Most General Position

joint work with

Jan Kynčl, Alexander Pilz, André Schulz

supported by the EuroGIGA project of the European Science Foundation

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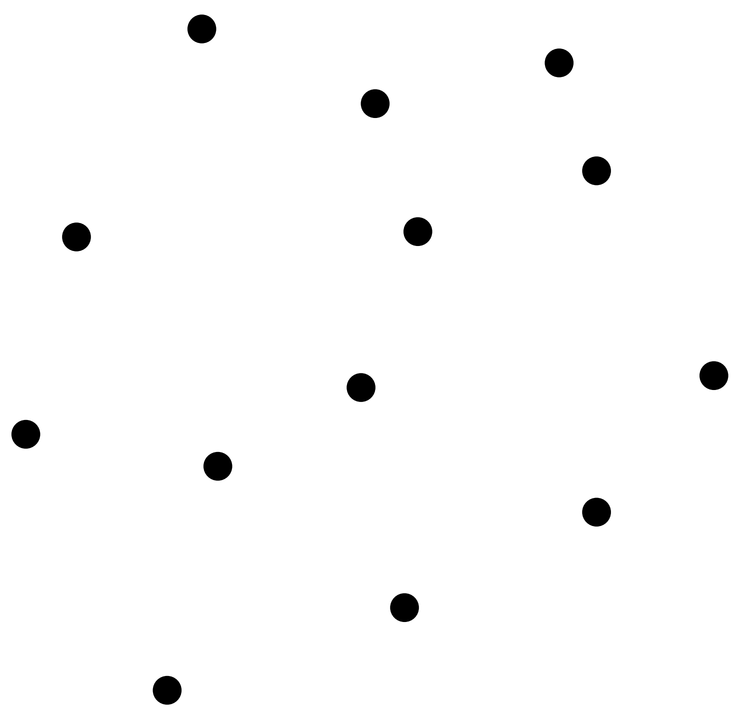

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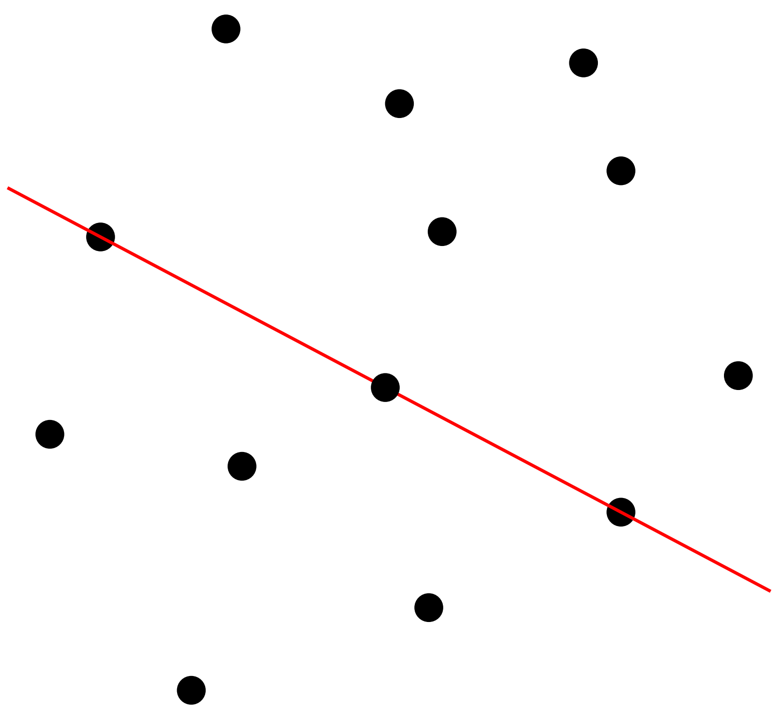

avoid

- collinear points

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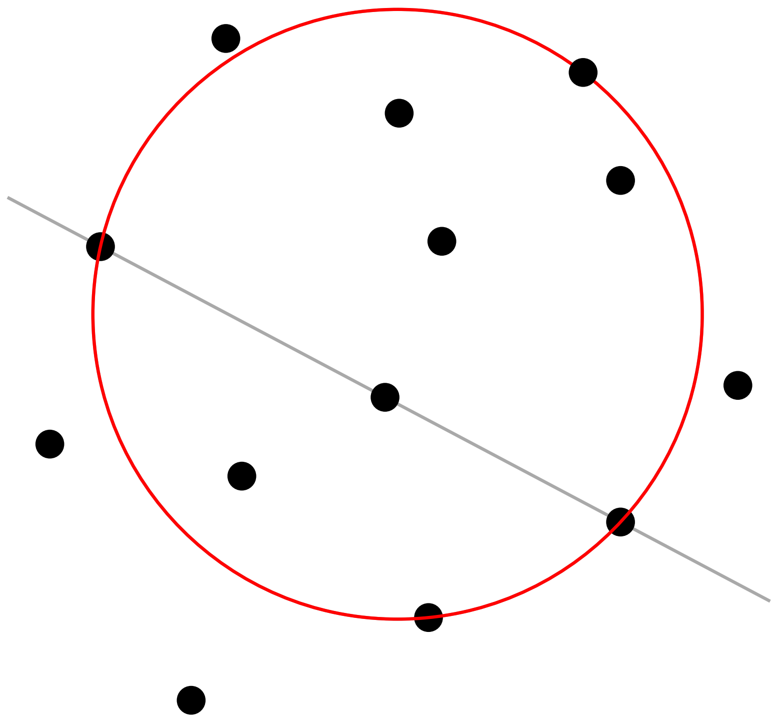

avoid

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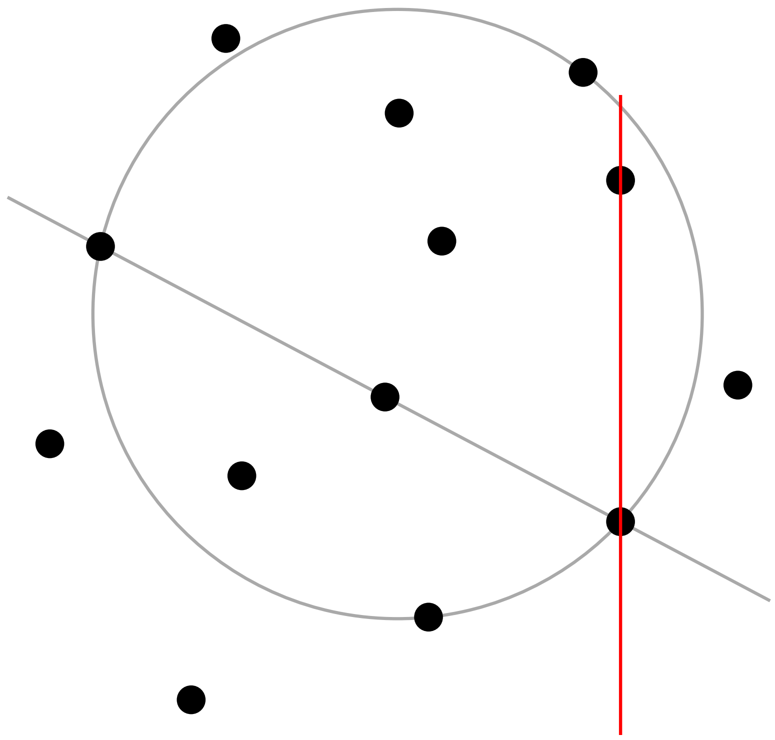
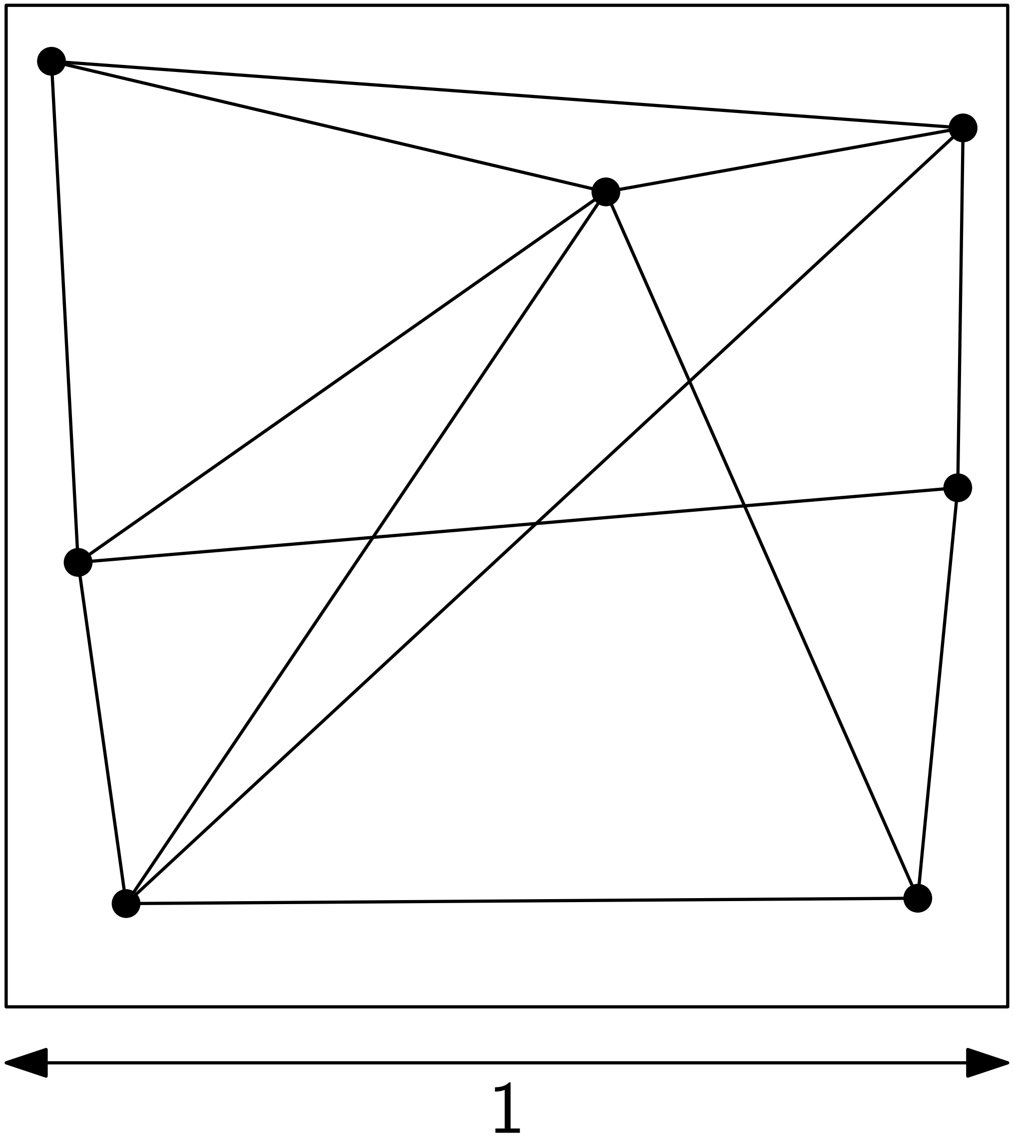


avoid

- collinear points
- cocircular points
- points with the same x-coordinate

Good Drawings

Find a (nonplanar) straight-line drawing of a given graph.



Good Drawings

Freie Universität Berlin

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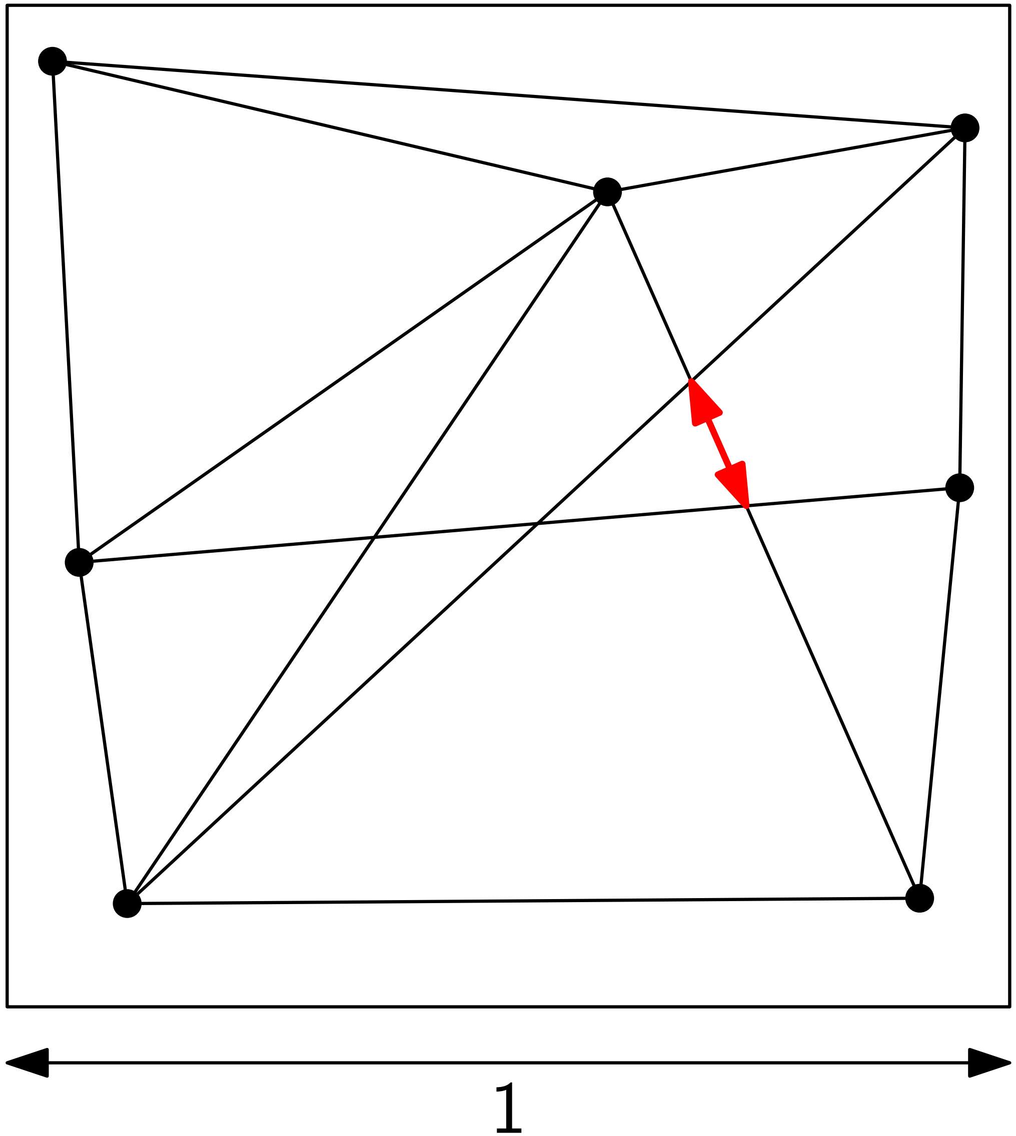

Maximize $\varepsilon =$

- the shortest edge

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Freie Universität Berlin

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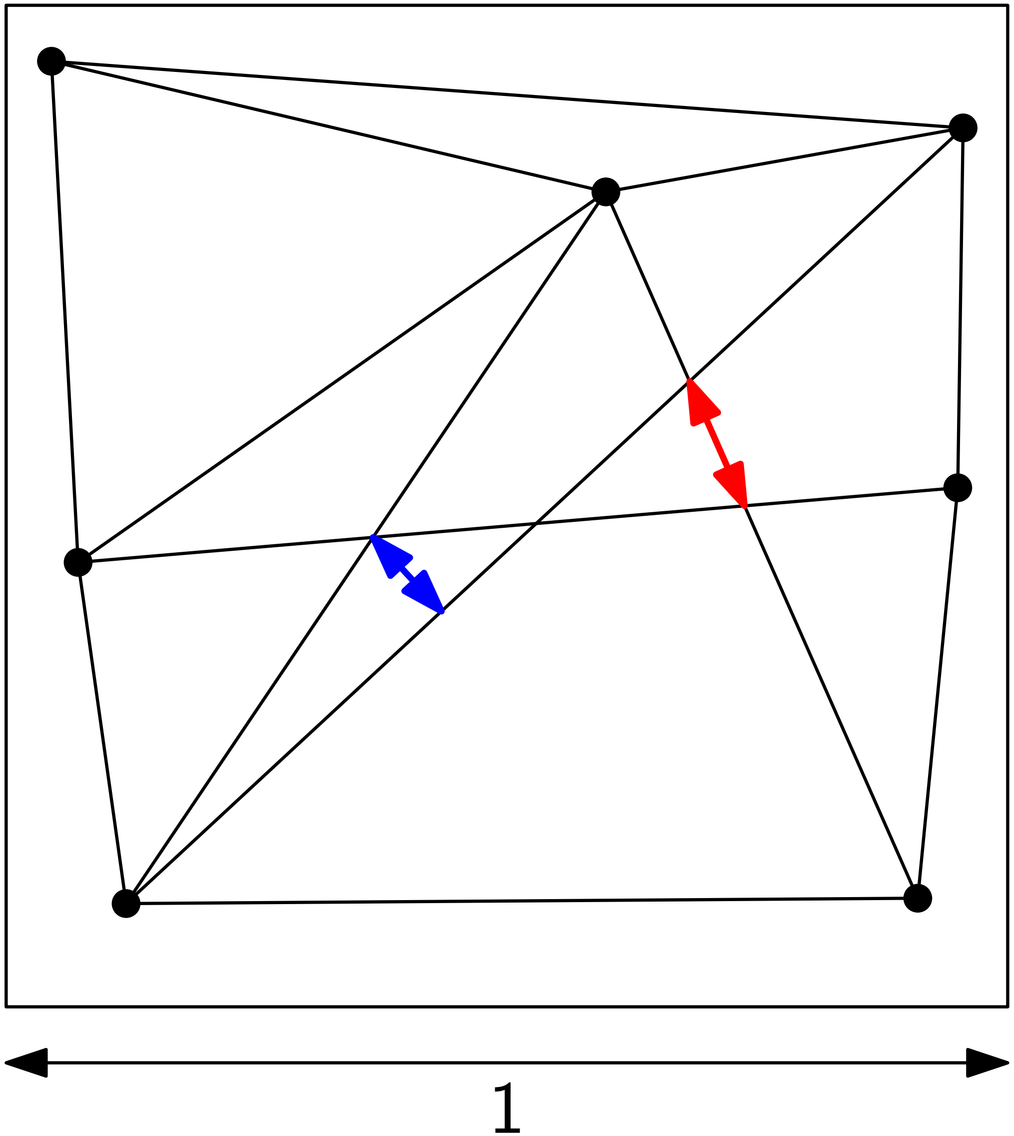

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Freie Universität Berlin

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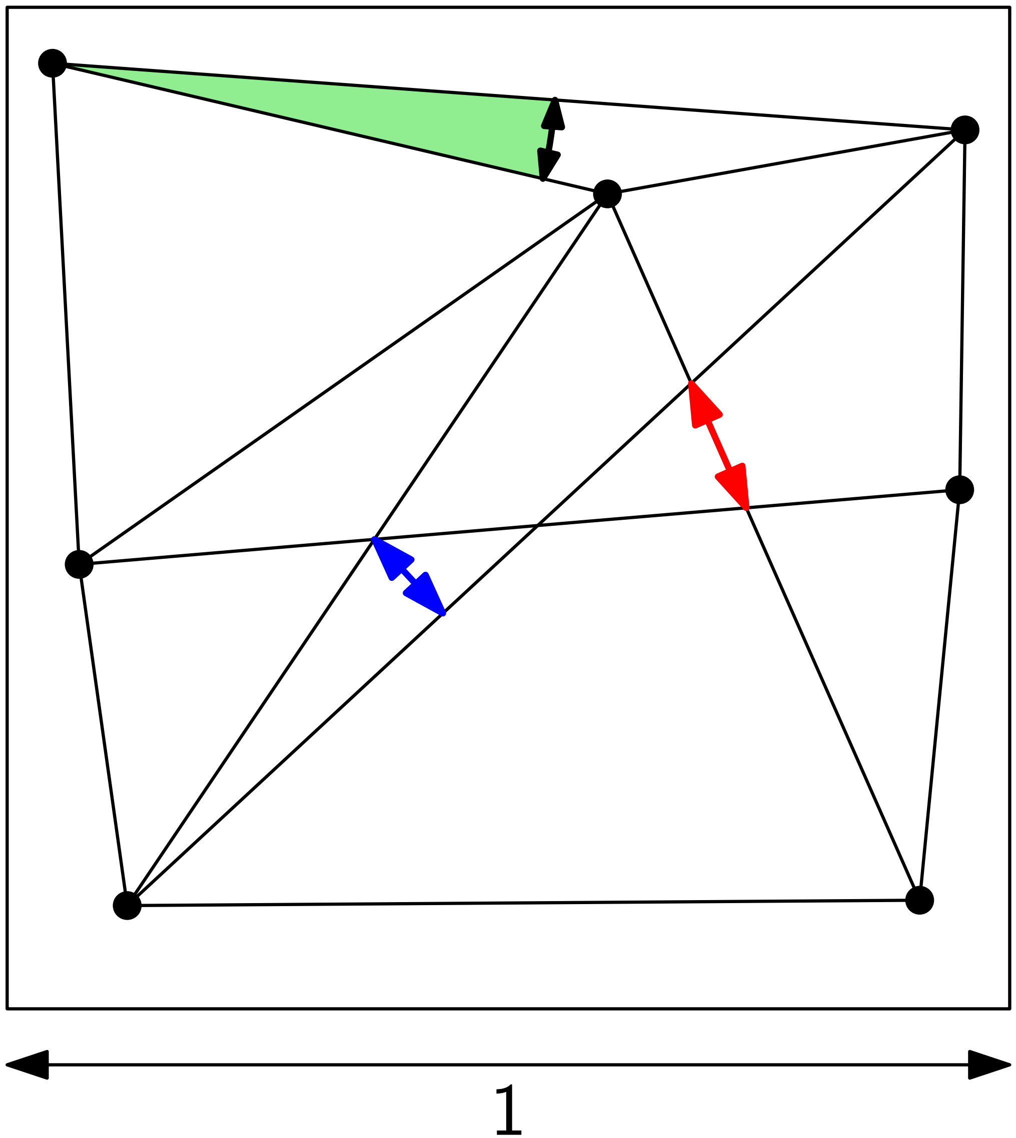

Maximize $\varepsilon =$

- the shortest edge
- the smallest feature distance
- the smallest angle

[realization as a computer game]

Good Drawings

Freie Universität Berlin

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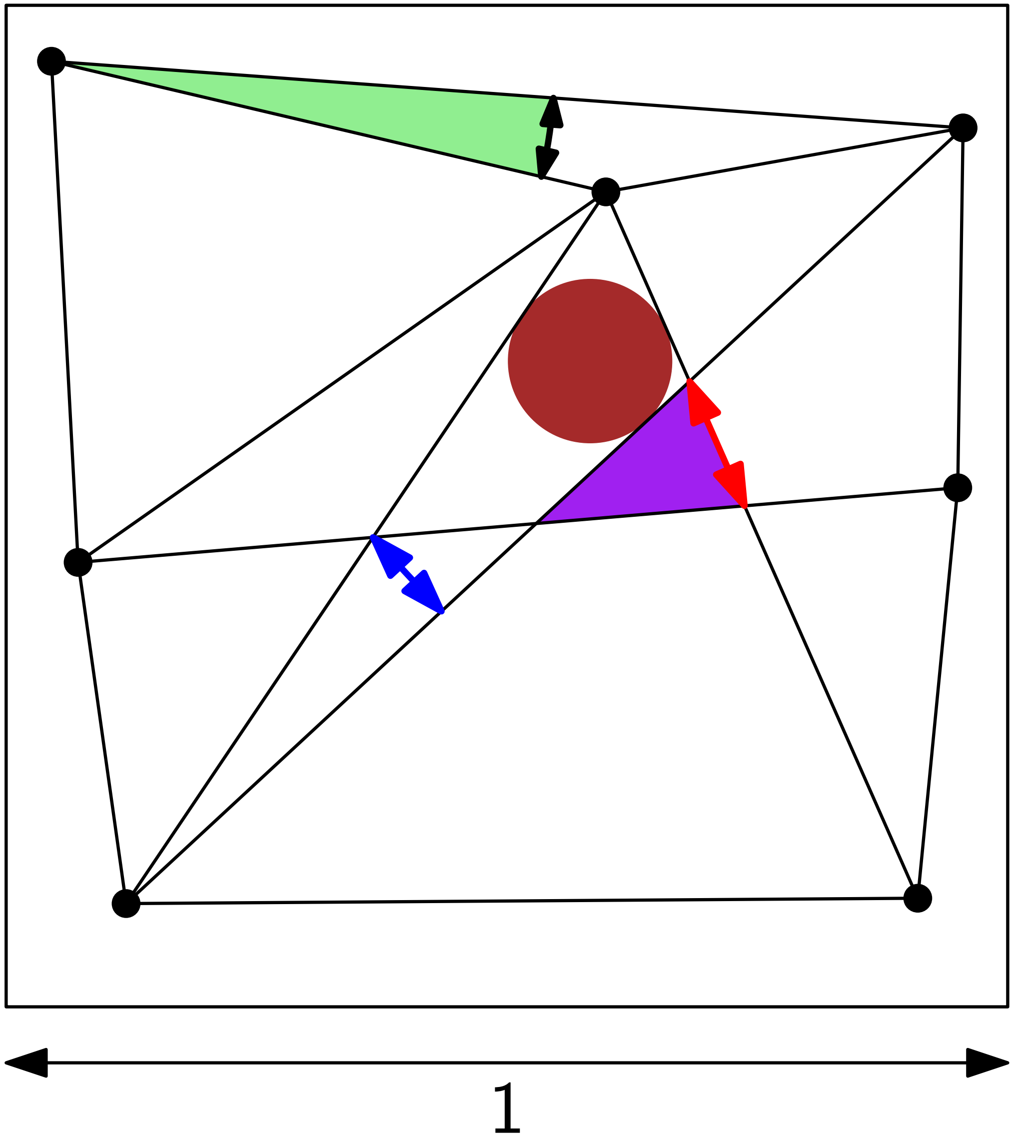


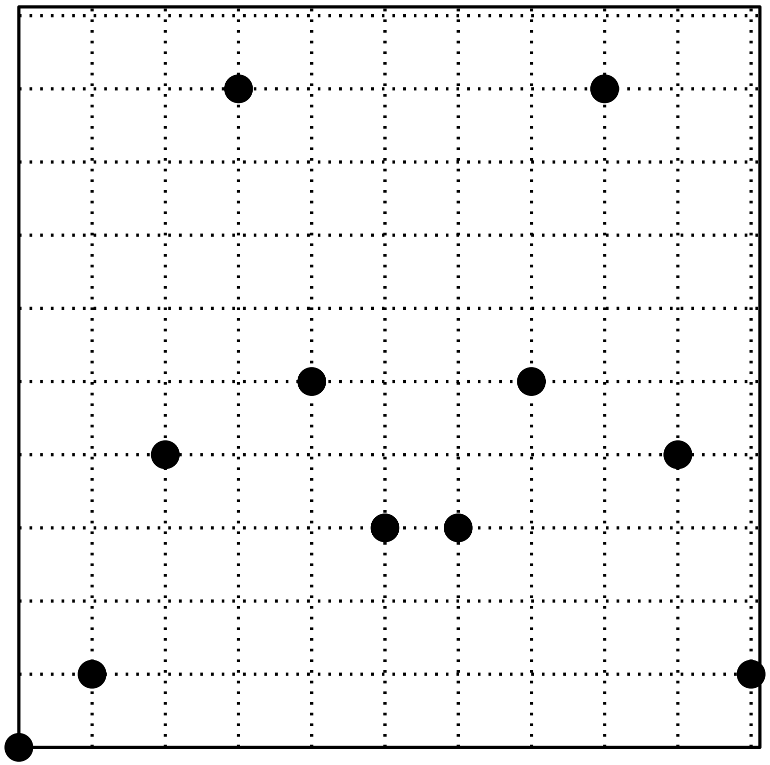
Maximize $\varepsilon =$

- the shortest edge
- the smallest feature distance
- the smallest angle
- the smallest face area
- the smallest incircle
- ...

[realization as a computer game]

Related Problem: The Moment Curve

Draw n points in general position, with small coordinates



Example: $p = 11$

$$P_i := \binom{i}{i^2 \bmod p},\ i = 0, \dots, p-1$$

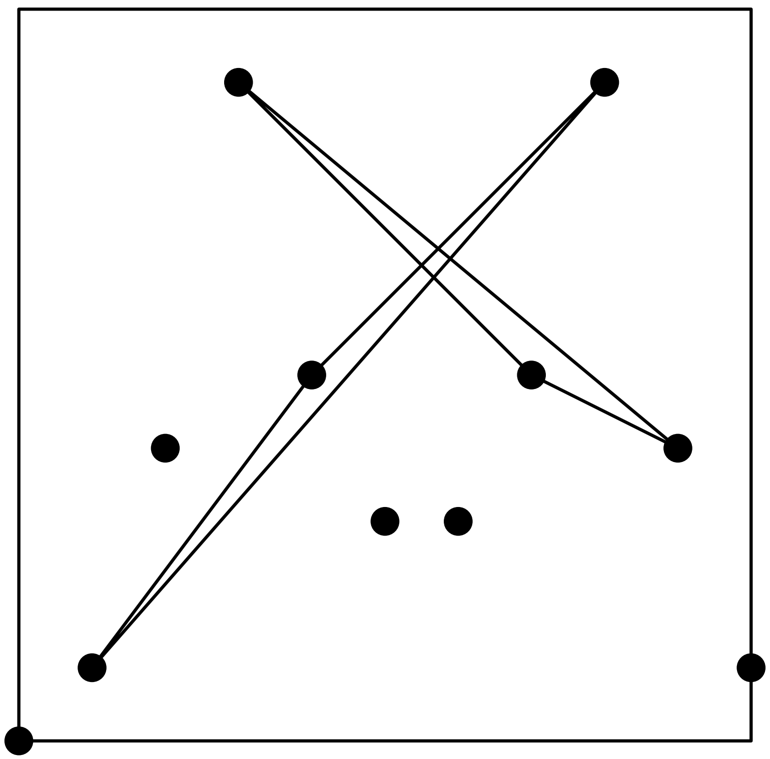
(p prime)

$$\begin{vmatrix} 1 & i & i^2 \bmod p \\ 1 & j & j^2 \bmod p \\ 1 & k & k^2 \bmod p \end{vmatrix}$$
$$\equiv (j-i)(k-i)(k-j) \mod p$$
$$\not\equiv 0 \mod p$$

Application:
Perturbation techniques for geometric algorithms
[Emiris & Canny 1995, Seidel 1998]

Related Problem: Heilbronn's Problem

Place n points in the unit square and maximize the smallest triangle area $\Delta_{\min}$



$$\Delta_{\min} \geq \Omega(1/n^2)$$

Related Problem: Heilbronn's Problem

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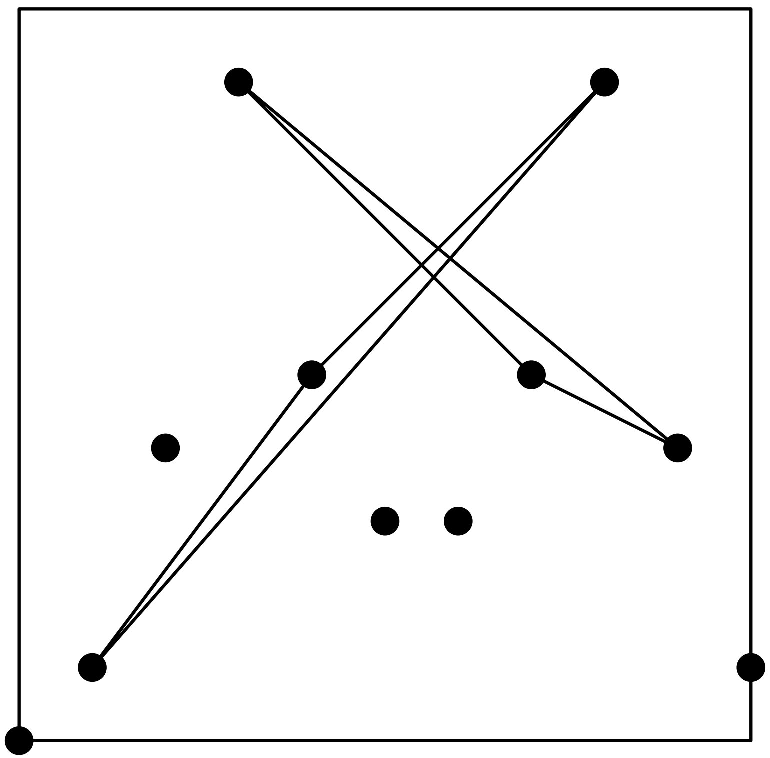


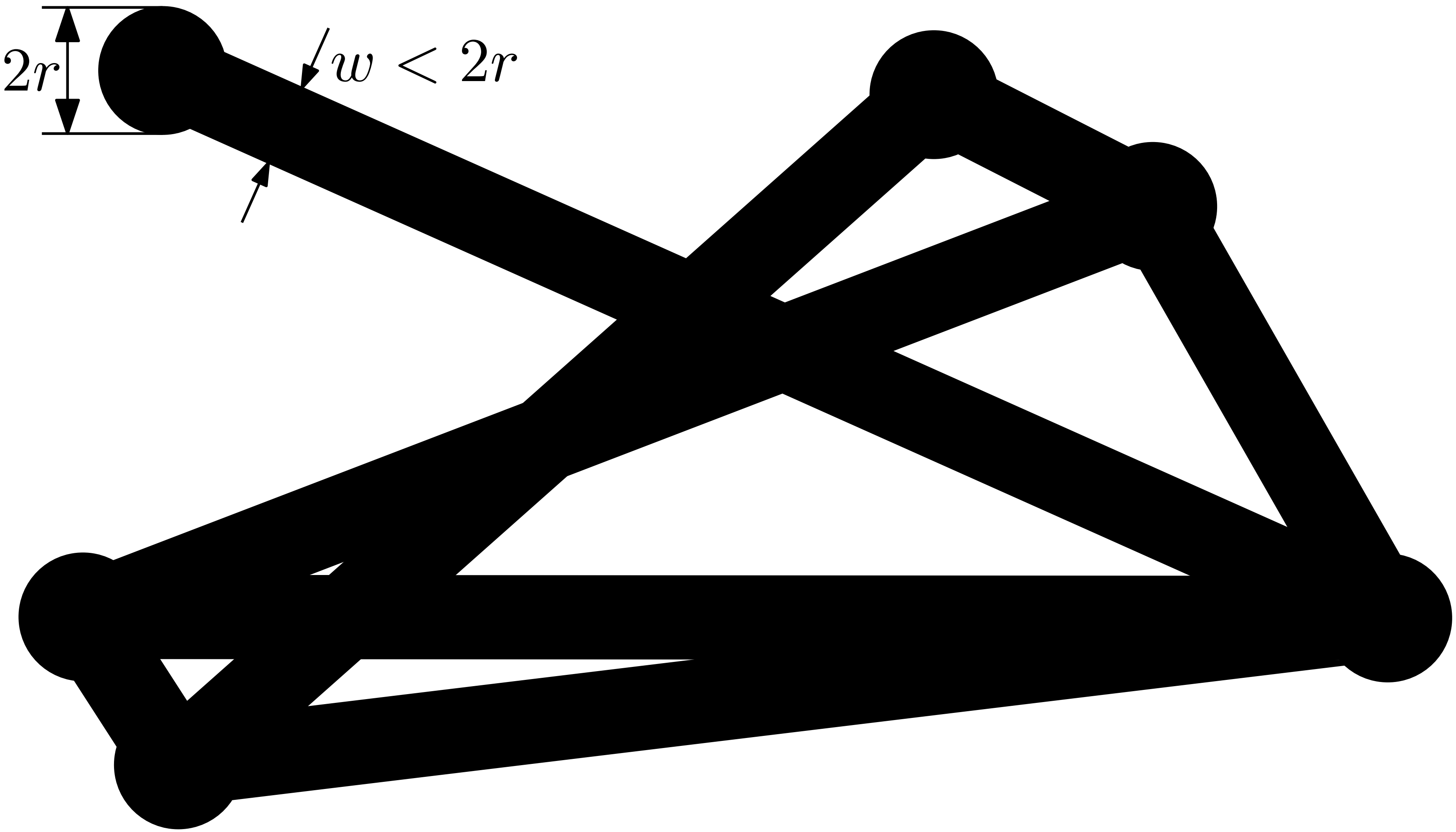
$$\Delta_{\min} \geq \Omega(1/n^2)$$

$$\frac{\ln n}{n^2} \leq \Delta_{\min} \leq \frac{1}{n^{8/7-\varepsilon}}$$

[Komlós, Pintz & Szemerédi 1981,1982]

Related Problem: Bold Drawings

[van Kreveld 2009]



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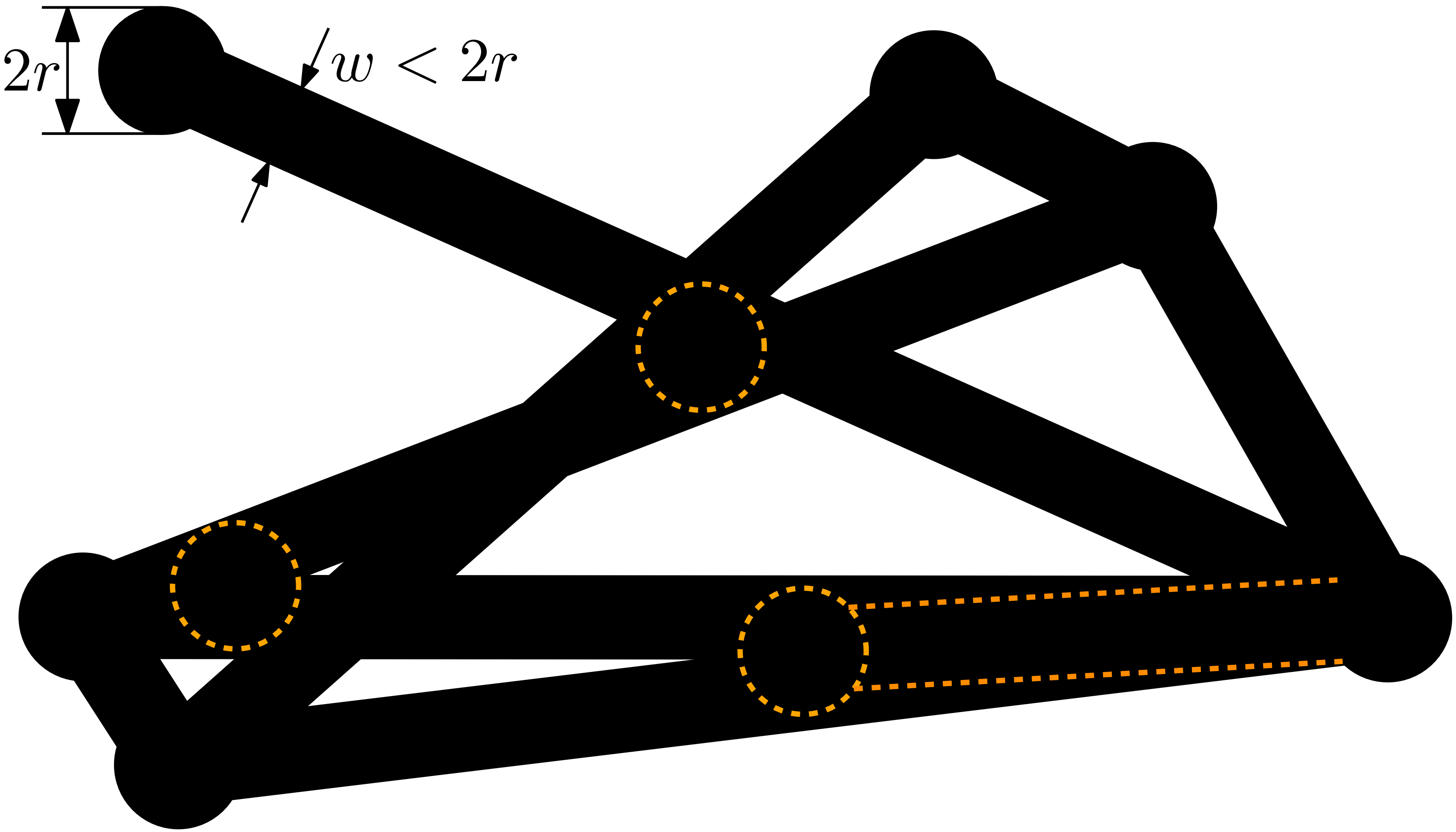
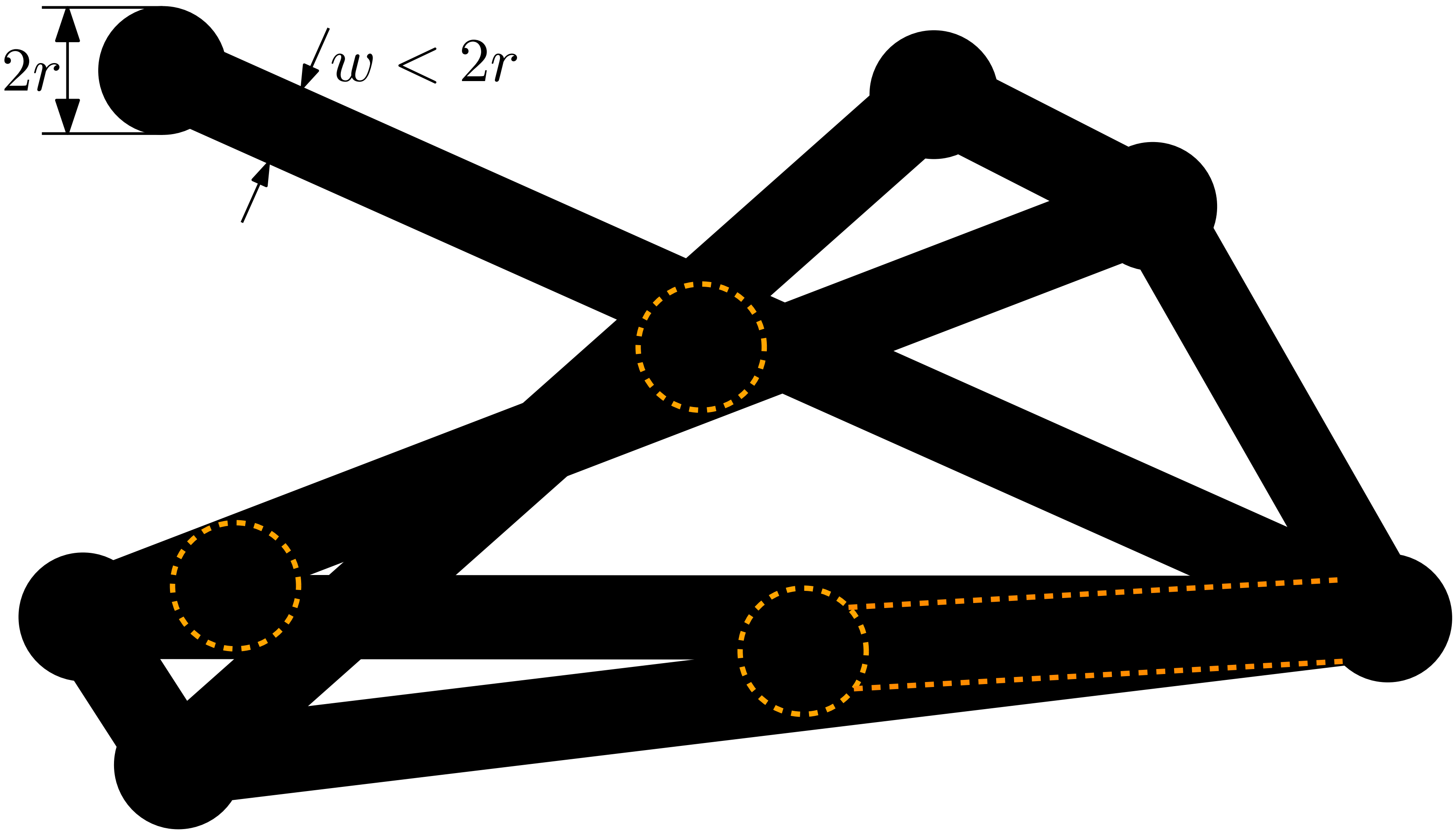
[van Kreveld 2009]



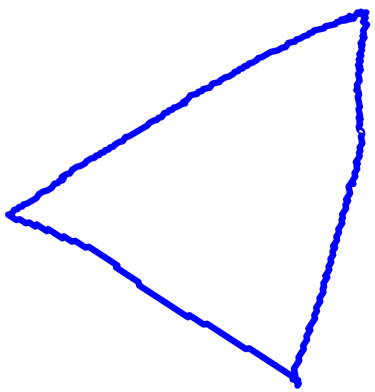
Related Problem: Bold Drawings

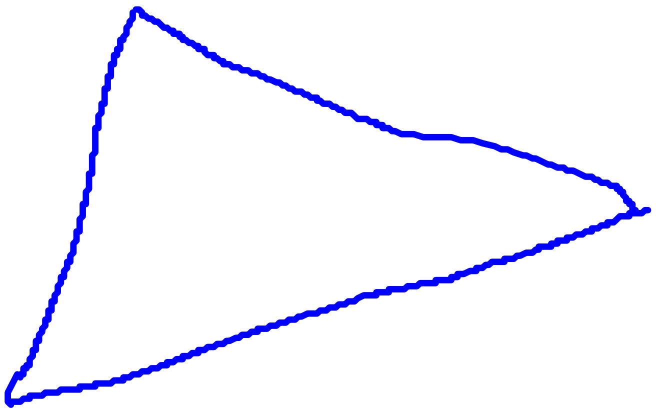
[van Kreveld 2009]



[Pach 2011] If $w < r$ then bold drawings of every K_n exist.

Related Problem: The General Triangle





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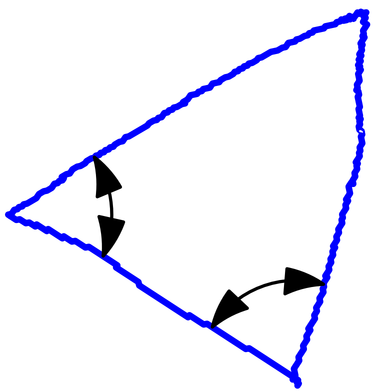
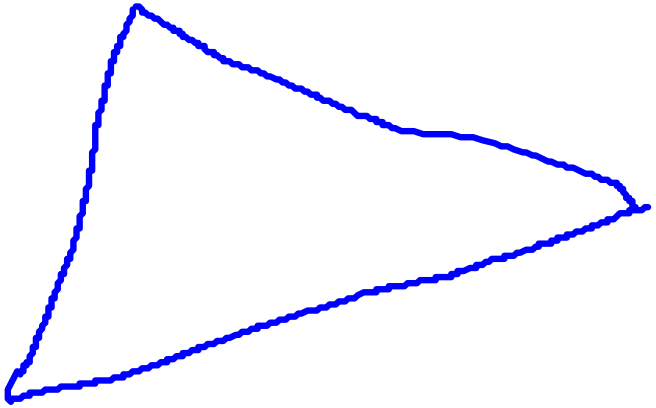


isosceles



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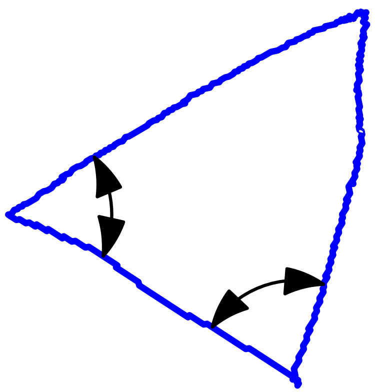

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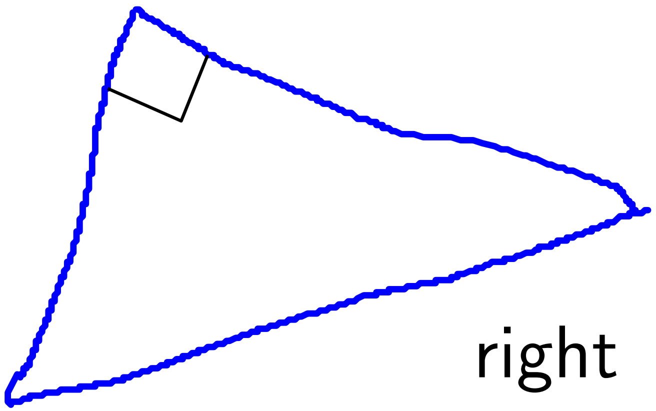
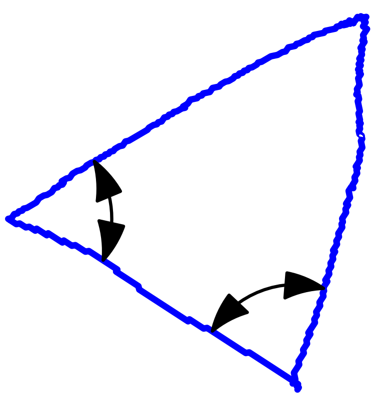
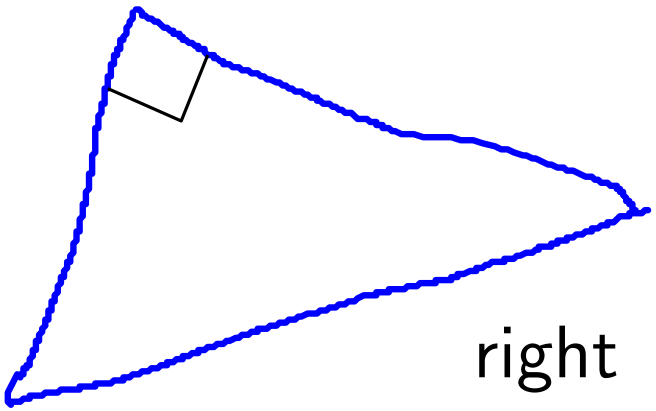


right

Related Problem: The General Triangle



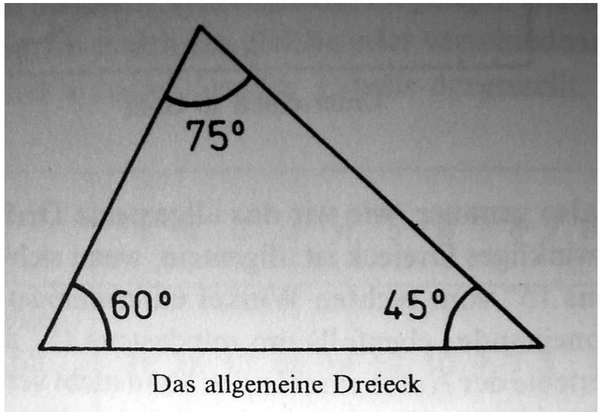
isosceles



right

Theorem. There is a unique (up to similarity) general triangle. Its angles are $45°$, $60°$, and $75°$.

[B. Tergan, Das allgemeine Dreieck, *J. Math. Did.* **1**, 1980]



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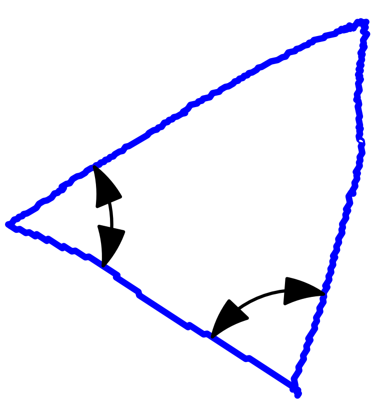
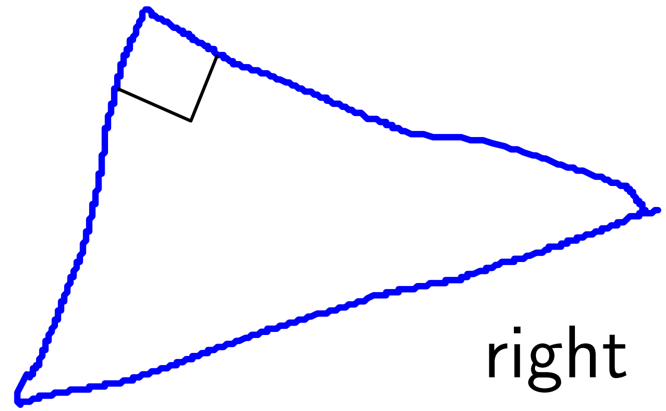


isosceles



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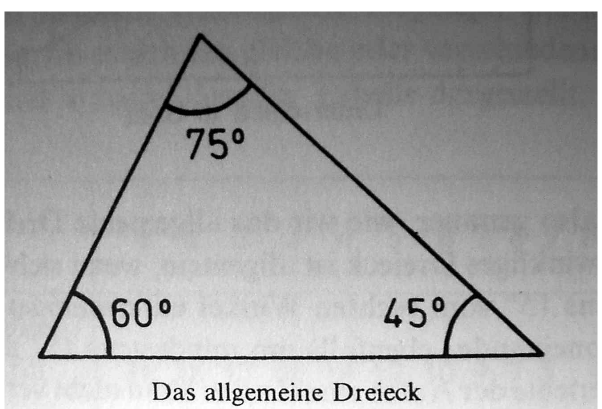
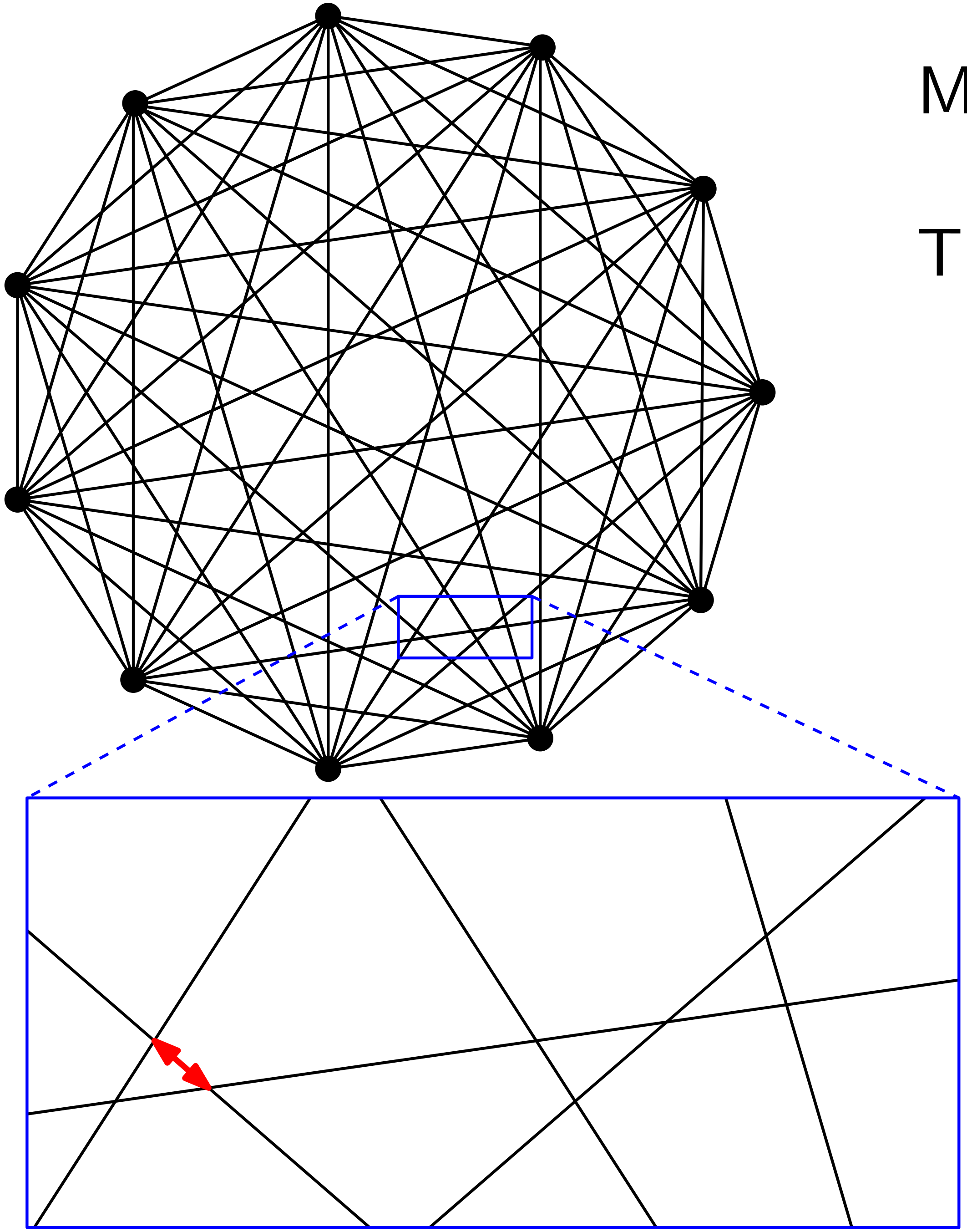


G. A. Jürgen, Das allgemeinste Dreieck, *Praxis d. Math.* **23**, 1981.
F. U. M. Weg, Das allgemeine Viereck, 1984.

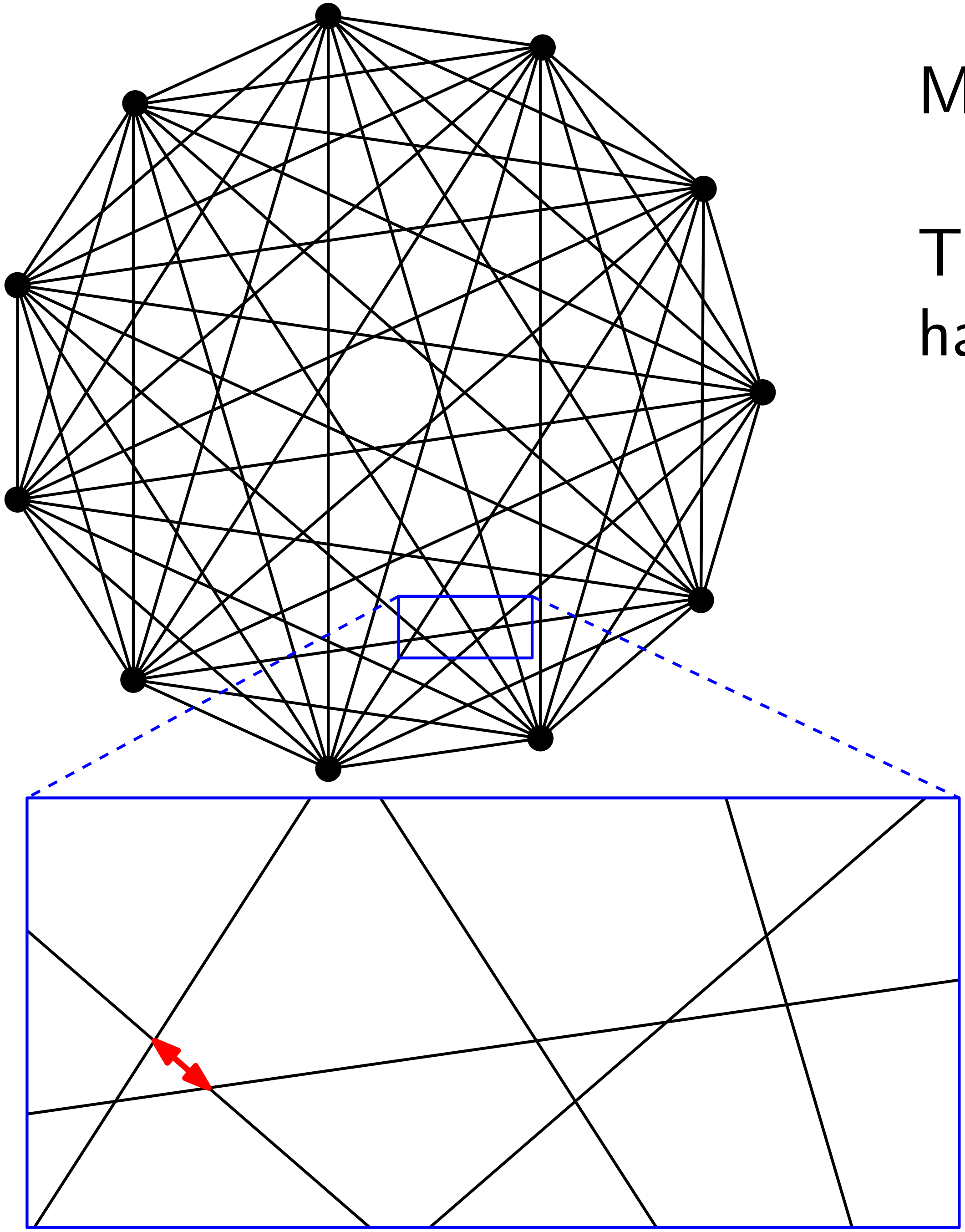
Drawing K_n



Maximize the shortest edge!

The regular n-gon (n odd)

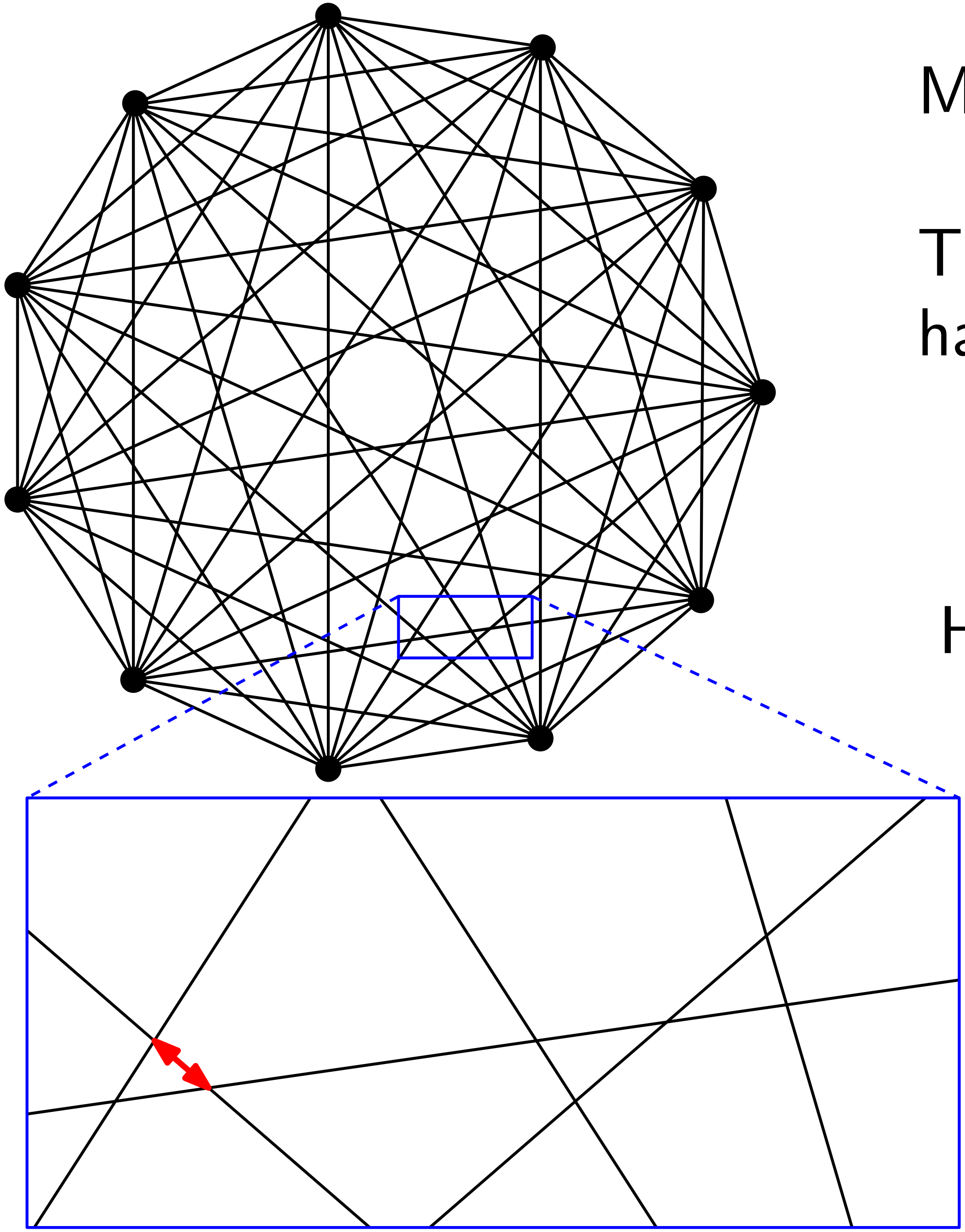
Drawing K_n



Maximize the shortest edge!

The regular n-gon (n odd) has no triple diagonal crossings.
[Heineken 1962, Bol 1936]

Drawing K_n



Maximize the shortest edge!

The regular n-gon (n odd) has no triple diagonal crossings.
[Heineken 1962, Bol 1936]

How long is the shortest edge?

Results:

deterministic
$$\varepsilon = O(1/n^{12.01})$$

randomized
$$\varepsilon = O(1/n^{5})$$

Shortest Edge in the Regular $K_n \approx \frac{1}{n^5}$

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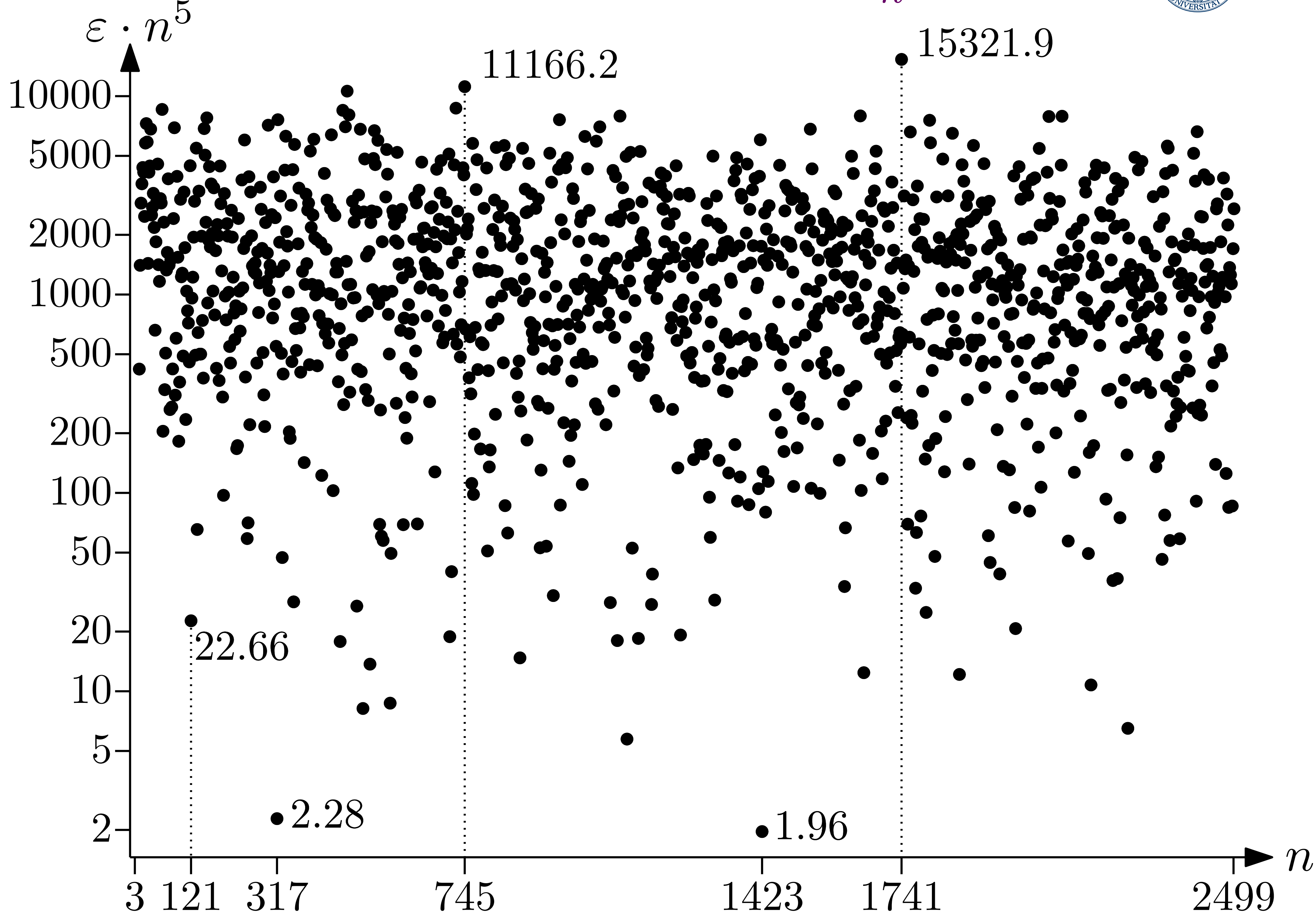

No Triple Diagonal Intersections

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The regular n-gon (n odd) has no triple diagonal crossings.
[Heineken 1962, Bol 1936]

$z = e^{2\pi i/n}$. Triple intersection of $\overline{z^0, z^r}$, $\overline{z^s, z^t}$, $\overline{z^u, z^v}$ $\implies$

$$\begin{vmatrix} 1 & z^0 z^r & z^0 + z^r \\ 1 & z^s z^t & z^s + z^t \\ 1 & z^u z^v & z^u + z^v \end{vmatrix} = 0$$

$\cdots \implies F(z) = 0$ with

$$F(z) = (1-z^t)(1-z^s)(1-z^{u+v-w}) - (1-z^u)(1-z^v)(1-z^{s+t-r})$$

z is primitive n-th root of unity $\implies F(z^2) = 0 \implies \cdots$

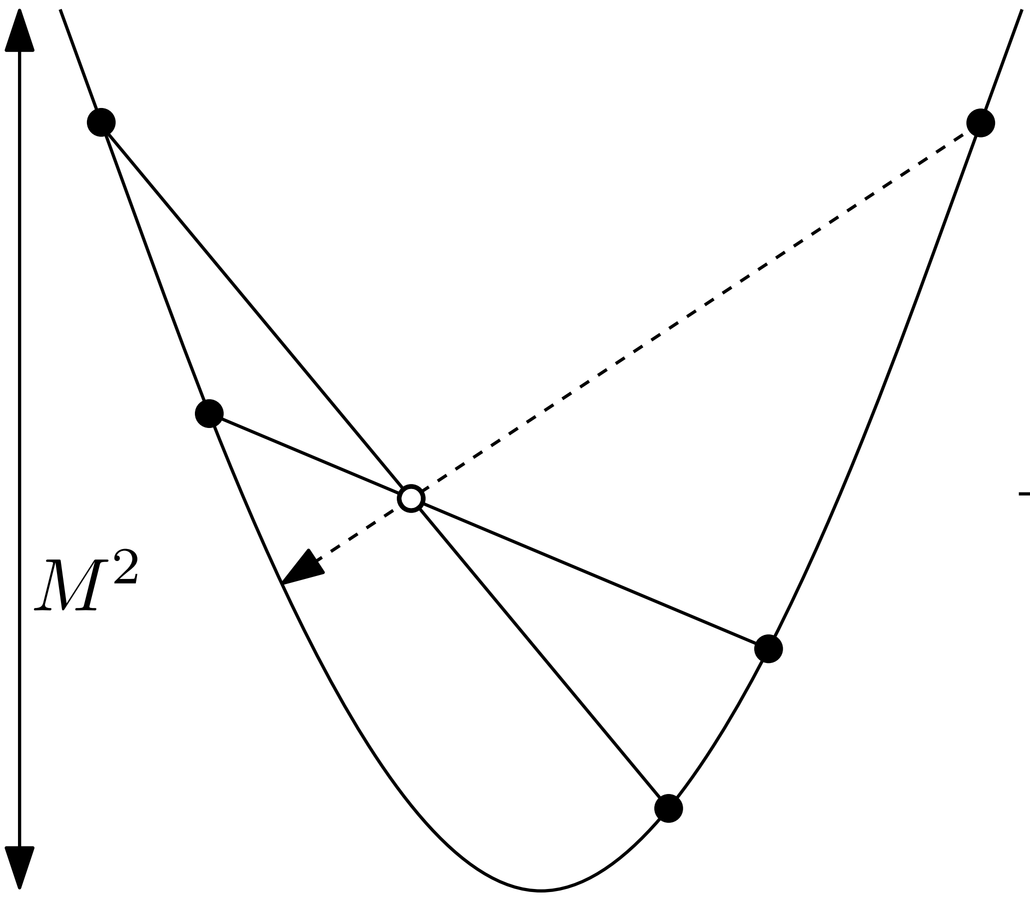
$\cdots \implies z^{u+v} = z^{s+t}$ $\square$

Separation bounds give something like $\varepsilon \ge 1/2^{O(n \log n)}$.

Incremental Construction

Freie Universität Berlin

Add points one by one on the parabola $\binom{i}{i^2}$, $i = -M, \ldots, M$



Every 5-tuple induces a "forbidden" placement for future points.

$M = n^5 \implies$ can place n points without triple crossings

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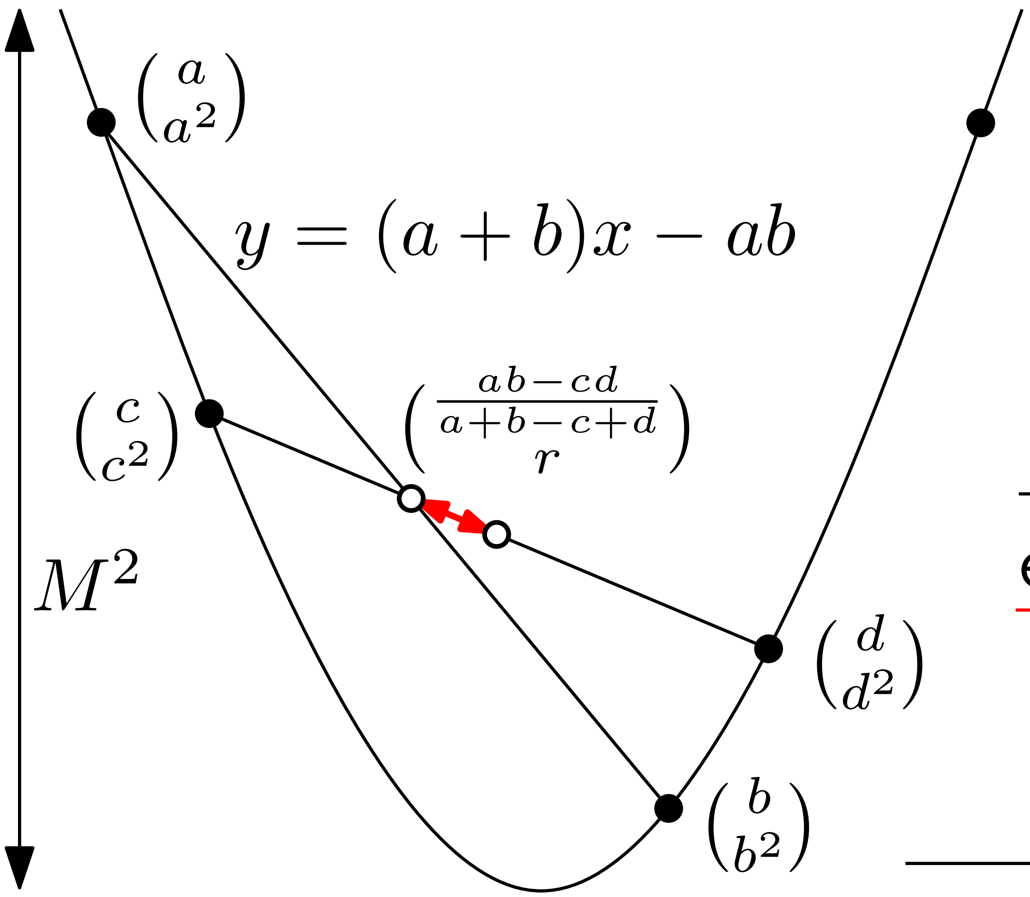

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---

edge length $\geq$ horizontal distance

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| \geq \frac{1}{|qq'|} \geq \frac{1}{4M^2}$$

$\implies \geq \Omega(1/M^3)$ after scaling

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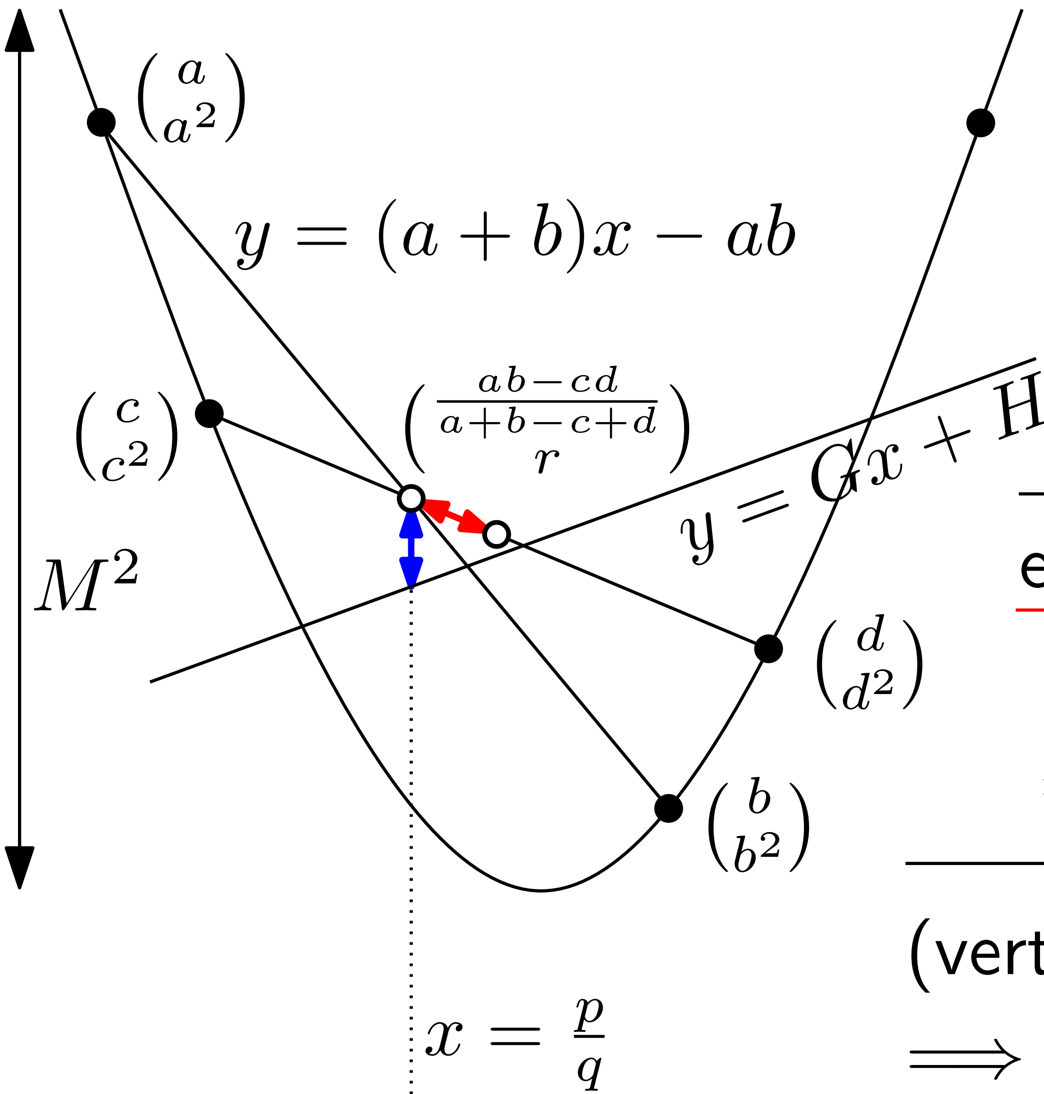

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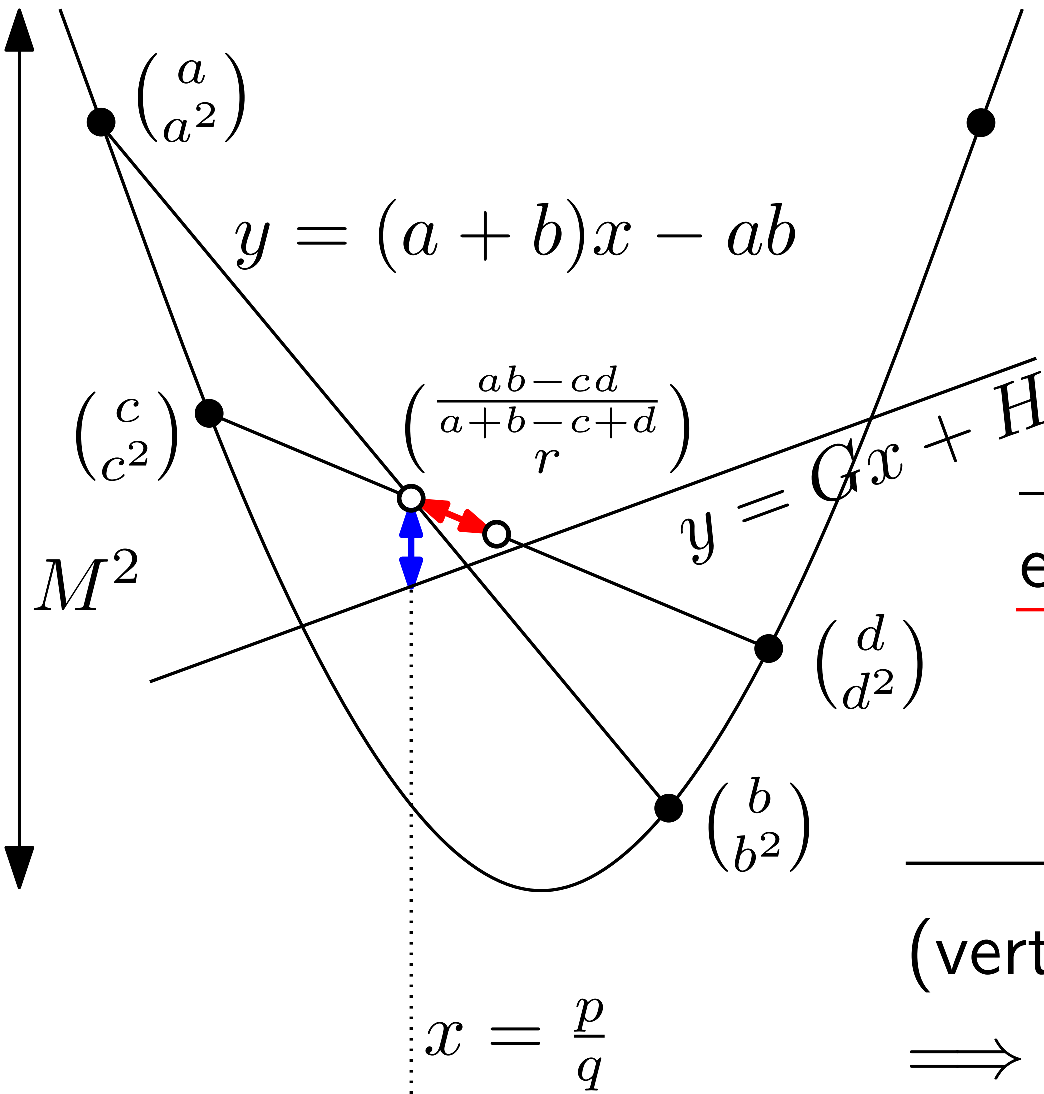

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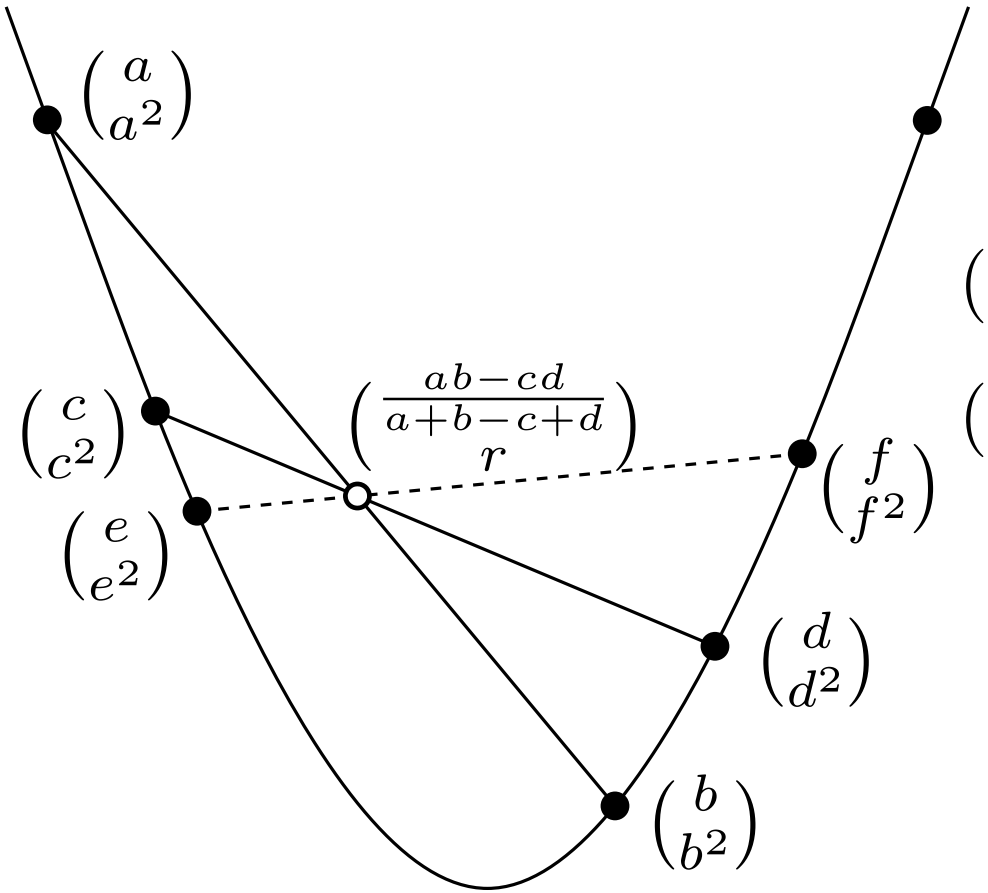


How many diagonals go through the point $\binom{p/q}{r}$?

$$(e - \tfrac{p}{q})(f - \tfrac{p}{q}) = (a - \tfrac{p}{q})(b - \tfrac{p}{q})$$

$$(qe - p)(qf - p) = R \text{ (const.)}$$

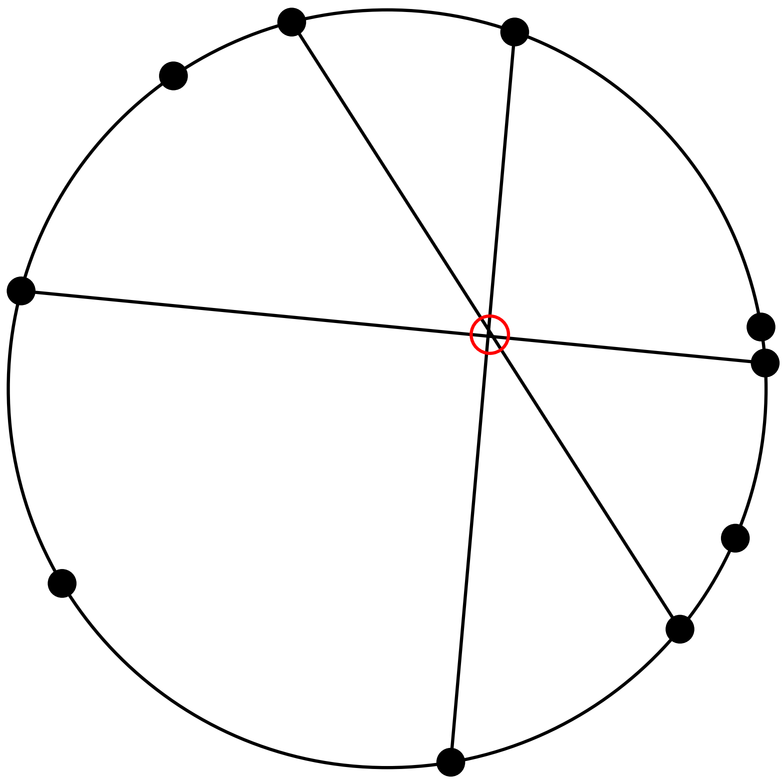
$$\# \text{ diagonals } \overline{\tbinom{e}{e^2}, \tbinom{f}{f^2}}$$
$$\le \# \text{ divisors of } R = O(M^2)$$
$$\le M^{O(1/\log\log M)}$$

Every 4-tuple eliminates $\le M^{O(1/\log\log M)}$ possibilities.

$\Longrightarrow M \ge n^4 \cdot M^{O(1/\log\log M)}$ is sufficient.

$\Longrightarrow M := n^{4+O(1/\log\log n)} \Longrightarrow \varepsilon = 1/n^{12+O(1/\log\log n)}$

Randomized Construction

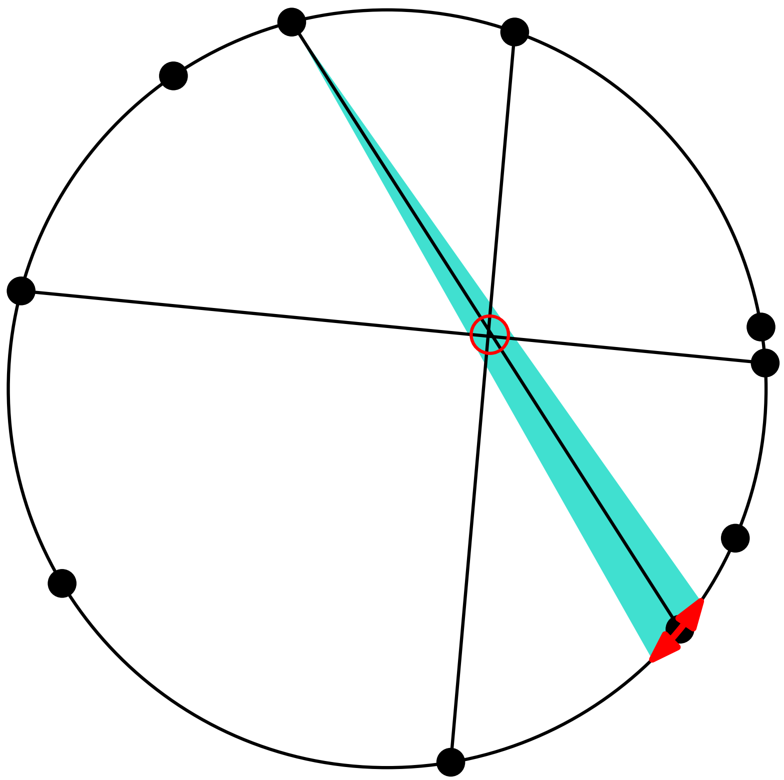


Place $2n$ random points on the circle

$E[\#$ triplets of diagonals closer than $\varepsilon] \leq n$

Remove $\leq n$ "bad" points

Randomized Construction



Place $2n$ random points on the circle

$E[\#$ triplets of diagonals closer than $\varepsilon] \leq n$

Remove $\leq n$ "bad" points

$P[$a random triplet of diagonals is closer than $\varepsilon] \leq O(\varepsilon)$

Want:

$E[\#$bad triplets$] = O(n^6) \cdot O(\varepsilon) \leq n$

$\implies \varepsilon = \text{const} \cdot \frac{1}{n^5}$.

Direct constructions?

Freie Universität Berlin

Points $\binom{i}{i^k}$, $i = 1, \ldots, n$? $\qquad k = 5$ works for $n \leq 350$.

Greedy algorithm on the parabola $\binom{i}{i^2}$, $i = 1, 2, \ldots$

Select $i =$1, 2, 3, 4, 5, 6, 7, 8, 14, 16, 24, 27, 30, 33, 50, 55, 61, 75, 77, 102, 103, 106, 140, 144, 150, 189, 201, 229, 257, 259, 268, 309, 317, 338, 353, 357, 368, 412, 456, 471, 514, 560, 563, 566, 568, 580, 594, 646, 655, 739, 778, 831, 865, 895, 921, 937, 1019, 1038, 1079, 1147, 1185, 1195, 1235, 1352, 1412, 1439, 1440, 1488, 1511, 1529, 1536, 1551, 1562, 1586, 1619, 1788, 1791, 1851, 1930, 1990, 2052, 2057, 2169, 2187, 2198, 2234, 2237, 2342, 2444, 2450, 2518, 2603, 2695, 2723, 2726, 2771, 2841, 2959, 3080, 3113, 3262, 3277, 3285, 3286, 3413, 3444, 3652, 3666, 3731, 3772, 3817, 3991, 4041, 4051, 4216, 4217, 4225, 4236, 4478, 4572, 4582, 4872, 4909, 4945, 5013, 5254, 5274, 5303, 5415, 5498, 5543, 5572, 5608, 5613, 5722, 5786, 5794, 5820, 6273, 6321, 6387, 6506, 6669, 6687, 6879, 7061, 7170, 7199, 7473, 7520, 7566, 7596, 7608, 7609, 7692, 7753, 7832, 8016, 8221, 8248, 8694, 8811, 8956, 8967, 8975, 9003, 9148, 9180, 9198, 9354, 9387, 9418, 9572, 9638, 9754, 10092, 10109, 10112, 10126, 10168, 10196, 10333,