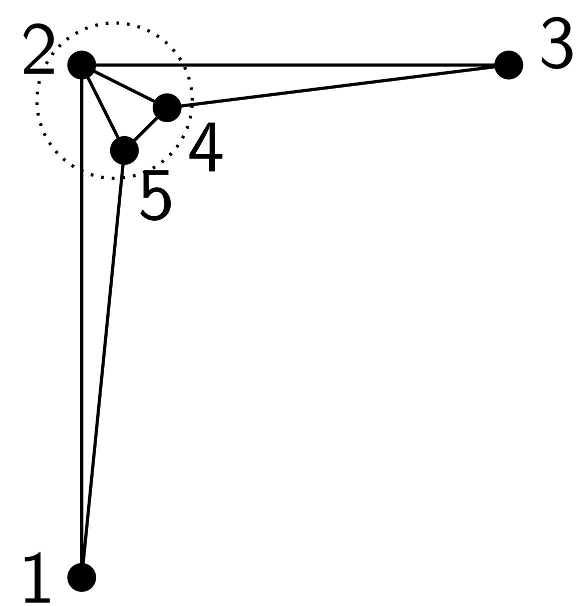
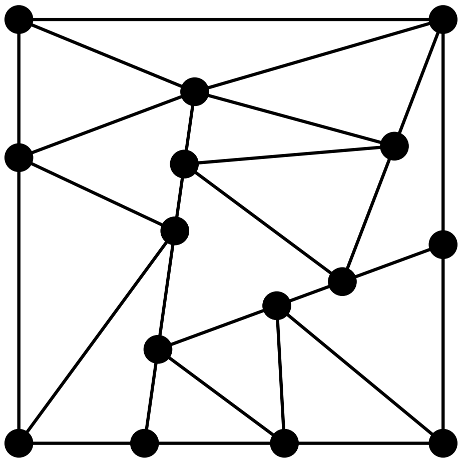


Area 0 does not enforce collinearity!





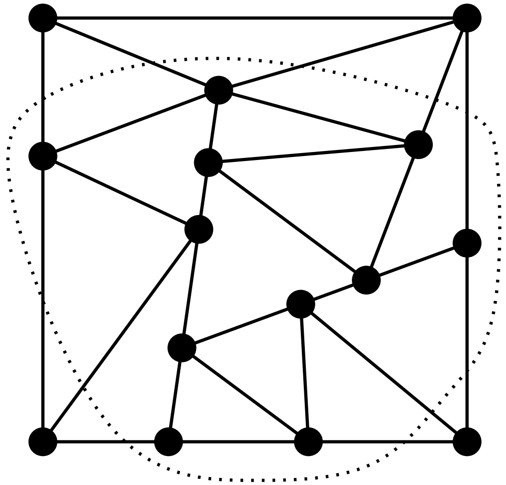
Area Deviation Polynomial



n triangles, areas $a_1, \ldots, a_n$

v unknown vertex positions (apart from the 4 fixed corners of the square)

Area Deviation Polynomial



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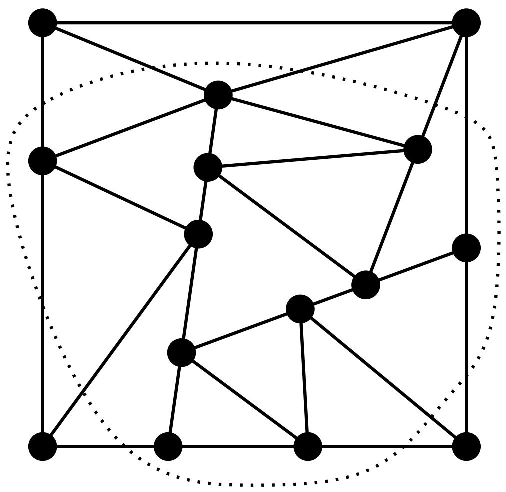

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$$T(\vec{x}) = \sum_{i=1}^{n} (a_i(\vec{x}) - \tfrac{1}{n})^2, \ \vec{x} \in \mathbb{R}^{2v}$$

. . . a degree-4 polynomial, RMS $= \sqrt{T(\vec{x})/n}$

Area Deviation Polynomial

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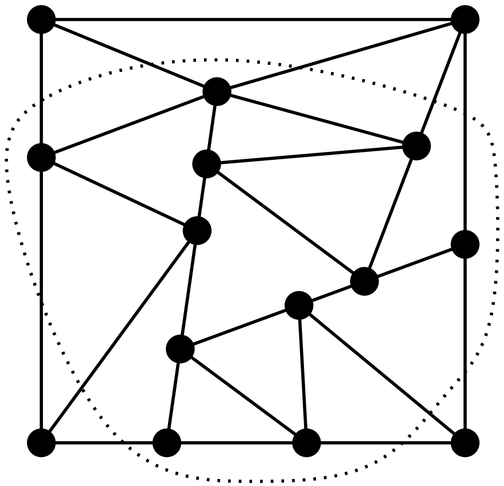

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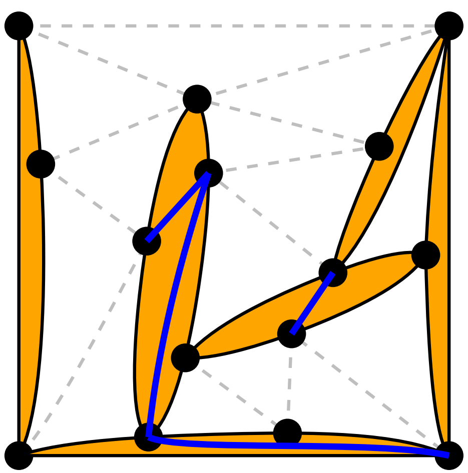
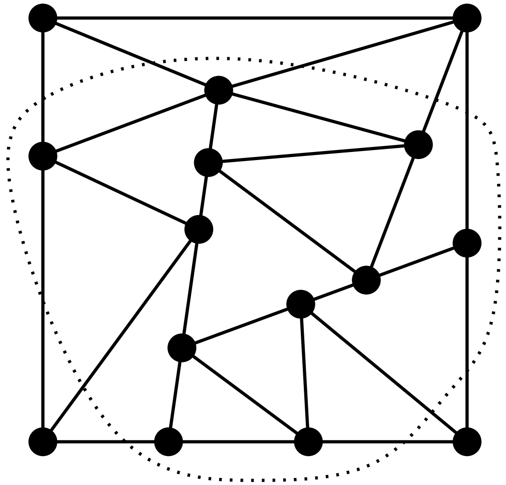


z zero-area triangles, areas $b_1, \ldots, b_z$

$$Z(\vec{x}) = \sum_{j=1}^{z} (b_j(\vec{x}) - 0)^2$$

Area Deviation Polynomial



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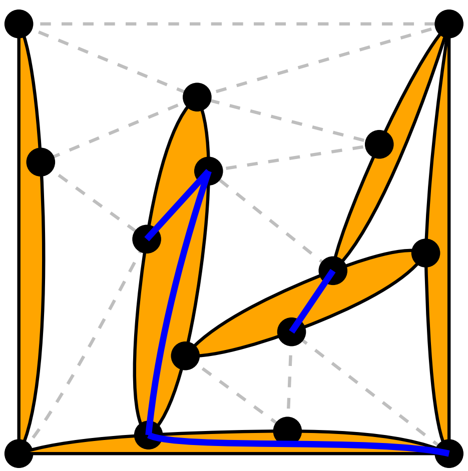
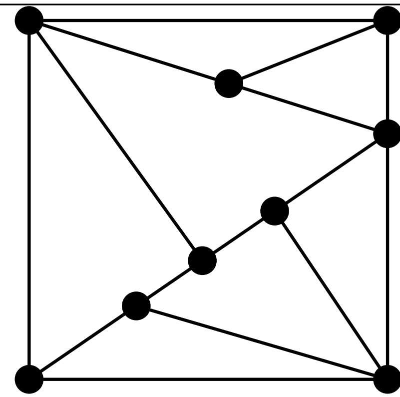


z zero-area triangles, areas $b_1, \ldots, b_z$

$$Z(\vec{x}) = \sum_{j=1}^{z} (b_j(\vec{x}) - 0)^2$$

$$T(\vec{x}) + Z(\vec{x}) \to \text{min!}, \ \vec{x} \in \mathbb{R}^{2v}$$

Lower-Bound Argument



$\min\{\,\mathrm{RMS}^2 \cdot n \mid \text{dissection}\,\}$

$= \min\{\, T(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v},\ \vec{x} \text{ is a dissection}\,\}$

$\geq \min\{\, T(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0\,\}$

$\geq \min\{\, T(\vec{x}) + Z(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0\,\}$

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> 0 (à la Sperner and Monsky)

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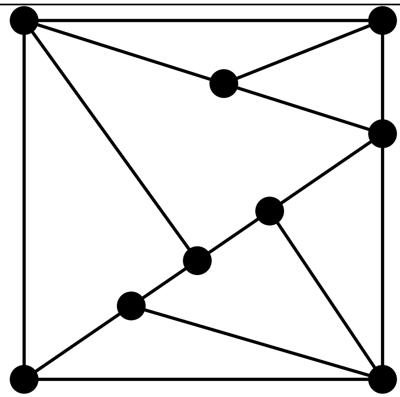
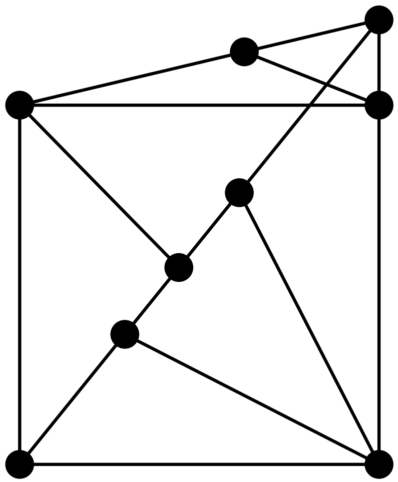


$\geq \min\{\, T(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0 \,\}$



$\geq \min\{\, T(\vec{x}) + Z(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v}, Z(\vec{x}) = 0 \,\}$

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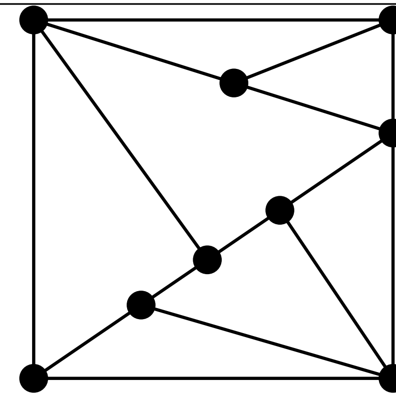

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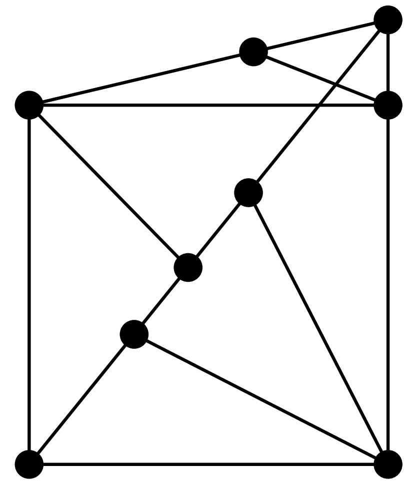

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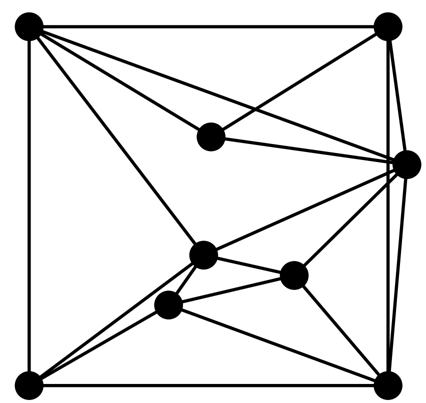

$\geq \min\{\, T(\vec{x}) + Z(\vec{x}) \mid \vec{x} \in \mathbb{R}^{2v} \,\}$

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Gap Theorems

"An algebraic number $\alpha \neq 0$ cannot be arbitarily close to 0."
(depending on the degree and the size of the coefficients)

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The DMM bound ("Davenport–Mahler–Mignotte")
[Emiris–Mourrain–Tsigaridas ,2010]

- polynomial $f(\vec{x})$ of degree d in k variables
- integer coefficients with $\leq \tau$ bits
- $f(x) > 0$ on the unit simplex in $\mathbb{R}^k$

$$\implies \min\{\, f(x) \mid x \in \text{unit simplex}\,\} \geq m_{\mathsf{DMM}}$$

$$\frac{1}{m_{\mathsf{DMM}}} = 2^{d(d-1)^{(k-1)}\left((d\log_2 k+\tau+1)(k+1)+(k^2+3k+1)\log_2 d+d+2k+1\right)} \times 2$$

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$$\frac{1}{m_{\mathsf{DMM}}} = 2^{d(d-1)^{(k-1)}}\Big((d \log_2 k + \tau + 1)(k+1) + (k^2 + 3k + 1)\log_2 d + d + 2k + 1\Big) \times 2$$

THEOREM: If the unit square is dissected into an odd number n of triangles, the range of areas is at least $1/2^{2^{\Omega(n)}}$.

Computer Experiments

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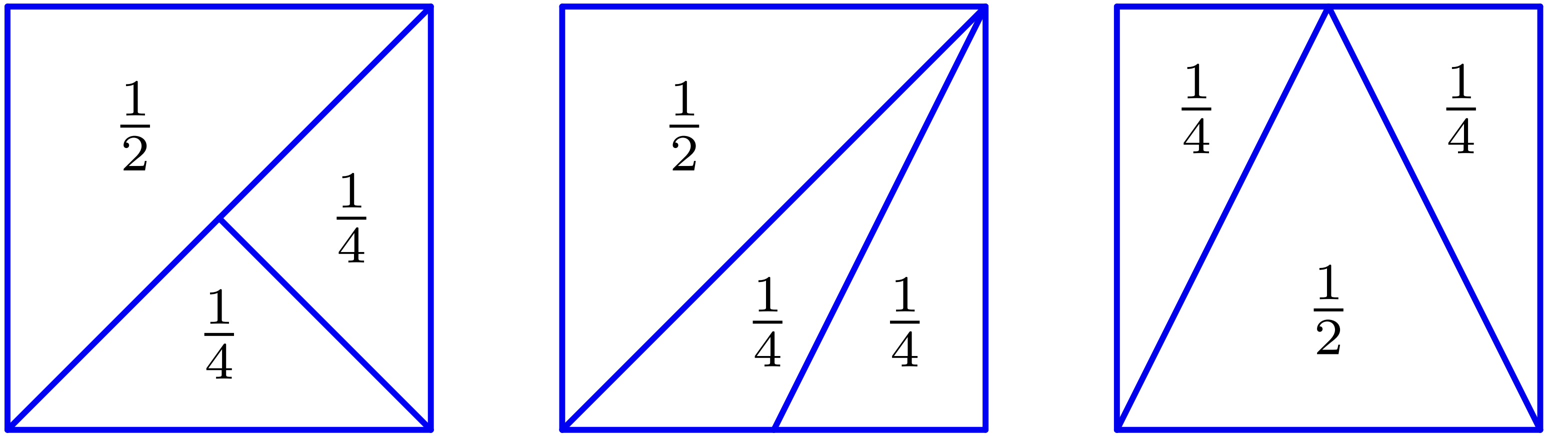
1. Generate all combinatorial types of triangulations/dissections
[`plantri` by Brinkmann and McKay]

2a. [Katja Mansow 2003] for triangulations:
Minimize the range numerically
[`minmax` command of MATLAB]

2b. For dissections: Minimize the squared error (RMS):
Find critical points of $T(\vec{x}) + \lambda Z(\vec{x})$.
$\rightarrow$ system of polynomial equations
[`Bertini` of Bates, Hauenstein, Sommese, Wampler]

Computer Experiments

$n = 3$, RMS $= 0.11786$, range $= 0.25$



Computer Experiments

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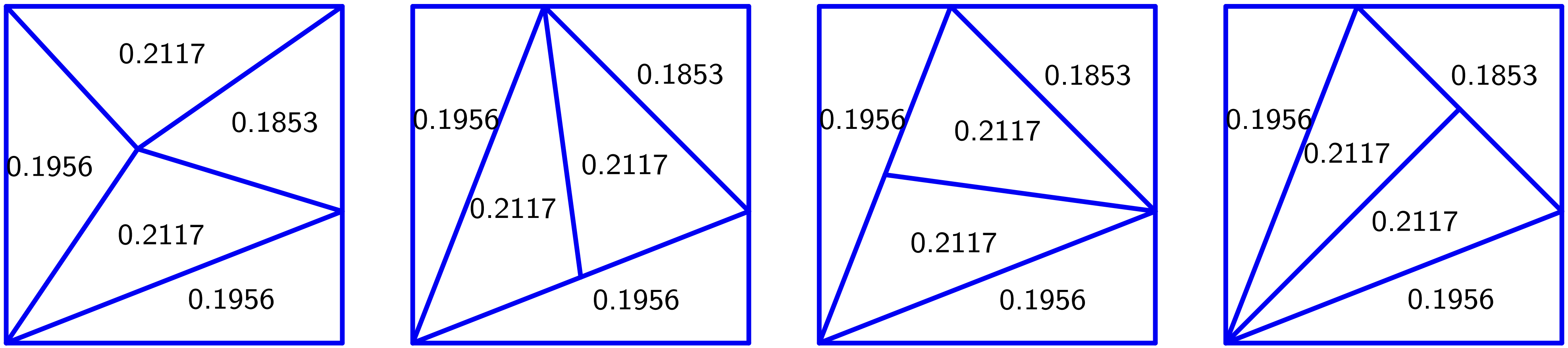
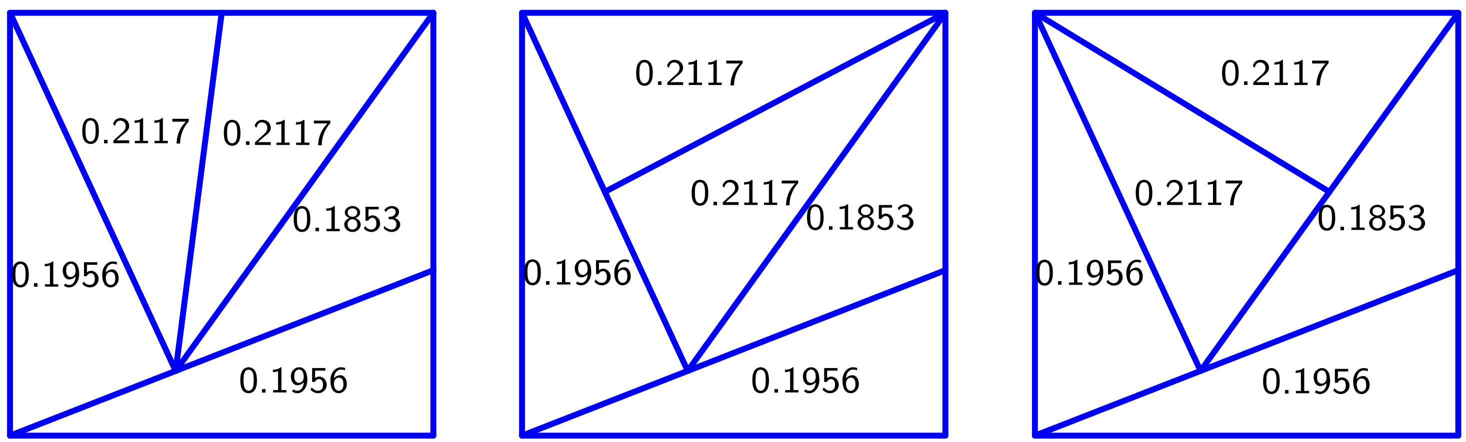
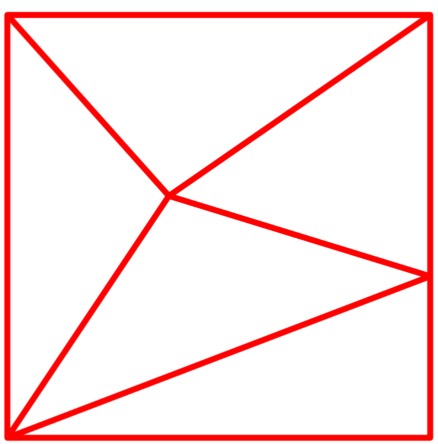
$n = 5$, RMS $= 0.01030$



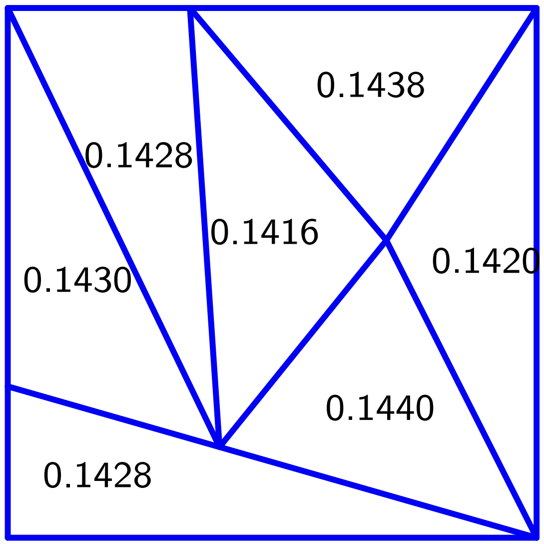


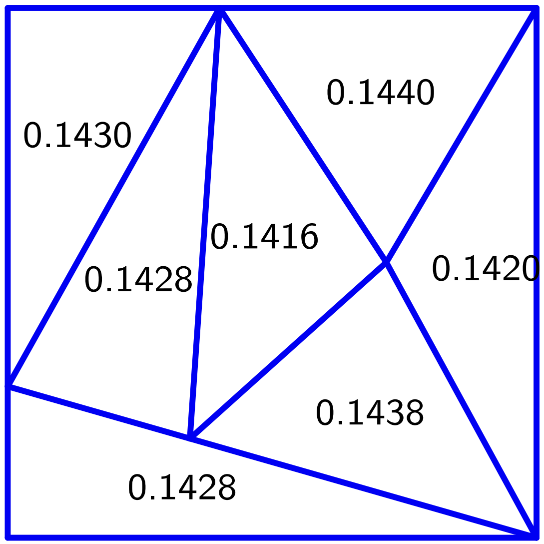


← the best *triangulation* found by Mansow:
range $= 0.0225$

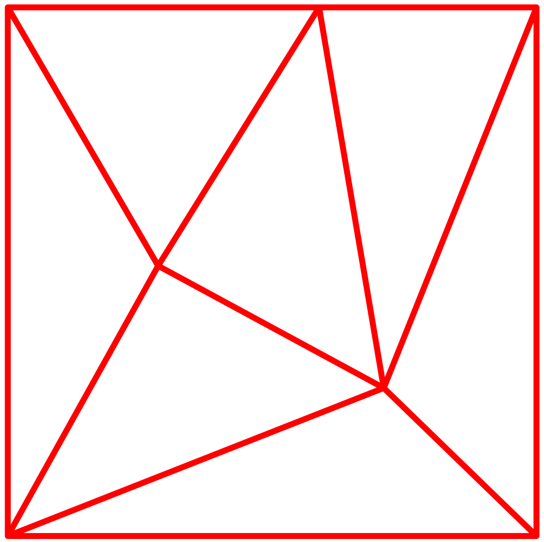
$n = 7$, RMS $= 0.000778$

the RMS-optimal solutions with at most 8 vertices:





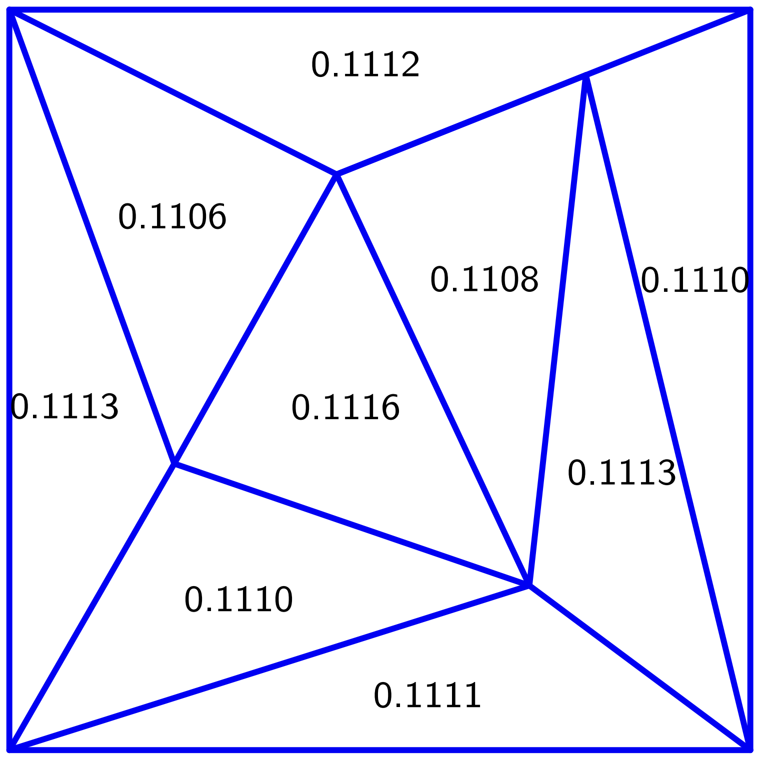
the best *triangulation* found by Mansow:
range $= 0.0031$



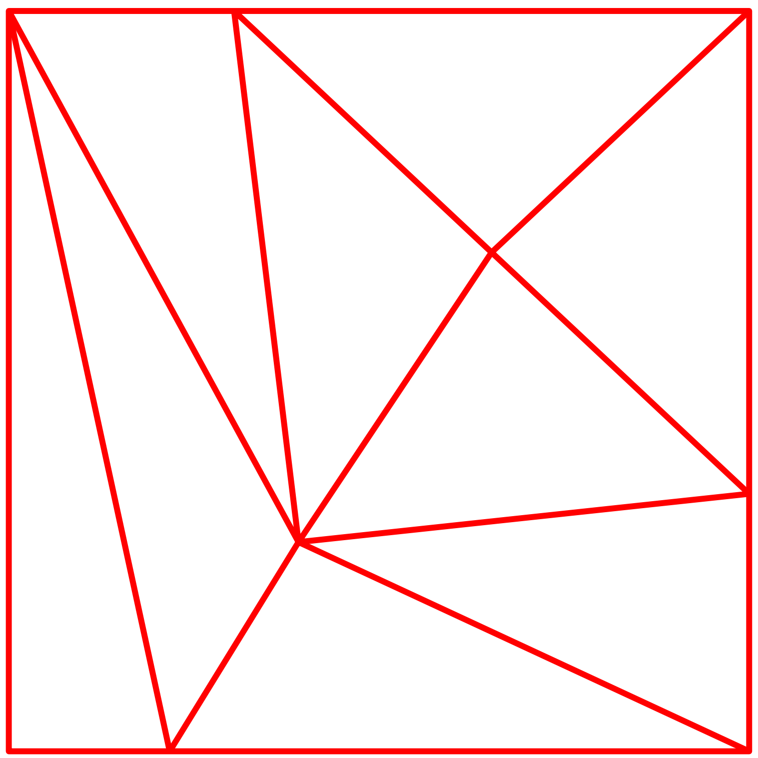
Computer Experiments

$n = 9$, RMS $= 0.000274$

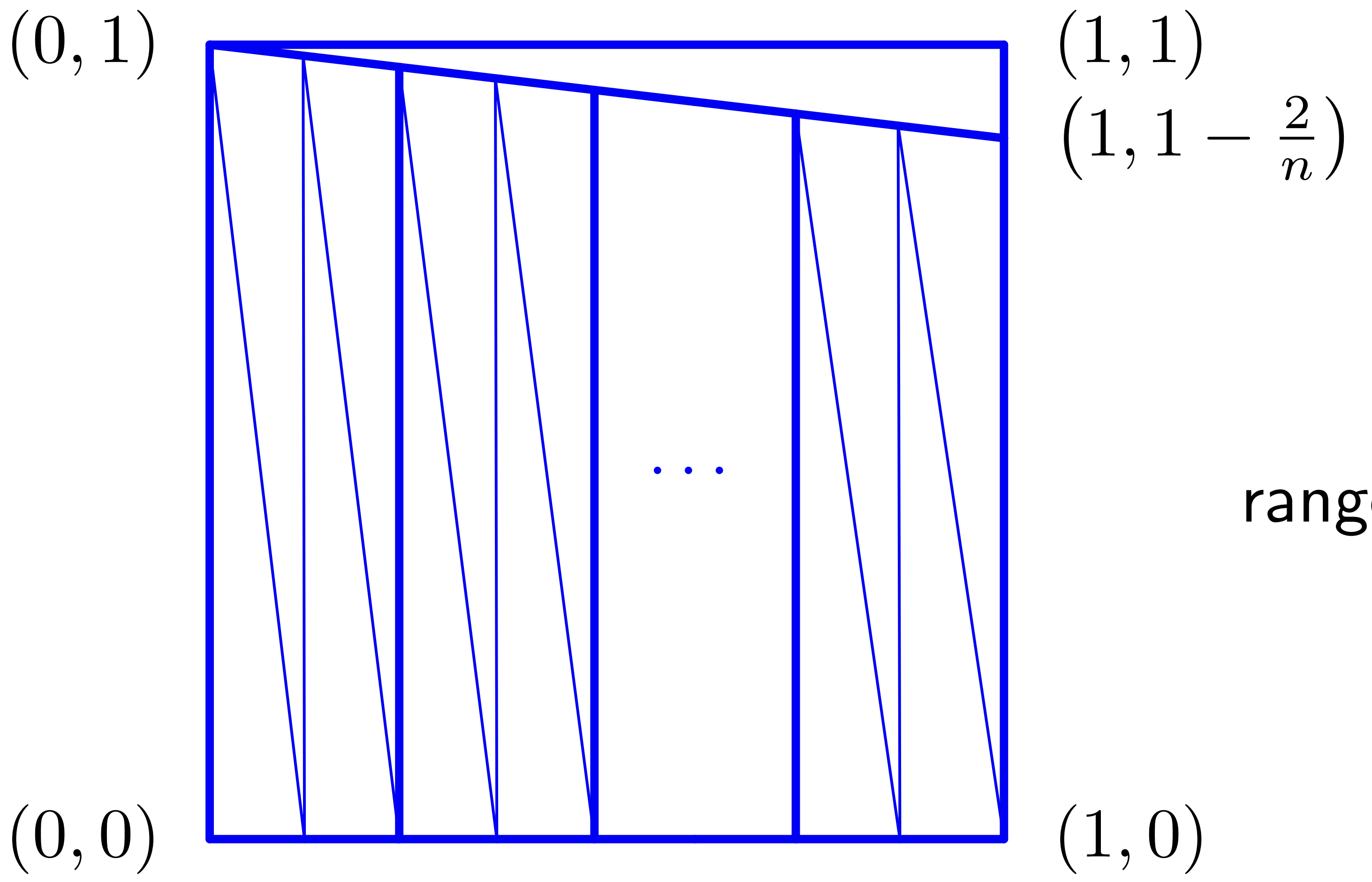
the RMS-optimal solution with at most 8 vertices:



best *triangulation* found by Mansow:
9 vertices, range $= 0.00014$!

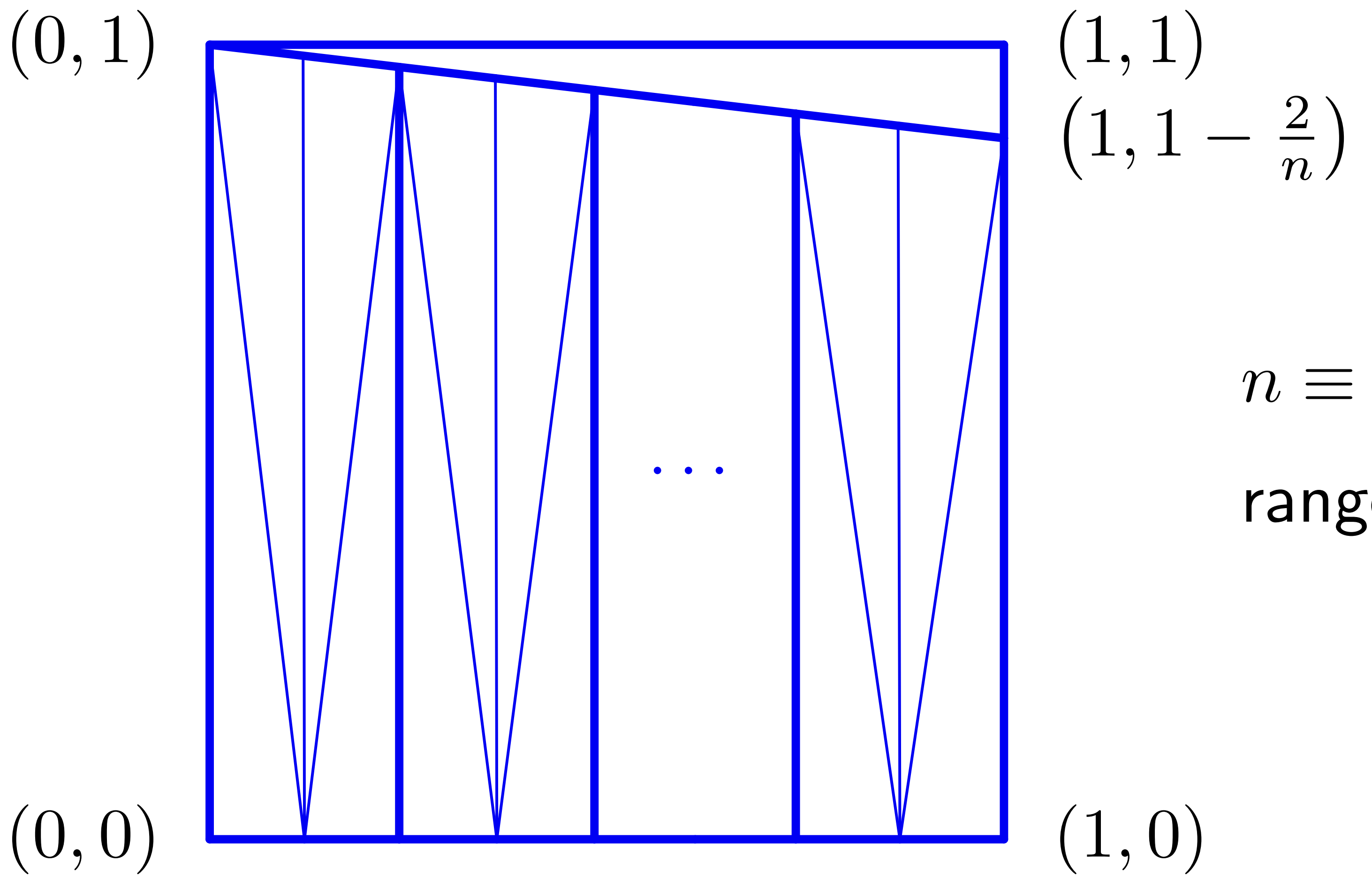


A Systematic Construction



range $= O(1/n^3)$

A Systematic Construction

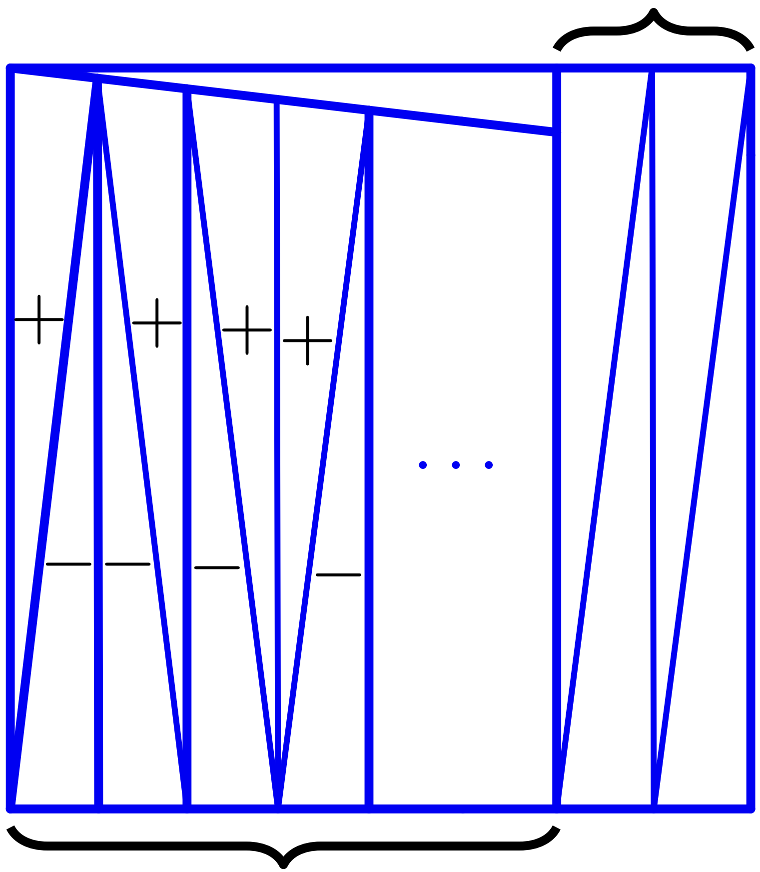


$n \equiv 1 \pmod 4$

range $= O(1/n^5)$

A Systematic Construction

$n - 2^{\lfloor \log_2 n \rfloor} - 1$ filler triangles



$2^{\lfloor \log_2 n \rfloor}$ triangles

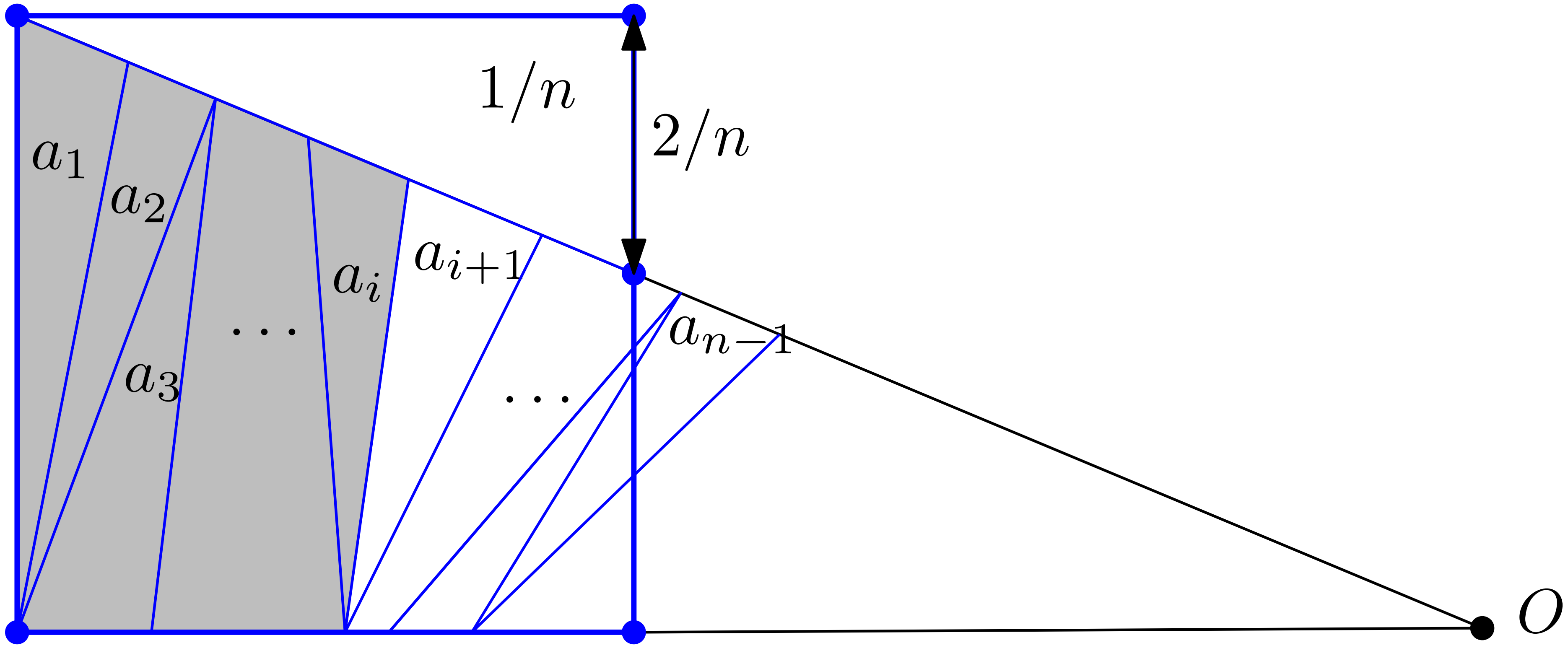
THEOREM:

$$\text{range} \le \frac{8n^2}{n^{\log_2 n}}\left(1 + O\left(\frac{\log n}{n}\right)\right)$$

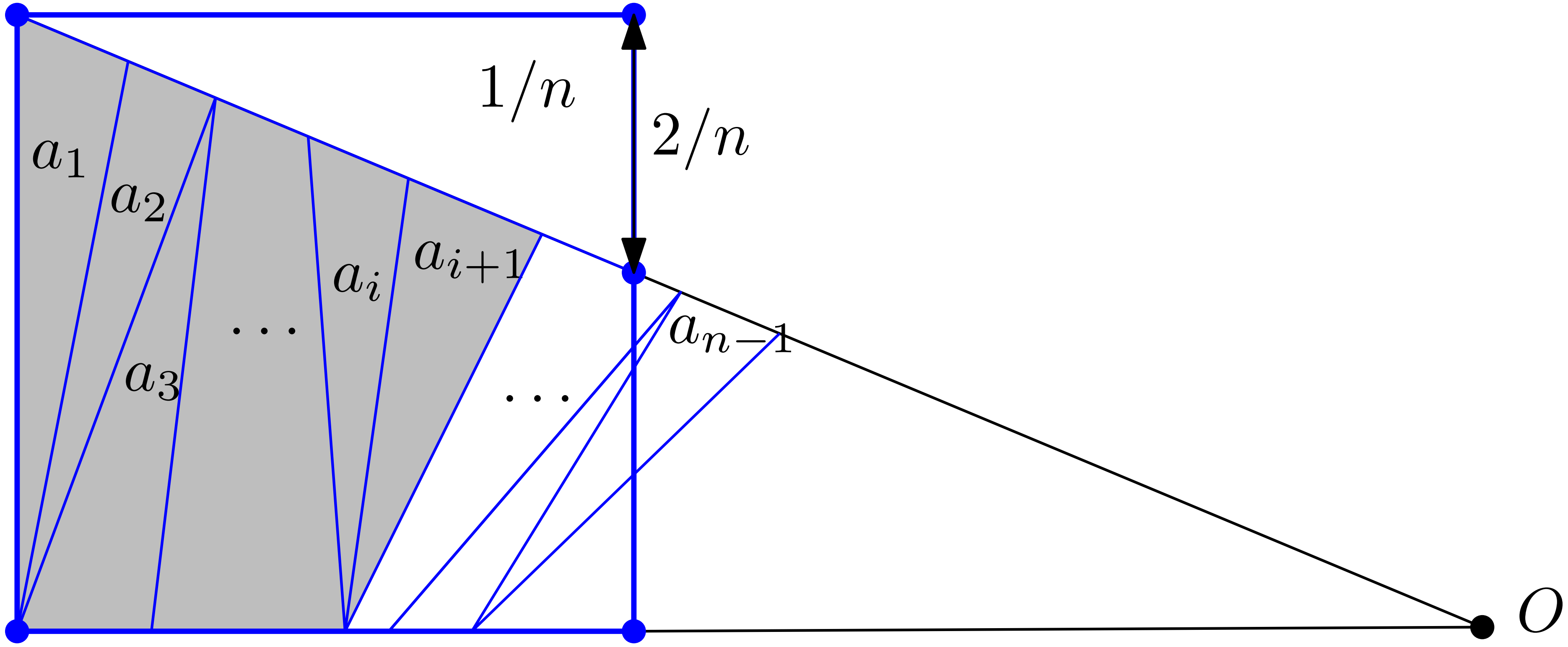
use the Thue-Morse sequence $s_1 s_2 s_3 \ldots =$

$+--+-++--++-+--+-++-+--++--+-++-\cdots$

Estimating the error



Estimating the error



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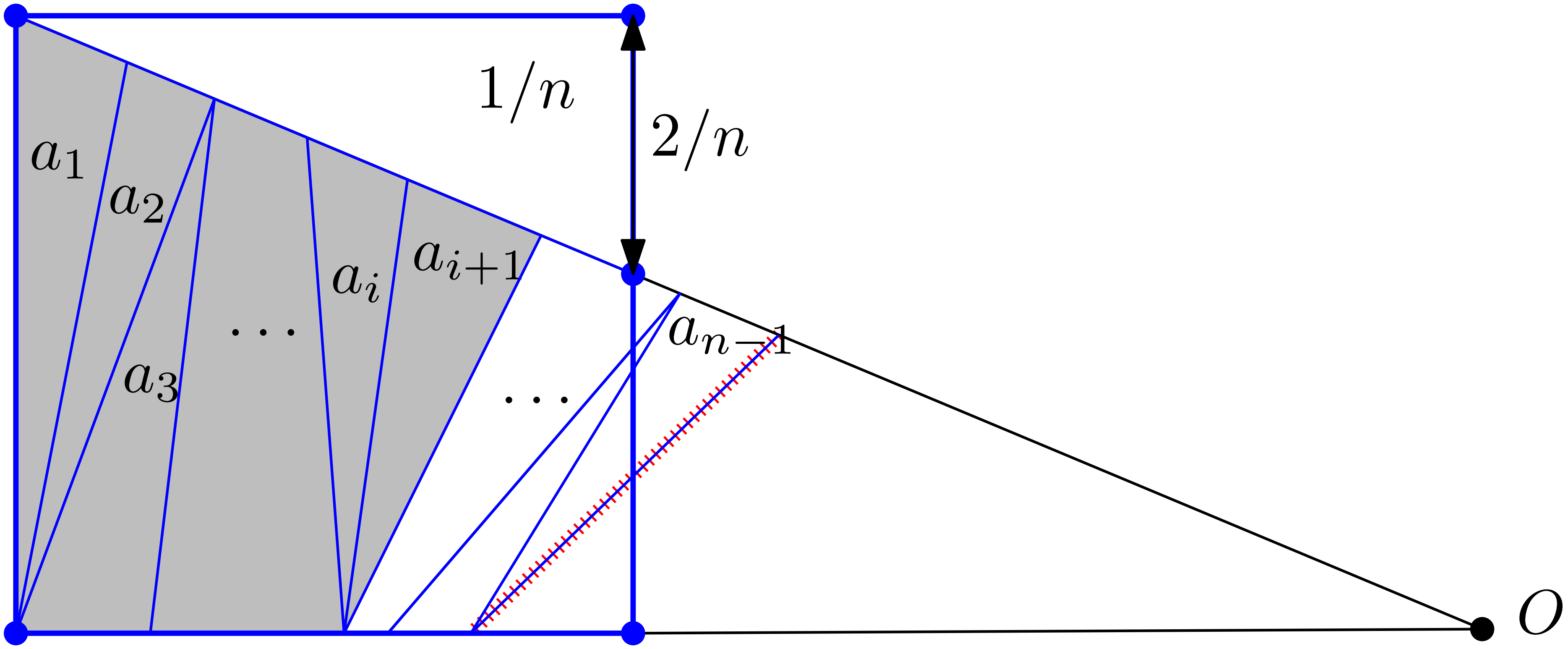
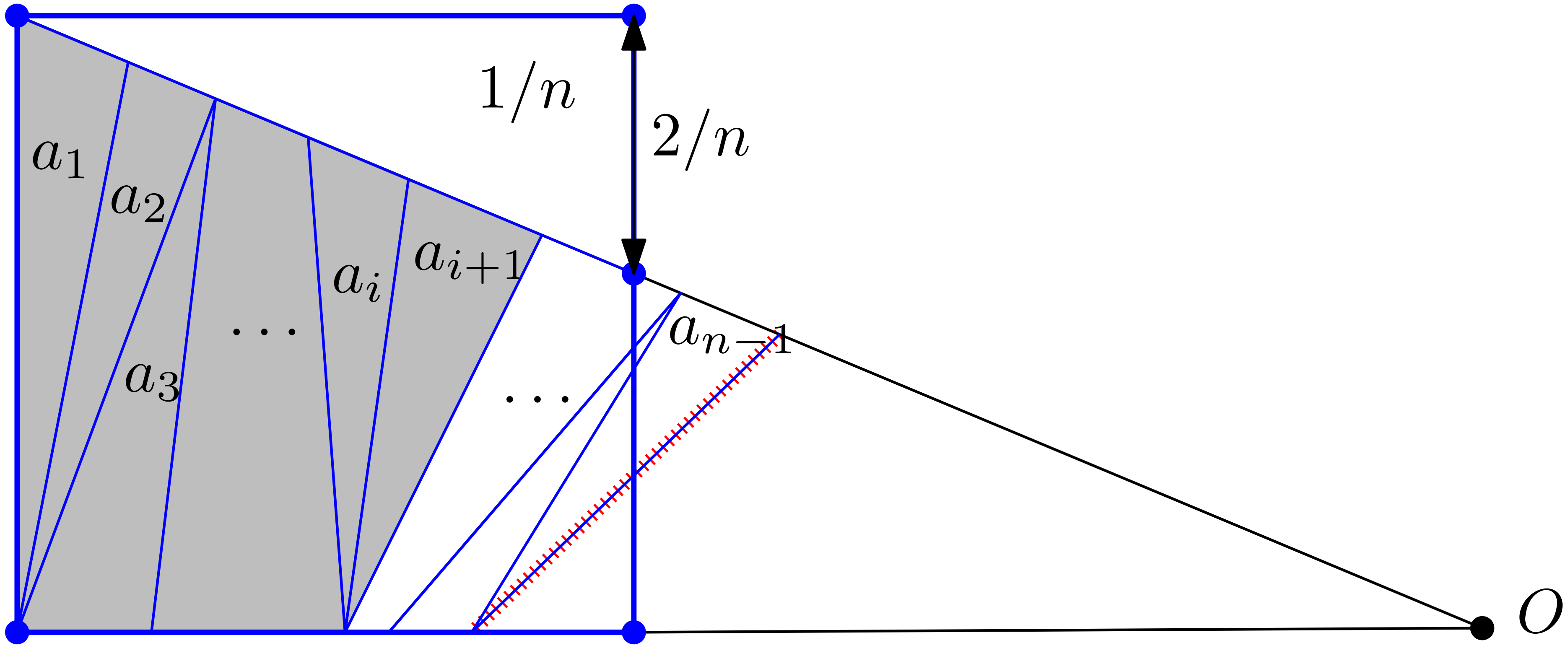


$$\prod_{i=1}^{n-1}\Big(\frac{1-iU}{1-(i-1)U}\Big)^{s_i} \overset{!}{\approx} 1, \qquad U := \tfrac{4}{n^2}$$

Estimating the error



$$\prod_{i=1}^{n-1} \Big(\frac{1 - iU}{1-(i-1)U} \Big)^{s_i} \overset{!}{\approx} 1, \qquad U := \frac{4}{n^2}$$

$$W := \sum_{i=1}^{n-1} s_i \ln \frac{1-iU}{1-(i-1)U} \overset{!}{\approx} 0$$

Estimating the error

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$$W := \sum_{i=1}^{n-1} s_i \ln \frac{1-iU}{1-(i-1)U} \overset{!}{\approx} 0$$

$$\sum_{i=1}^{n-1} s_i \ln \frac{1-iU \pm \varepsilon}{1-(i-1)U \pm \varepsilon} \overset{!}{=} 0$$

W small $\to \varepsilon$ small. $\implies$ Concentrate on small W!

$$\sum_{i=1}^{n-1} s_i \ln(1-iU)$$

$$= \sum_{i=1}^{n-1} s_i \left(-iU - \frac{i^2}{2}U^2 - \frac{i^3}{3}U^3 - \cdots \right)$$

Try to cancel the first powers of U

Annihilate Powers

$1^0 - 2^0 - 3^0 + 4^0 - 5^0 + 6^0 + 7^0 - 8^0 - 9^0 + 10^0 + 11^0 - 12^0 + 13^0 - 14^0 - 15^0 + 16^0 =$

$1 - 2 - 3 + 4 - 5 + 6 + 7 - 8 - 9 + 10 + 11 - 12 + 13 - 14 - 15 + 16 =$

$1^2 - 2^2 - 3^2 + 4^2 - 5^2 + 6^2 + 7^2 - 8^2 - 9^2 + 10^2 + 11^2 - 12^2 + 13^2 - 14^2 - 15^2 + 16^2 =$

$1^3 - 2^3 - 3^3 + 4^3 - 5^3 + 6^3 + 7^3 - 8^3 - 9^3 + 10^3 + 11^3 - 12^3 + 13^3 - 14^3 - 15^3 + 16^3 =$

$1^4 - 2^4 - 3^4 + 4^4 - 5^4 + 6^4 + 7^4 - 8^4 - 9^4 + 10^4 + 11^4 - 12^4 + 13^4 - 14^4 - 15^4 + 16^4 =$

Theorem (E. Prouhet 1851)

If f is a polynomial of degree $< k$ then

$$\sum_{i=1}^{2^k} s_i \cdot f(i) = 0,$$

for the Thue-Morse sequence $s_1, s_2, s_3, \ldots$.

Upper bound

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THEOREM:

For every n, there is a dissection with

$$\text{range} \le \frac{8n^2}{n^{\log_2 n}} \cdot \left(1 + O\left(\frac{\log n}{n}\right)\right)$$

Systematic is not Always Best

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n	optimal sign sequence s	$\varepsilon = \pm$range$/2$
3*	$+-$	-0.16667
5*	$+--+$	$+0.01250$
7	$+-+--+$	-0.00010248
9*	$+--+-++-$	-0.00016360
11	$+-+-+--+-+$	-4.1201×10^{-6}
13	$+--++--+-++-$	$+5.9928\times10^{-6}$
15	$+-+-+-+--+-+-+$	-5.2871×10^{-7}
17*	$++-+--+--+-+--++$	-3.4708×10^{-8}
19	$+--+--+-++++--+-+-$	$+4.2052\times10^{-8}$
21	$+-+--+-++---++++---+$	-5.5778×10^{-9}
23	$++--+--++--+-++--+-+-+$	$+3.5359\times10^{-9}$
25	$+++-+-----+++----++++--+$	-7.457×10^{-10}
27	$++--+-+-+--+-++--+--++-++-$	-1.266×10^{-10}
29	$++---+-+-++---+++-+---++++--$	$+9.026\times10^{-12}$
31	$+++-+--+-+-+----+-++--+-+-+-++$	$+2.446\times10^{-12}$
33*	$+--+--++--++++-+--+++-----+-++-+$	-1.423×10^{-12}
35		$+1.777\times10^{-13}$
37		$+1.100\times10^{-14}$

Systematic is not Always Best

Example: $n = 33$

Best sequence:

$$+--+--++--++++-+--+++-----+-++-+$$

$$\varepsilon = -1.4232\times 10^{-12}$$

Thue-Morse:

$$+--+-++--++-+--+-++-+--++--+-++-$$

$$\varepsilon = 1.0615 \times 10^{-10}$$

Guarantee from theorem: $|\varepsilon| \leq 6.6565 \times 10^{-5}$

Heuristic Explanation

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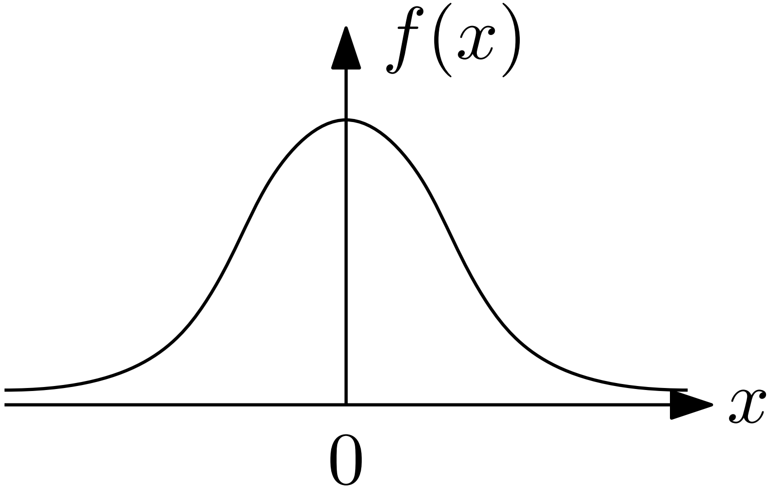
$$W := \sum_{i=1}^{n-1} s_i \cdot \ln \frac{1 - iU}{1 - (i-1)U} \overset{!}{\approx} 0$$

RANDOM $s_i = \pm 1$

W: approximately Gaussian with $\mu = 0$ and $\sigma \approx U\sqrt{n} \sim n^{-3/2}$.

Take $N = 2^{n-1}$ random samples from this distribution.



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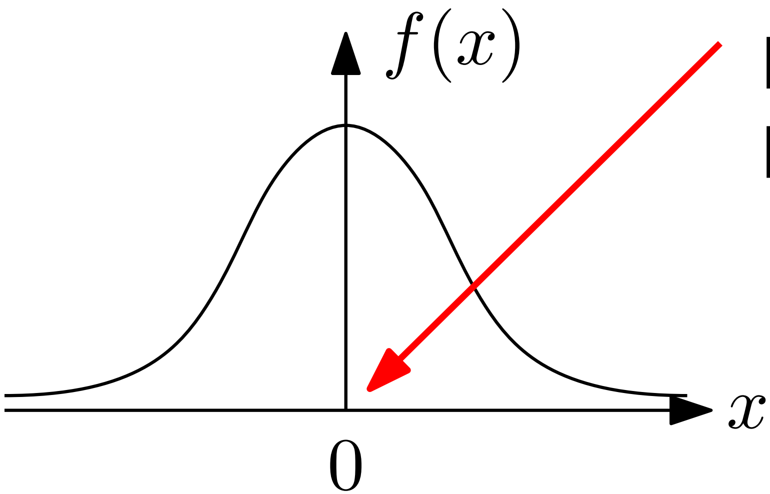
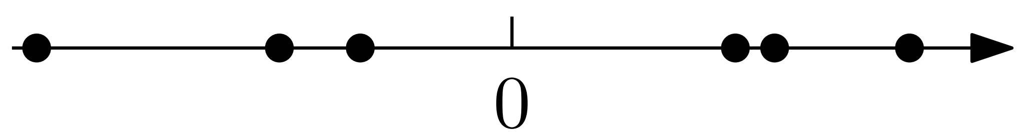


Near $x = 0$, these samples are like a Poisson distribution with density

$$\lambda = N \cdot f(0) \sim 2^n / n^{3/2}$$



$\rightarrow$ Smallest absolute value $= 1/2\lambda \sim n^{3/2}/2^n$

Triangulations?

So far: Ideas for systematic computer experiments.
No general analysis.

The Tarry-Escott Problem

$$\sum_{i=1}^{2^k} s_i \cdot i^d = 0, \text{ for } d = 0, 1, \ldots, k-1$$

Can you annihilate the first k powers with a SHORTER sign sequence?

The Tarry-Escott Problem: Find two distinct sets of integers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that

$$\alpha_1^d + \cdots + \alpha_n^d = \beta_1^d + \cdots + \beta_n^d, \text{ for all } d = 0, 1, 2, \ldots, k-1$$

Try to make n as small as possible.