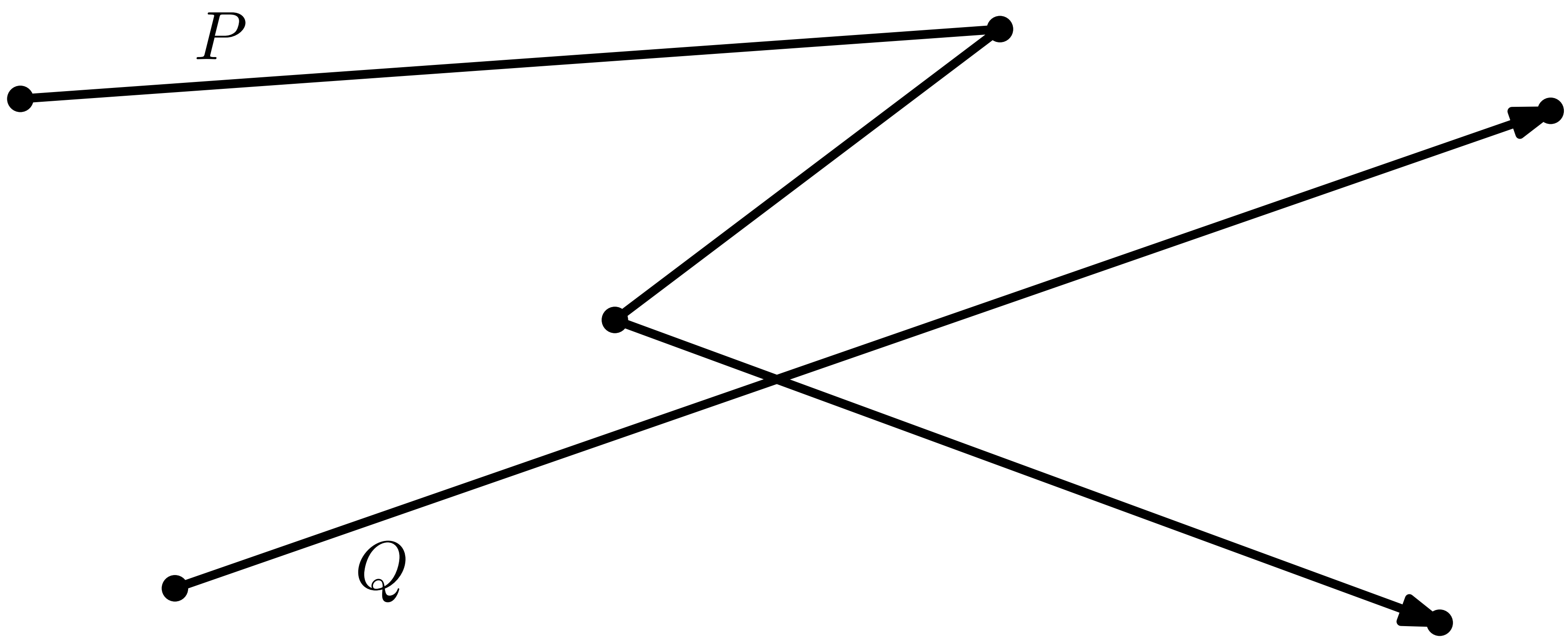
Lexicographic Fréchet Matching

Günter Rote

Freie Universität Berlin

Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves



Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$

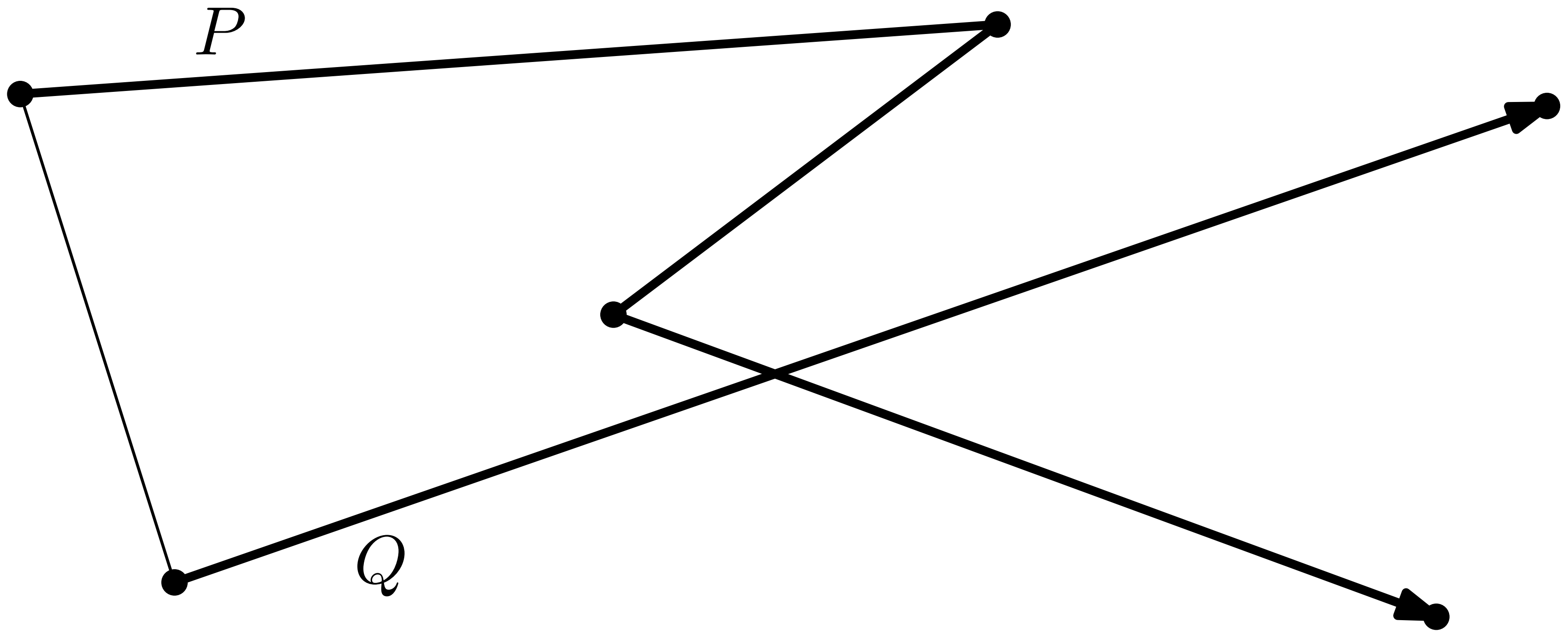
$Q\colon [0, L_Q] \to \mathbb{R}^2$

two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$\beta\colon [0, M] \to [0, L_Q]$

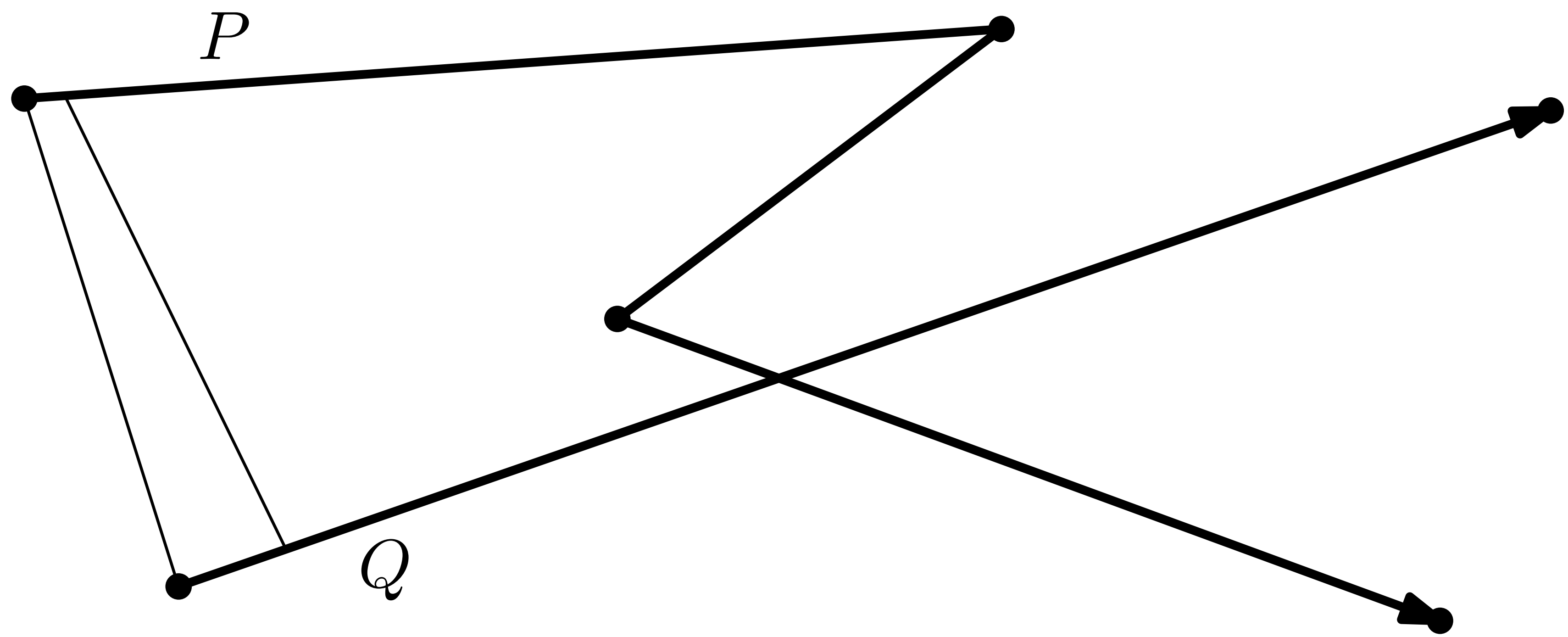
joint parametrization $\to$ "matching"



Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$

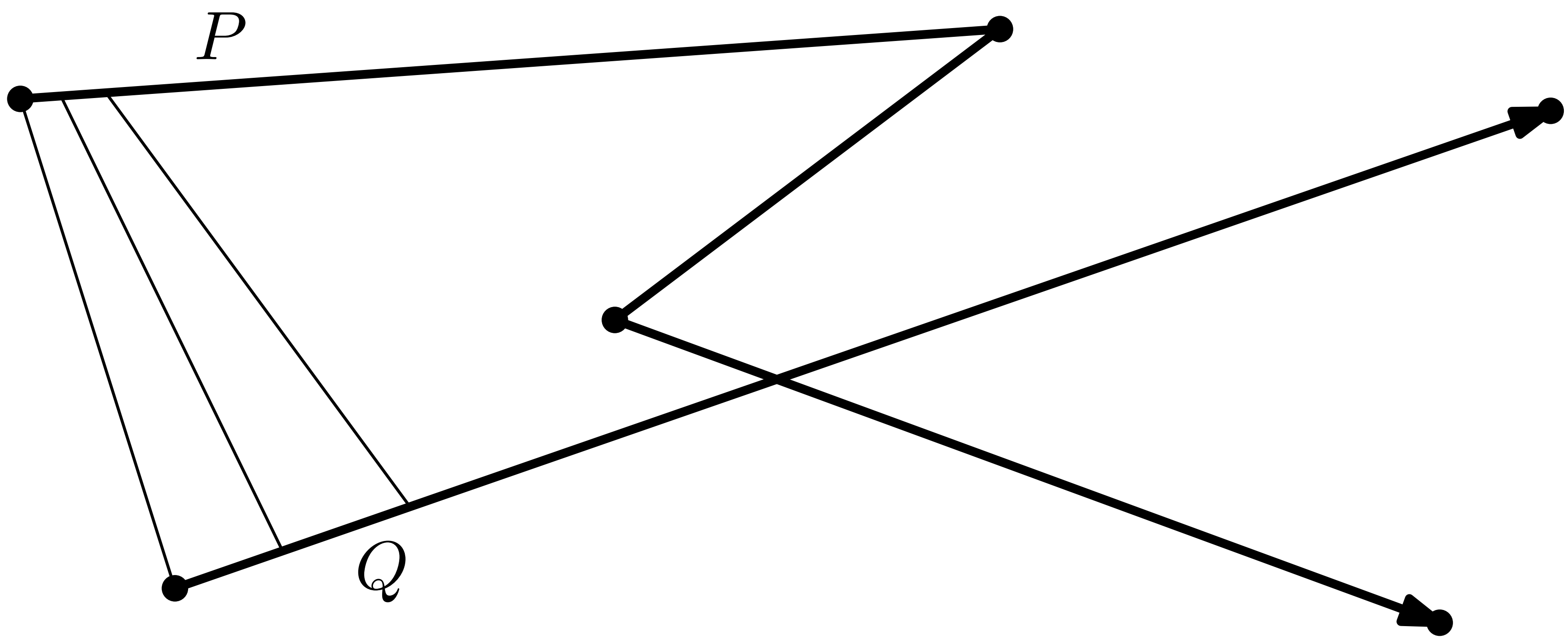
$Q\colon [0, L_Q] \to \mathbb{R}^2$

two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$\beta\colon [0, M] \to [0, L_Q]$

joint parametrization $\to$ "matching"

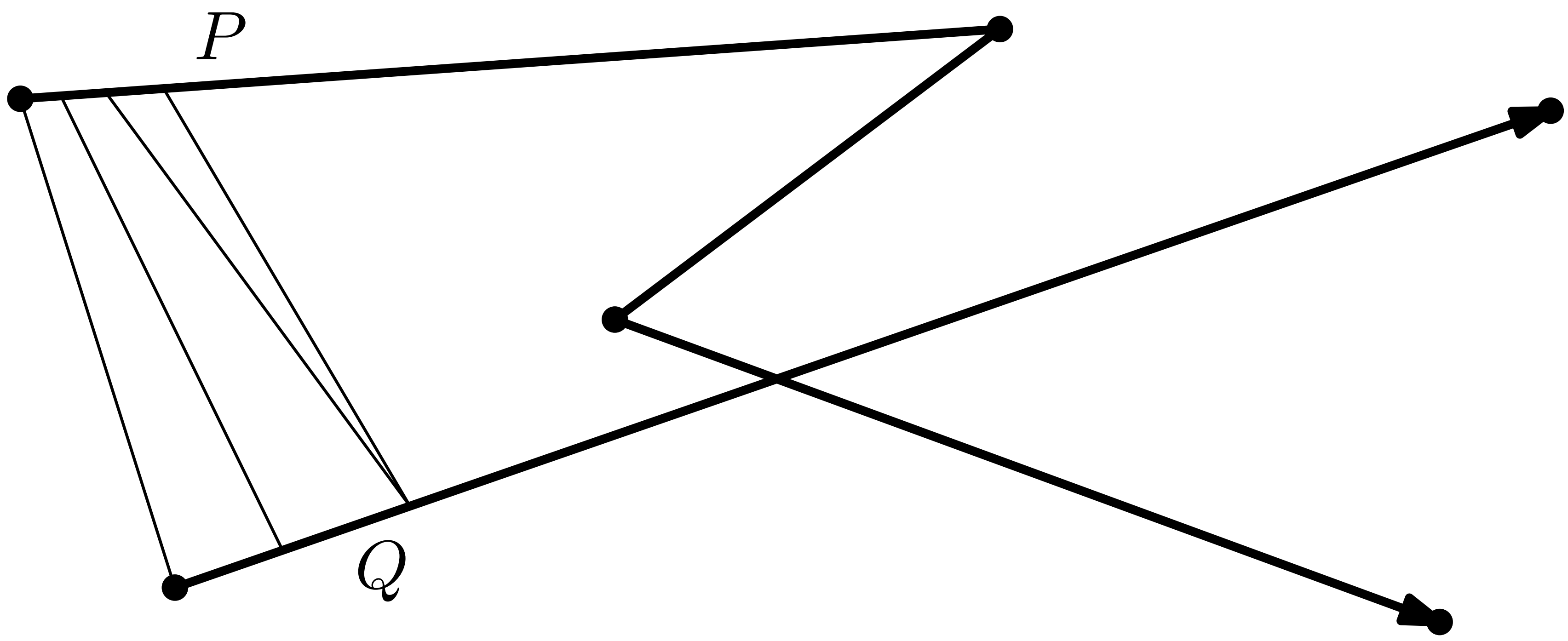


Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"

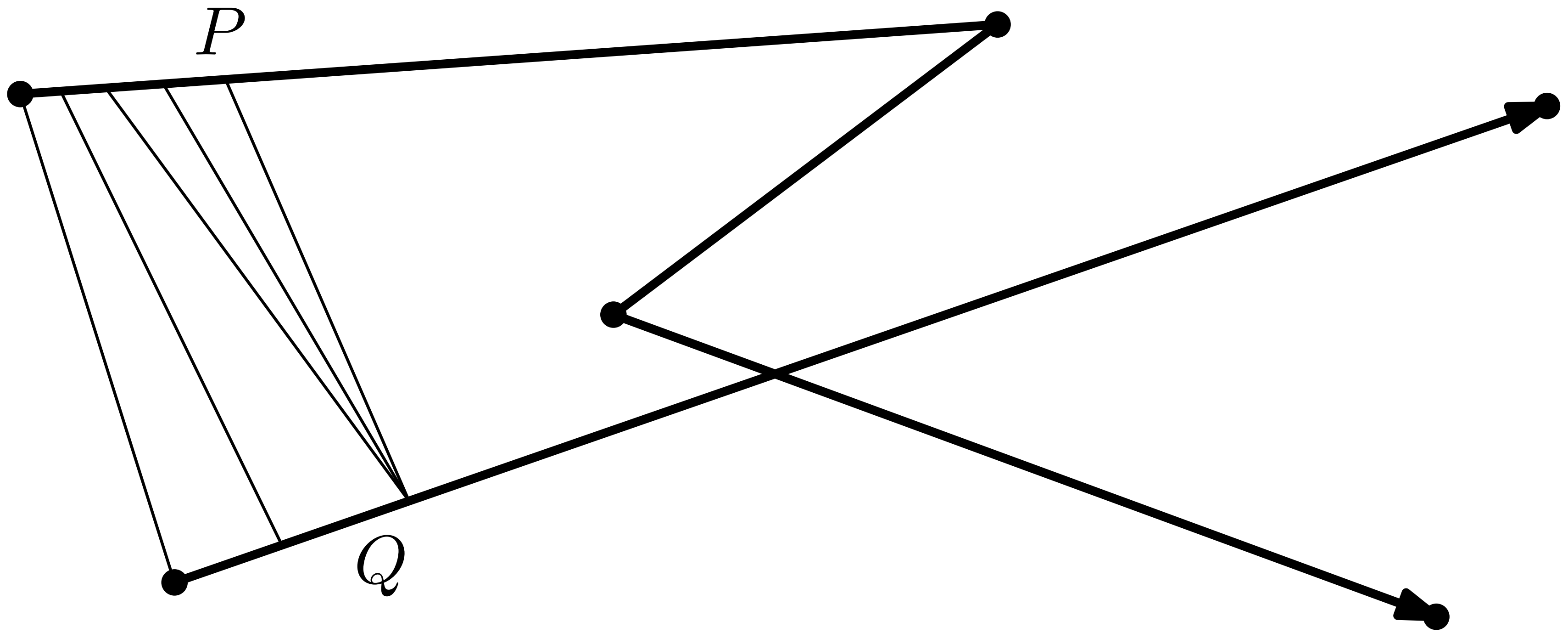


Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

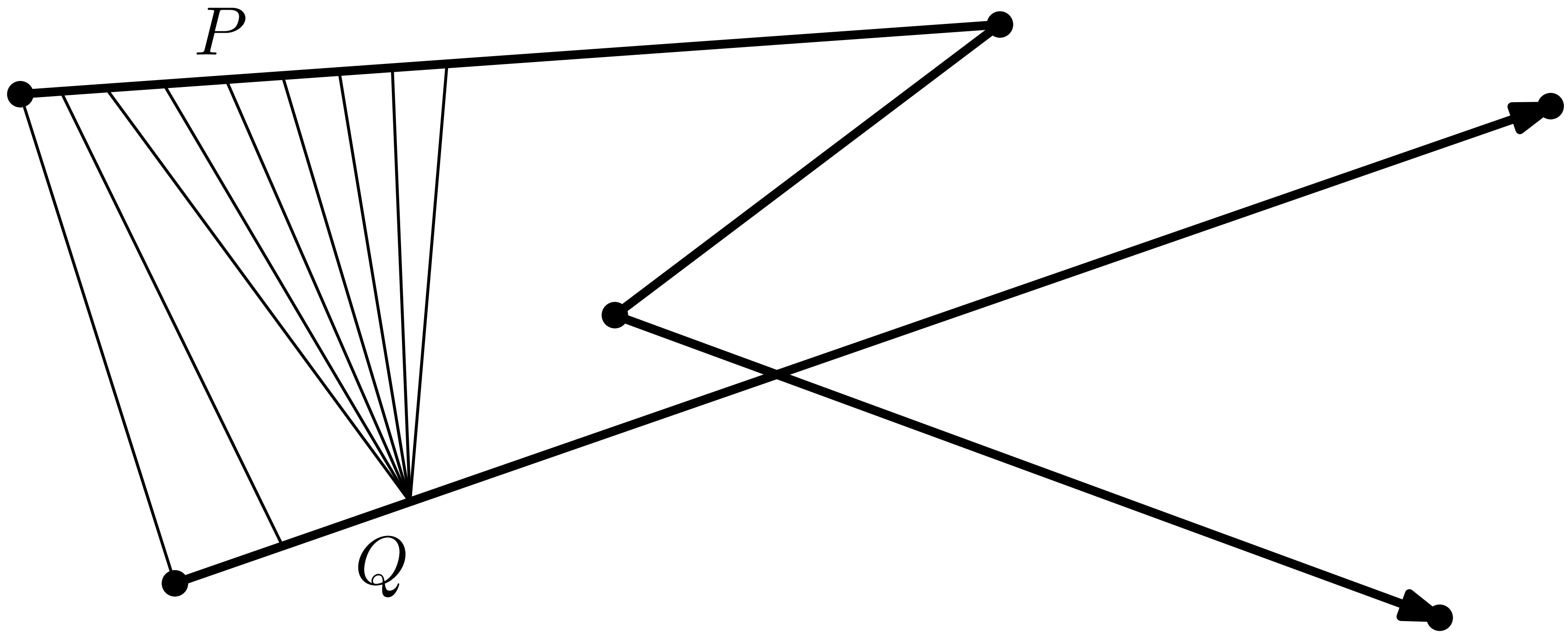
$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$

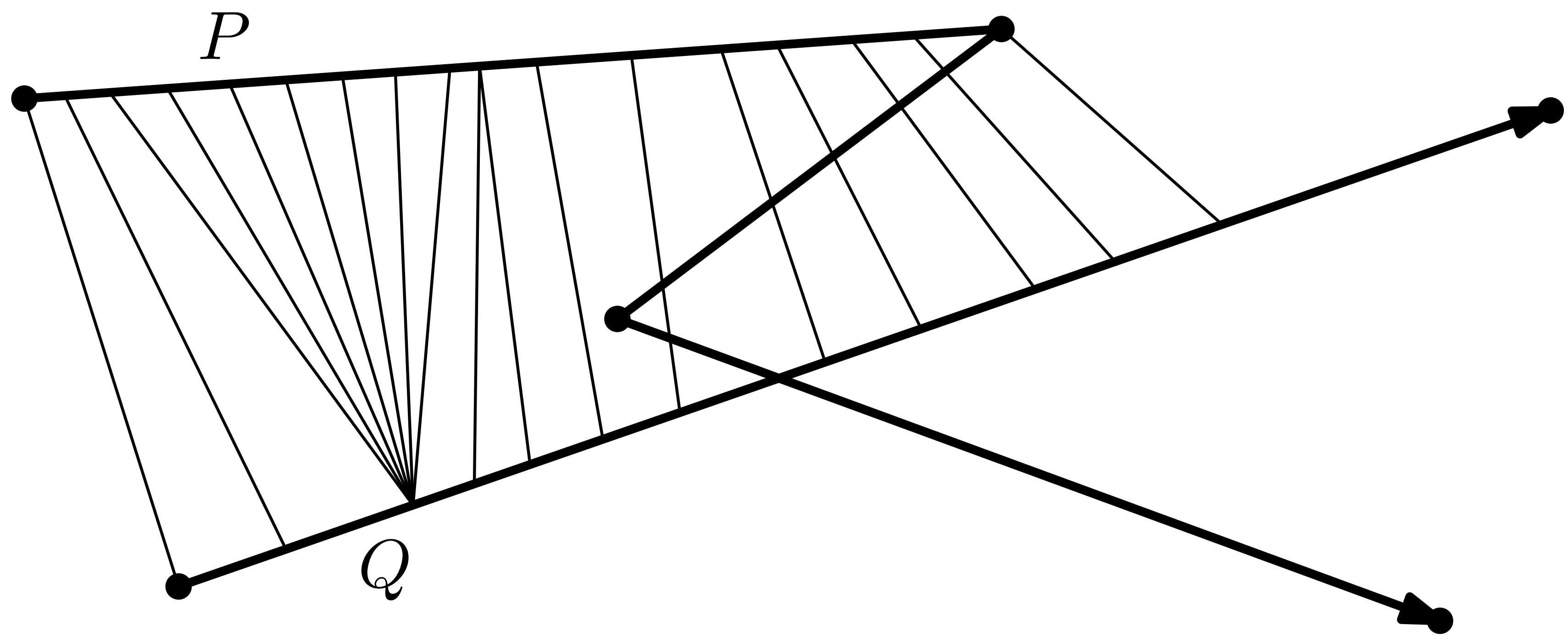
$Q\colon [0, L_Q] \to \mathbb{R}^2$

two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$\beta\colon [0, M] \to [0, L_Q]$

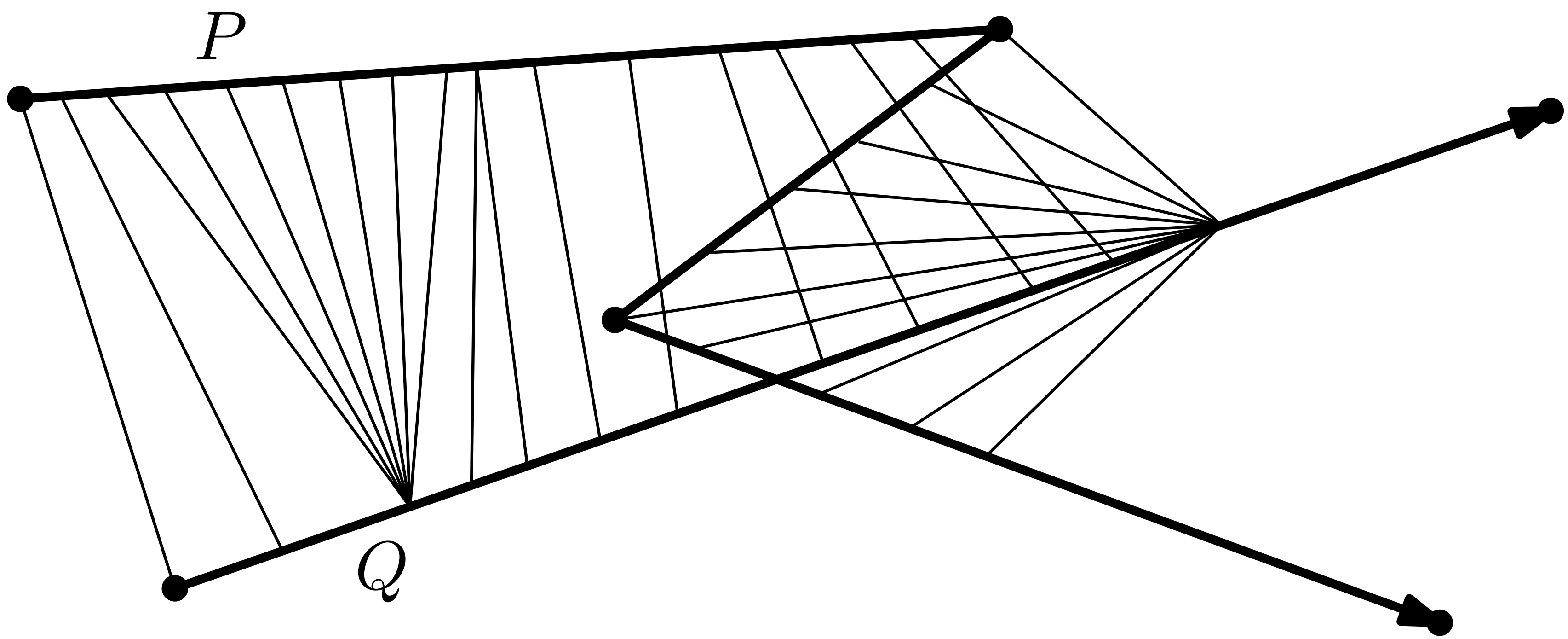
joint parametrization $\to$ "matching"



Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$

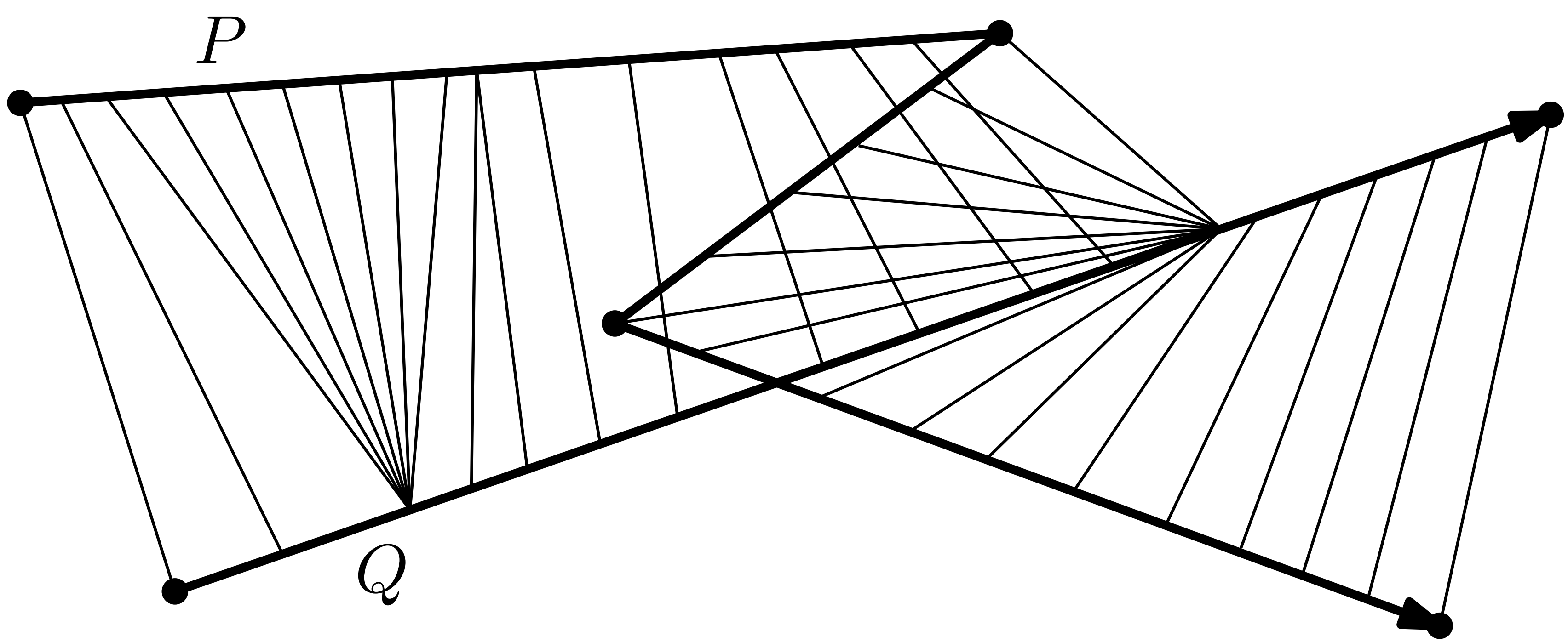
$Q\colon [0, L_Q] \to \mathbb{R}^2$

two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$\beta\colon [0, M] \to [0, L_Q]$

joint parametrization $\to$ "matching"



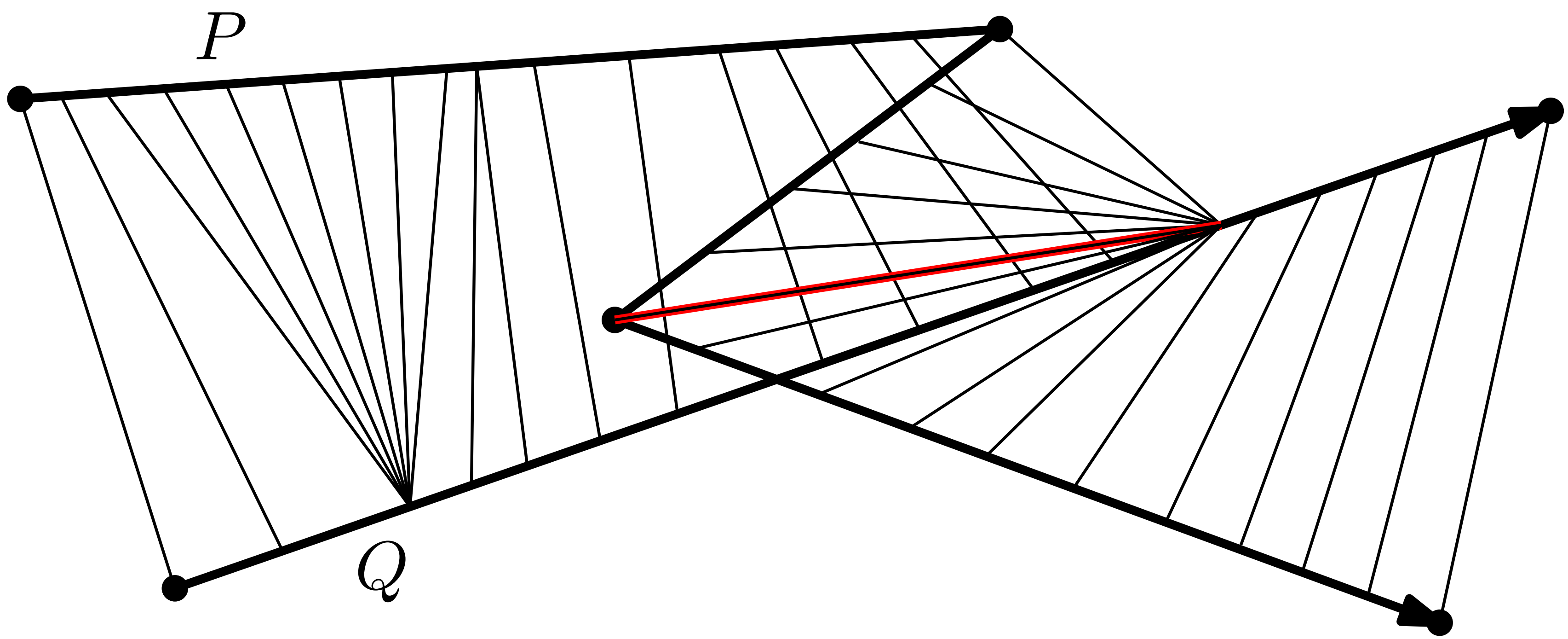
Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$ $\quad$ $\alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$Q\colon [0, L_Q] \to \mathbb{R}^2$ $\quad$ $\beta\colon [0, M] \to [0, L_Q]$

two curves $\quad$ joint parametrization $\to$ "matching"



Fréchet distance [Alt Godau 1995]:

$\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| : 0 \le t \le M \,\} \to$ MIN!

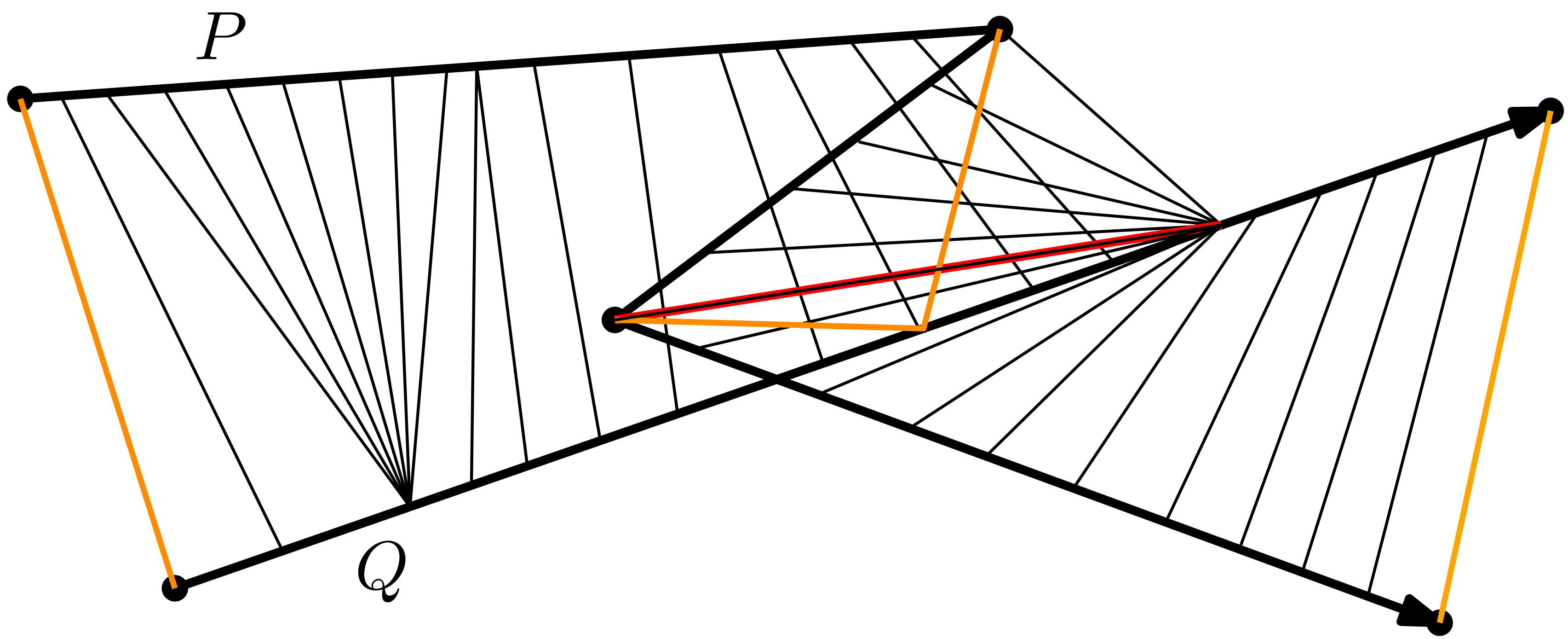
Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$ $\qquad \alpha\colon [0, M] \to [0, L_P]$, monotone bijections

$Q\colon [0, L_Q] \to \mathbb{R}^2$ $\qquad \beta\colon [0, M] \to [0, L_Q]$

two curves $\qquad$ joint parametrization $\to$ "matching"



Fréchet distance [Alt Godau 1995]:

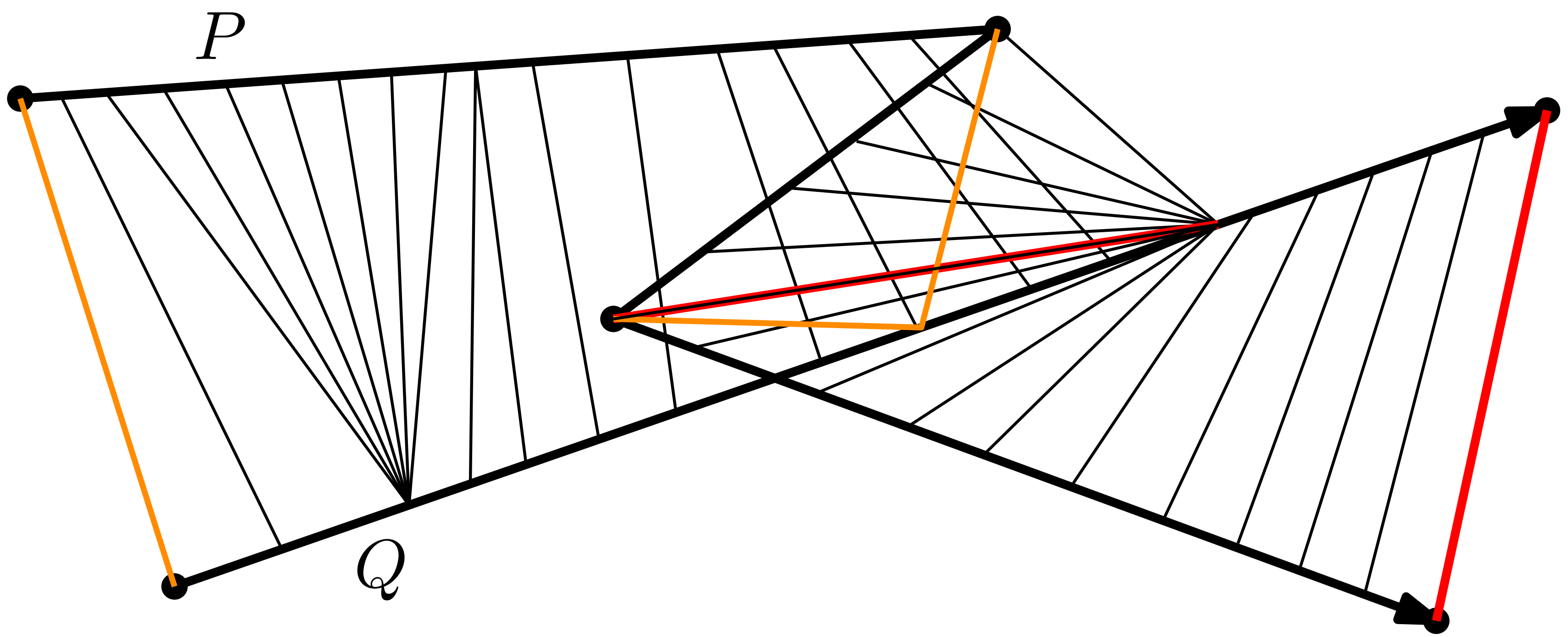
$\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| : 0 \le t \le M \,\} \to$ MIN!

Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



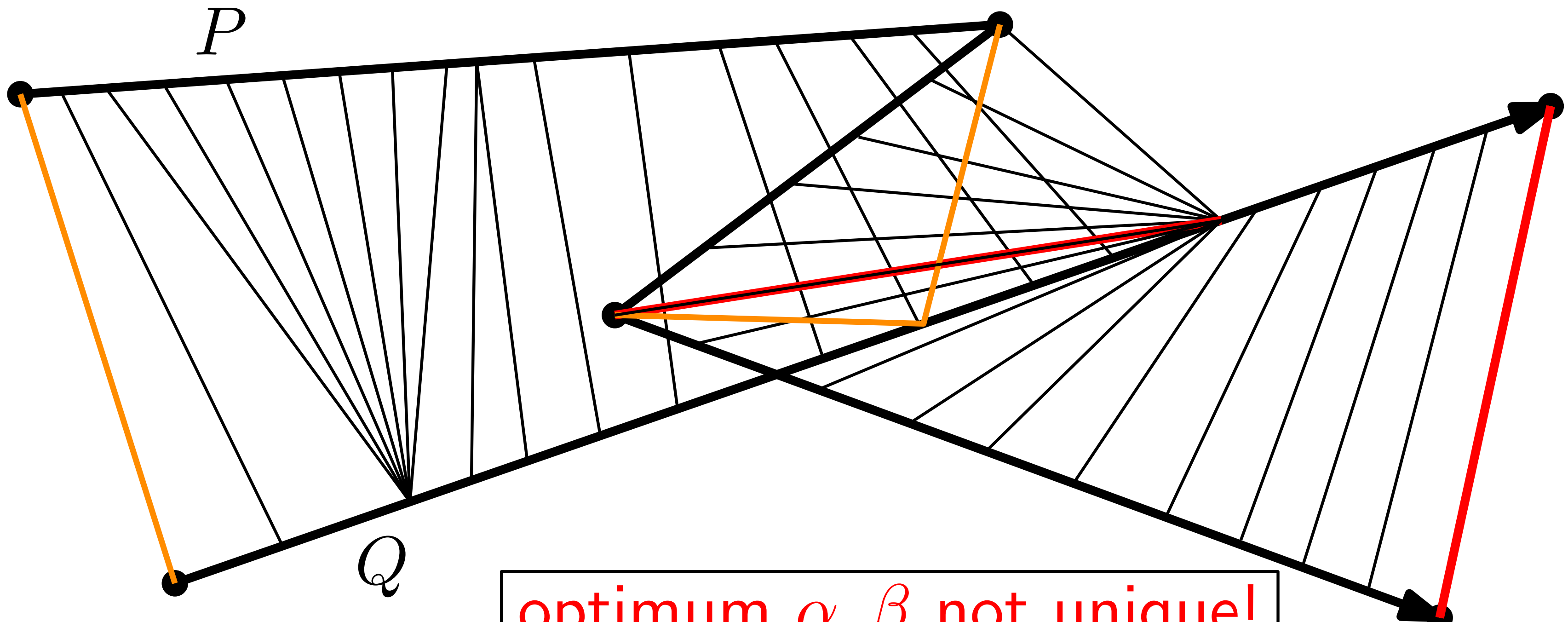
Fréchet distance [Alt Godau 1995]:
$\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| : 0 \le t \le M \,\} \to$ MIN!

Matching between two Curves

Freie Universität Berlin

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



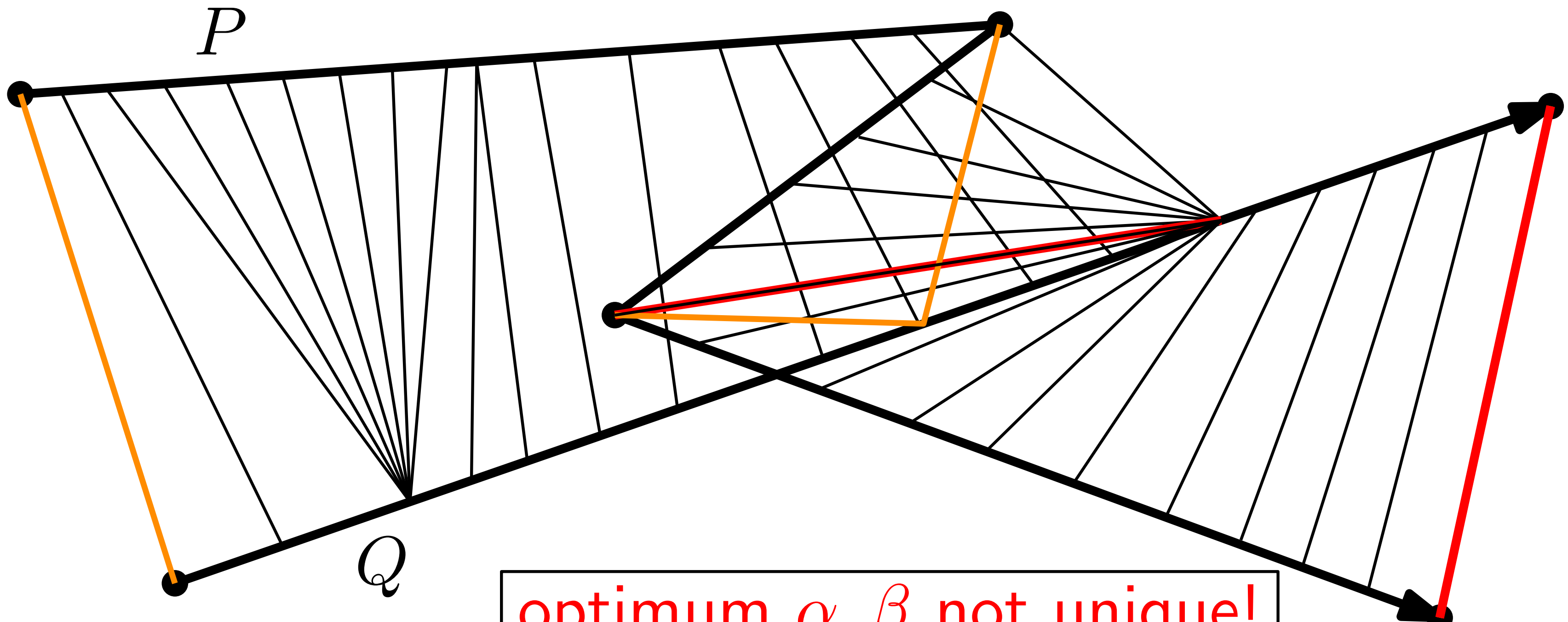
optimum α, β not unique!

Fréchet distance [Alt Godau 1995]:
$\max\{\, \|P(\alpha(t)) - Q(\beta(t))\| : 0 \le t \le M \,\} \to$ MIN!

Matching between two Curves

$P\colon [0, L_P] \to \mathbb{R}^2$
$Q\colon [0, L_Q] \to \mathbb{R}^2$
two curves

$\alpha\colon [0, M] \to [0, L_P]$, monotone bijections
$\beta\colon [0, M] \to [0, L_Q]$
joint parametrization $\to$ "matching"



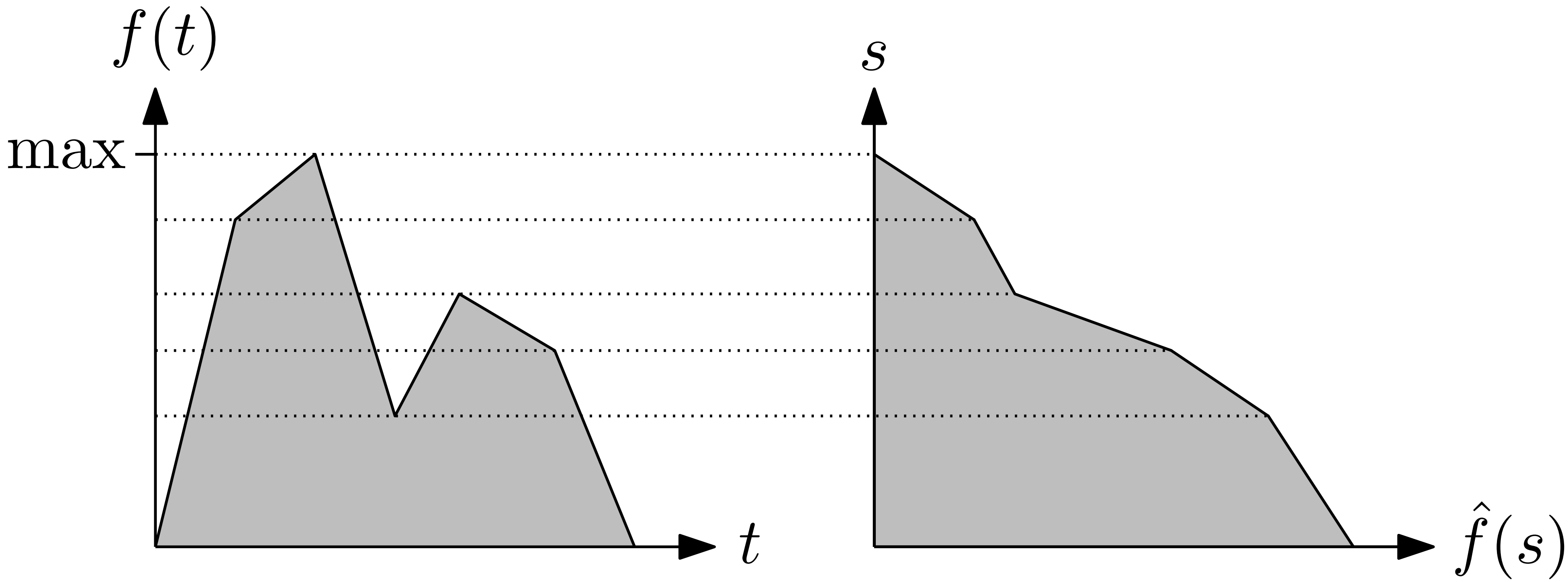
optimum α, β not unique!

Fréchet distance [Alt Godau 1995]:

$$\max\{\ \underbrace{\|P(\alpha(t)) - Q(\beta(t))\|}_{f(t)\ =\ \text{distance function}} : 0 \le t \le M\ \} \to \text{MIN!}$$

Comparison of Distance Functions

Goal: a finer criterion than $\max\{ f(t) : 0 \leq t \leq M \}$



profile function $\hat{f} \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$:

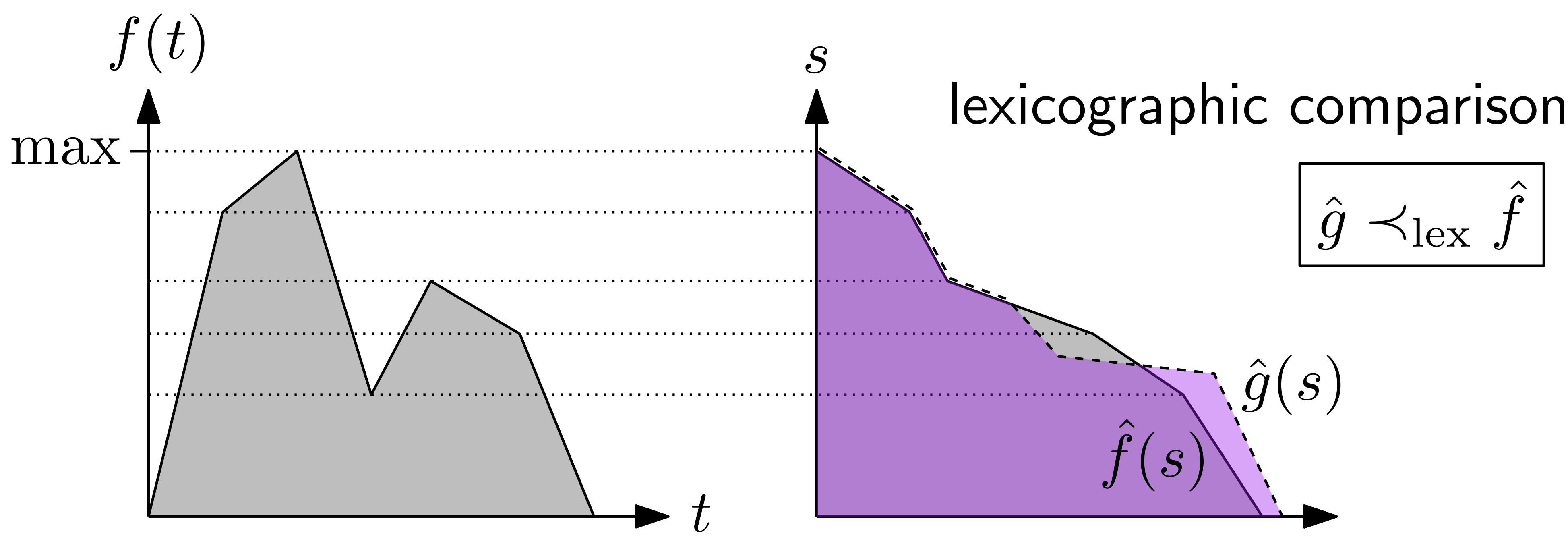
$$\hat{f}(s) = \text{the amount of time that } f(t) \text{ is at least } s$$

$$= \mu(\{ t \mid f(t) \geq s \}) \qquad (\mu = \text{Lebesgue measure})$$

Comparison of Distance Functions

Freie Universität Berlin

Goal: a finer criterion than $\max\{ f(t) : 0 \le t \le M \}$



profile function $\hat{f} \colon \mathbb{R} \to \mathbb{R}_{\ge 0}$:

$\hat{f}(s) =$ the amount of time that $f(t)$ is at least s

$= \mu(\{ t \mid f(t) \ge s \})$ ($\mu =$ Lebesgue measure)

Normalization

Freie Universität Berlin

Main Assumption:
The speed at which the curves P and Q are traversed by the parametrizations α and β is bounded by 1.

Normalization

Main Assumption:
The speed at which the curves P and Q are traversed by the parametrizations α and β is bounded by 1.

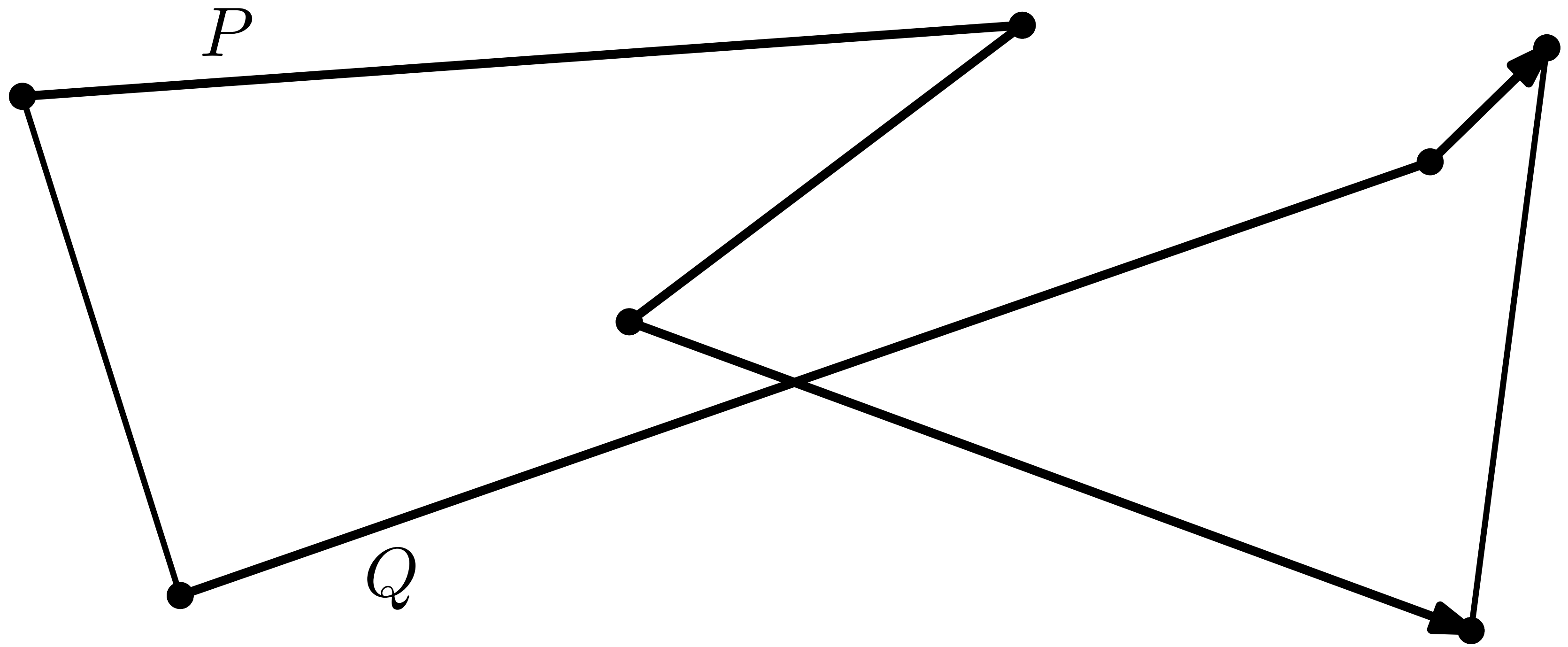
Assume arc-length parametrization for P and Q.

PROBLEM STATEMENT:
Minimize the profile $\hat{f}$ of the distance function

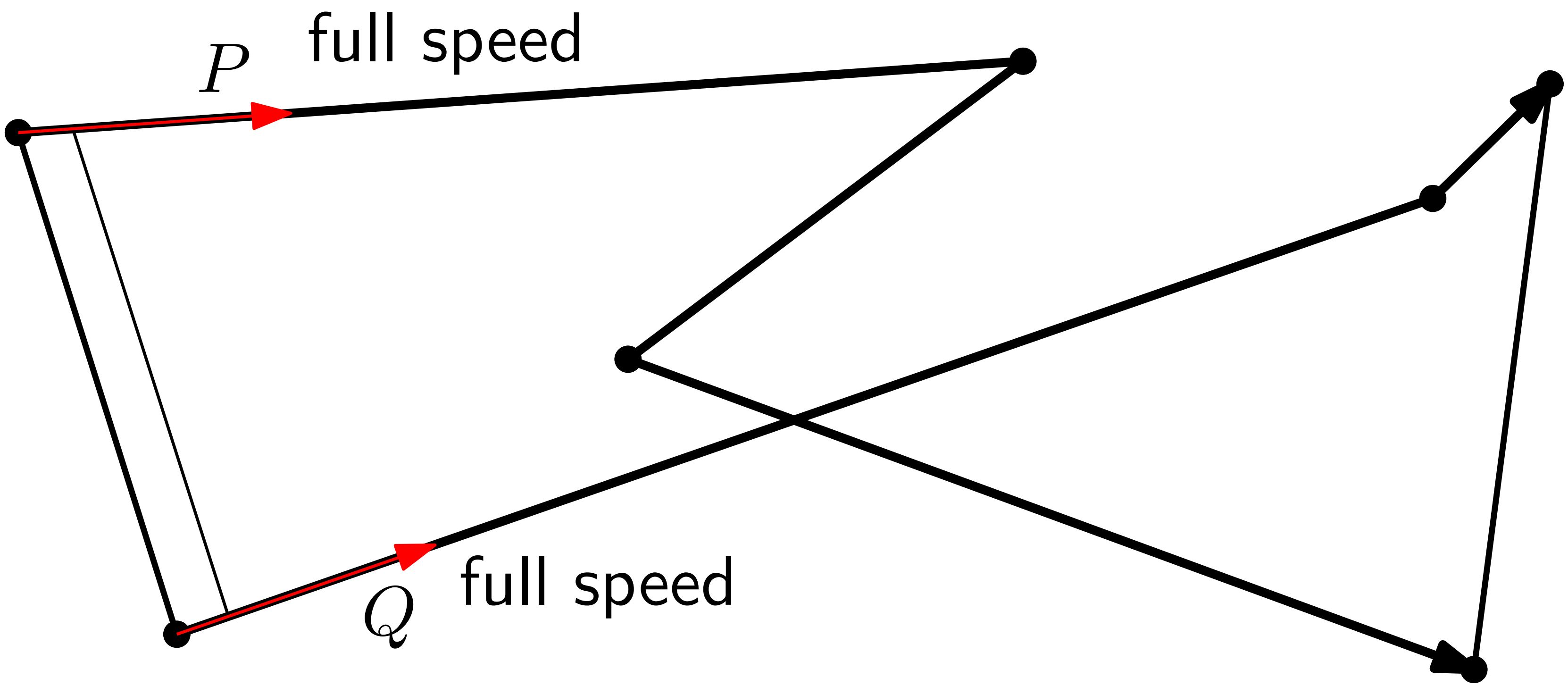
$$f(t) = \|P(\alpha(t)) - Q(\beta(t))\|$$

with respect to $\prec_{\mathrm{lex}}$
under the constraints $\alpha'(t), \beta'(t) \leq 1$.

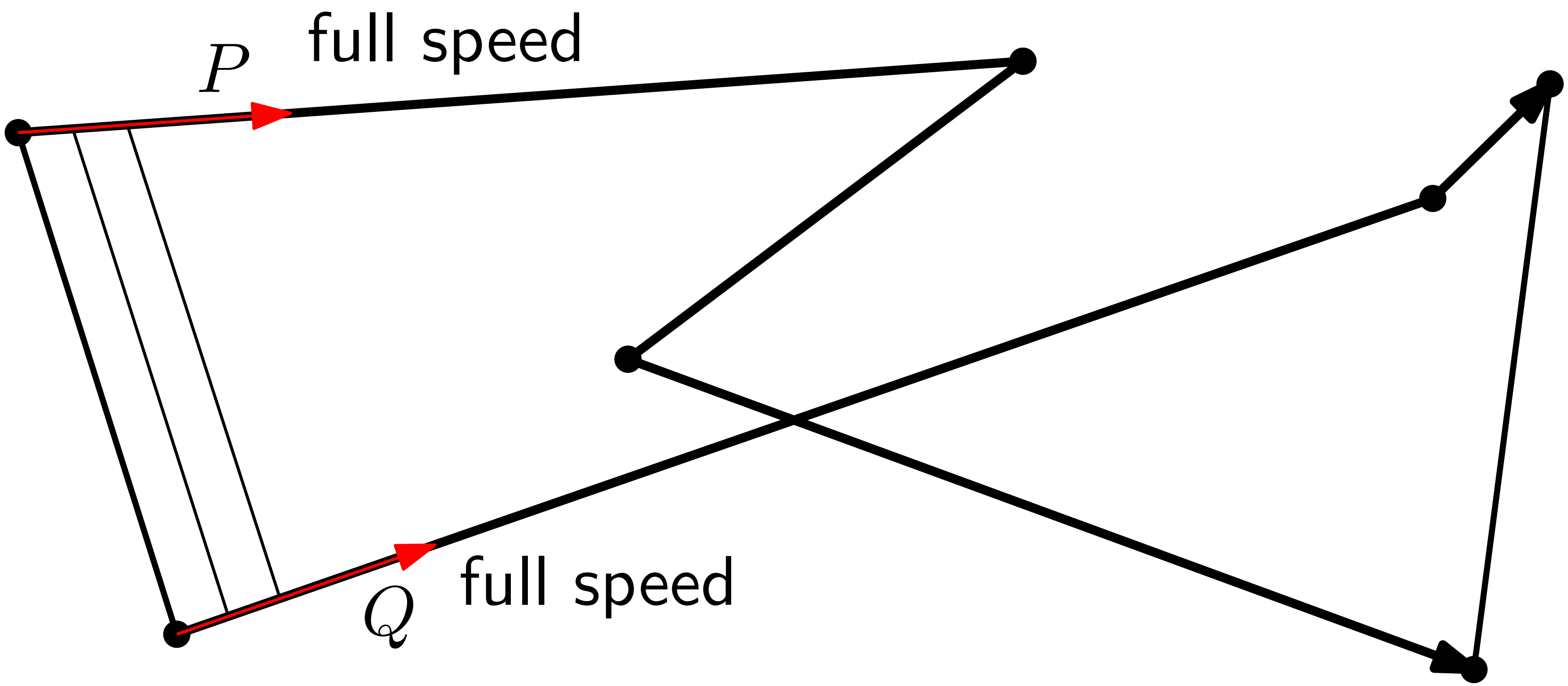
Example



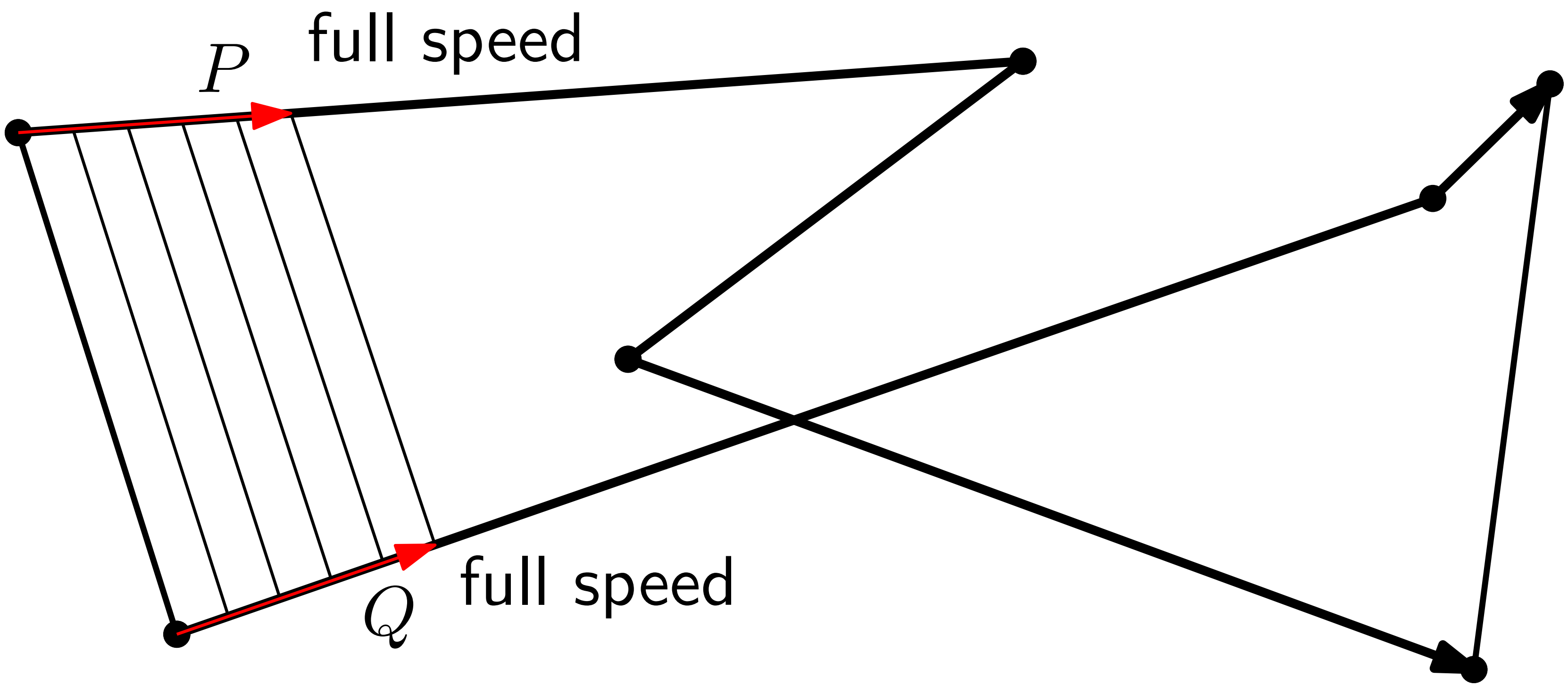
Example



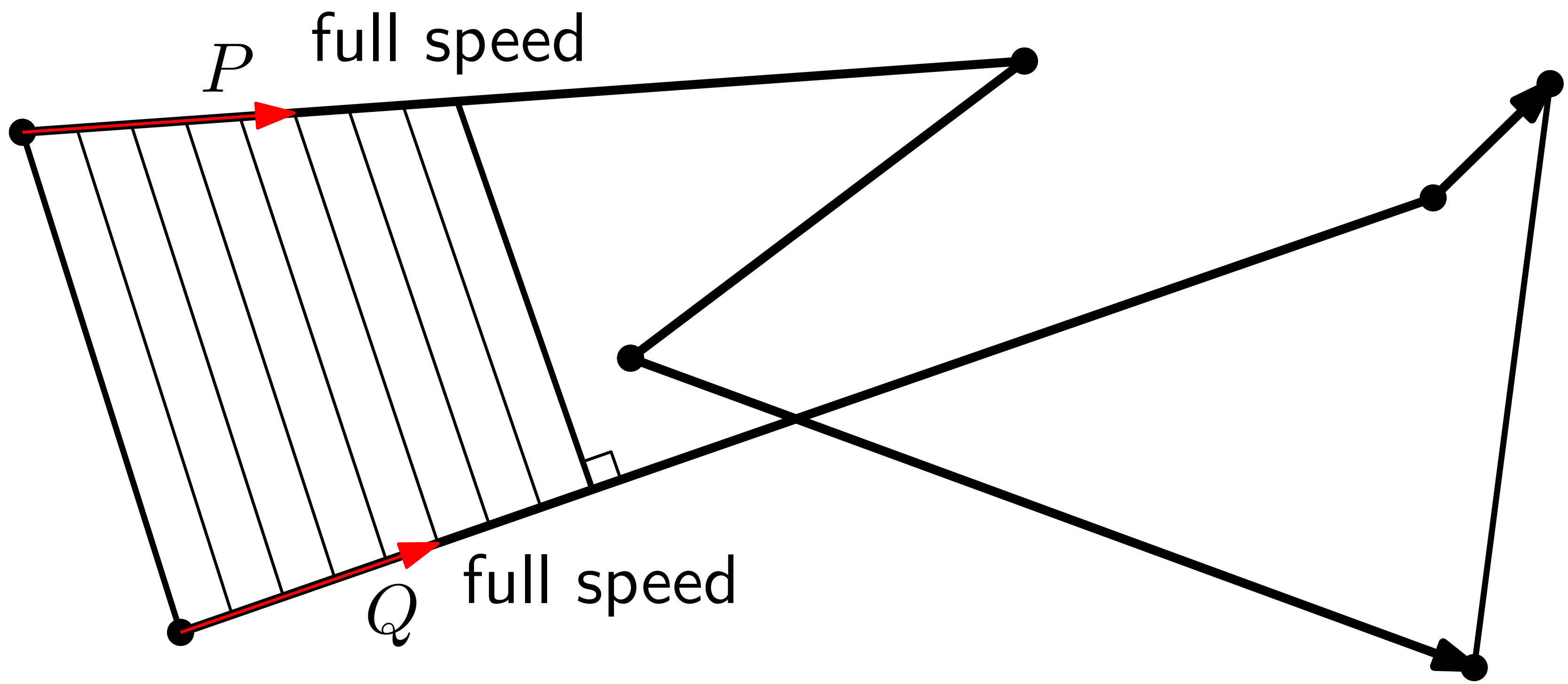
Example



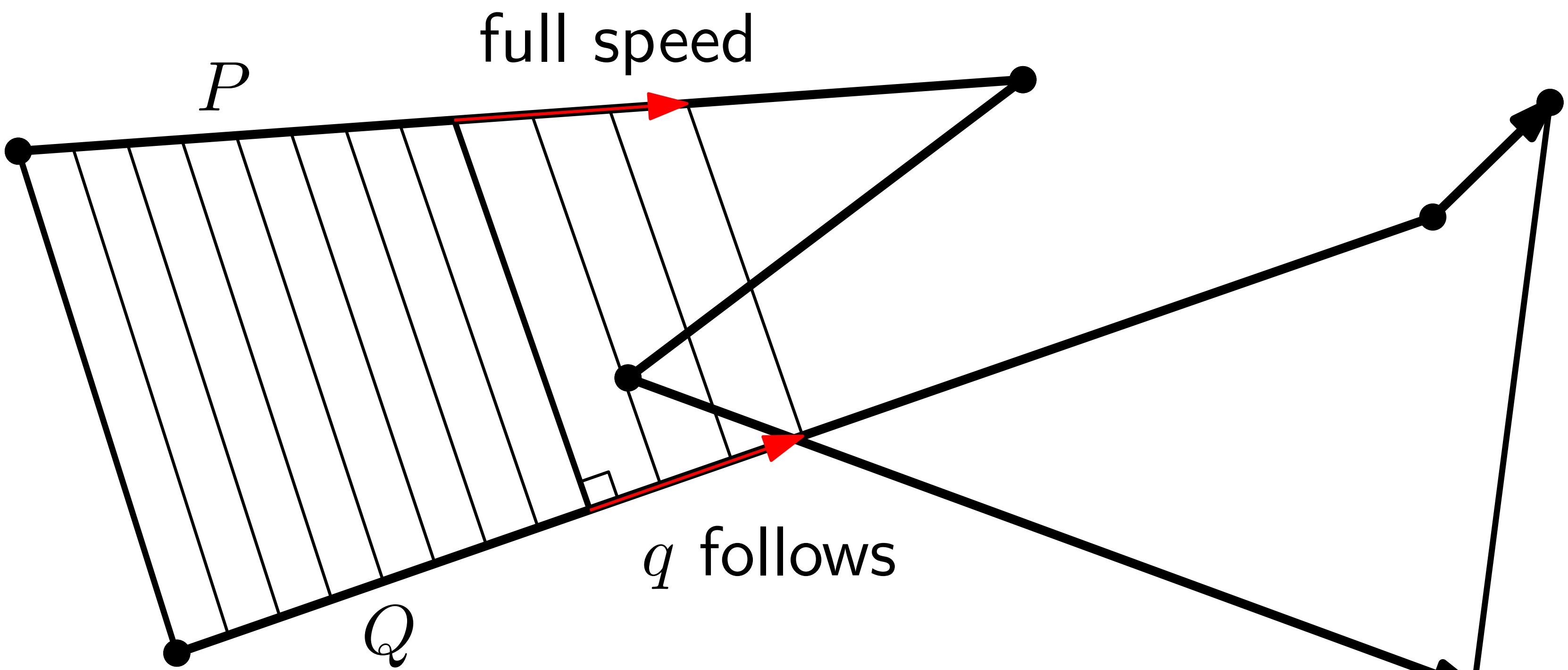
Example



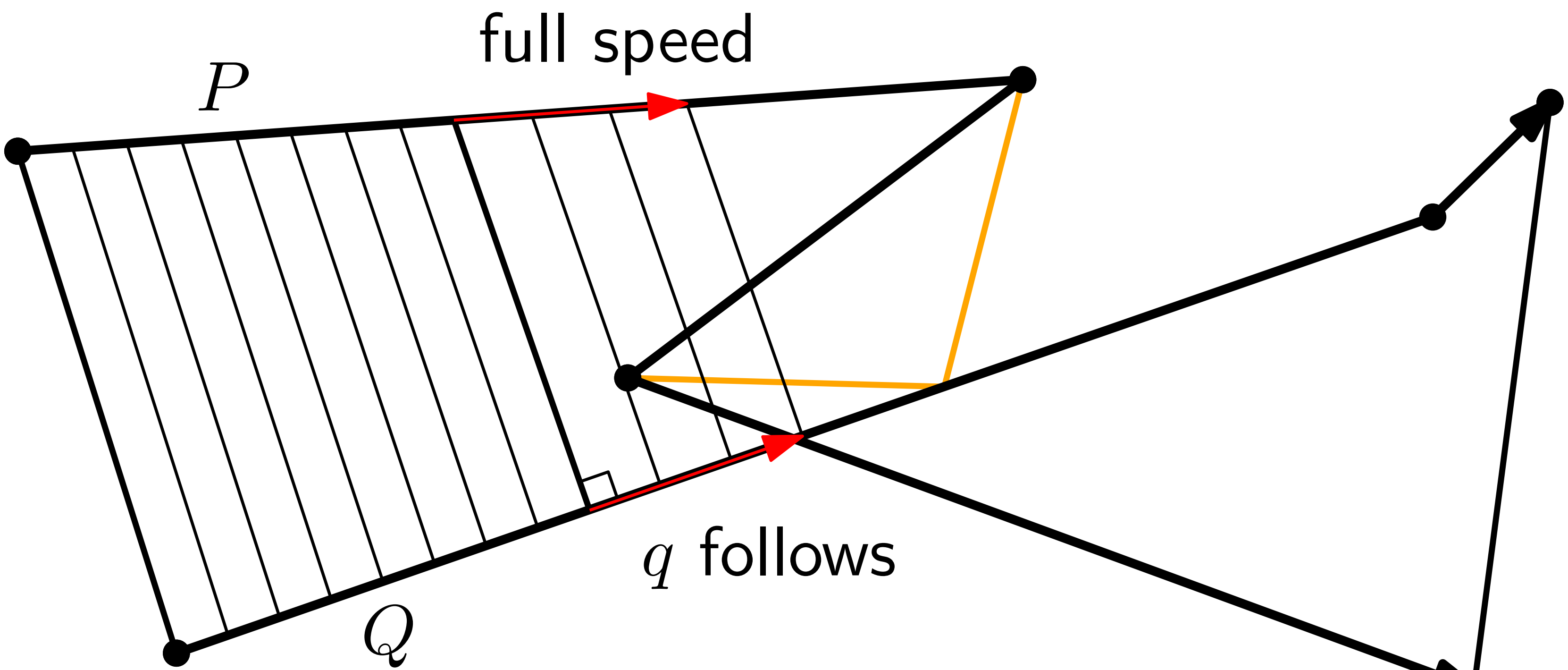
Example



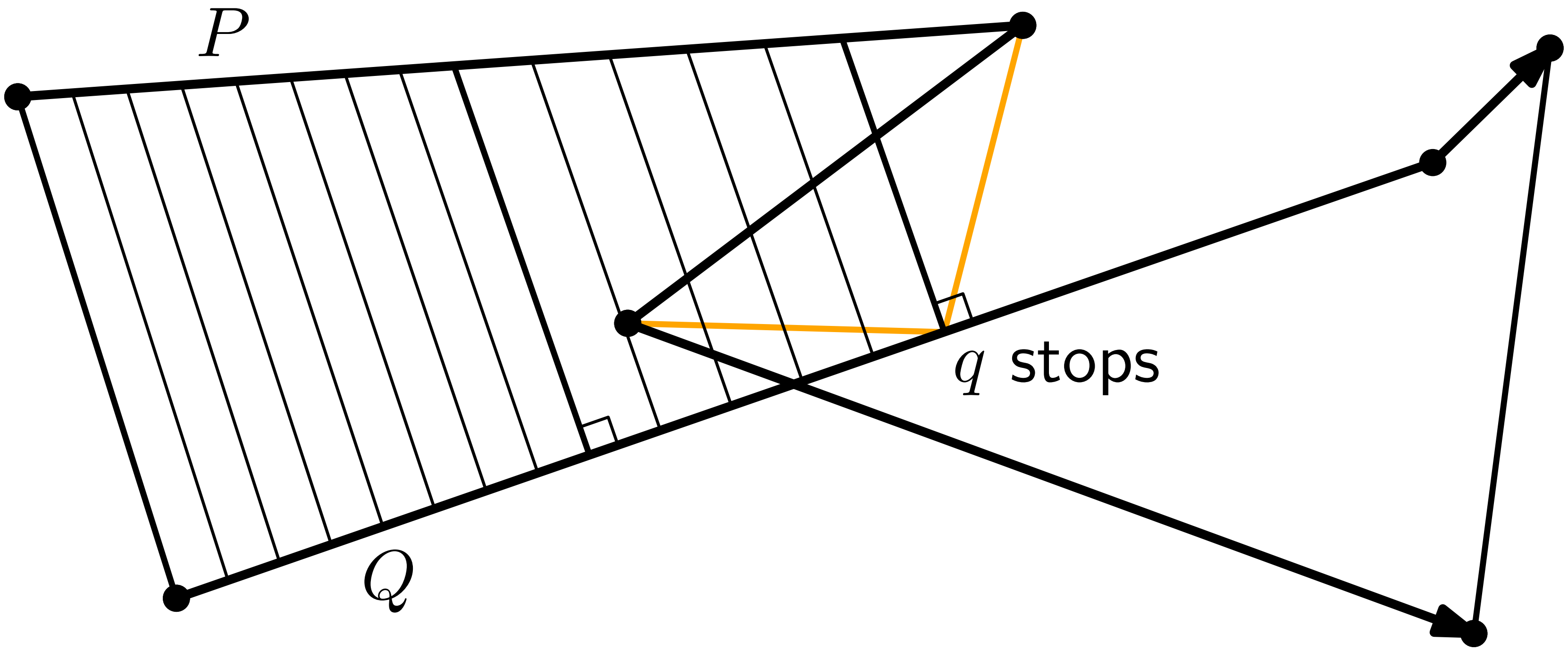
Example



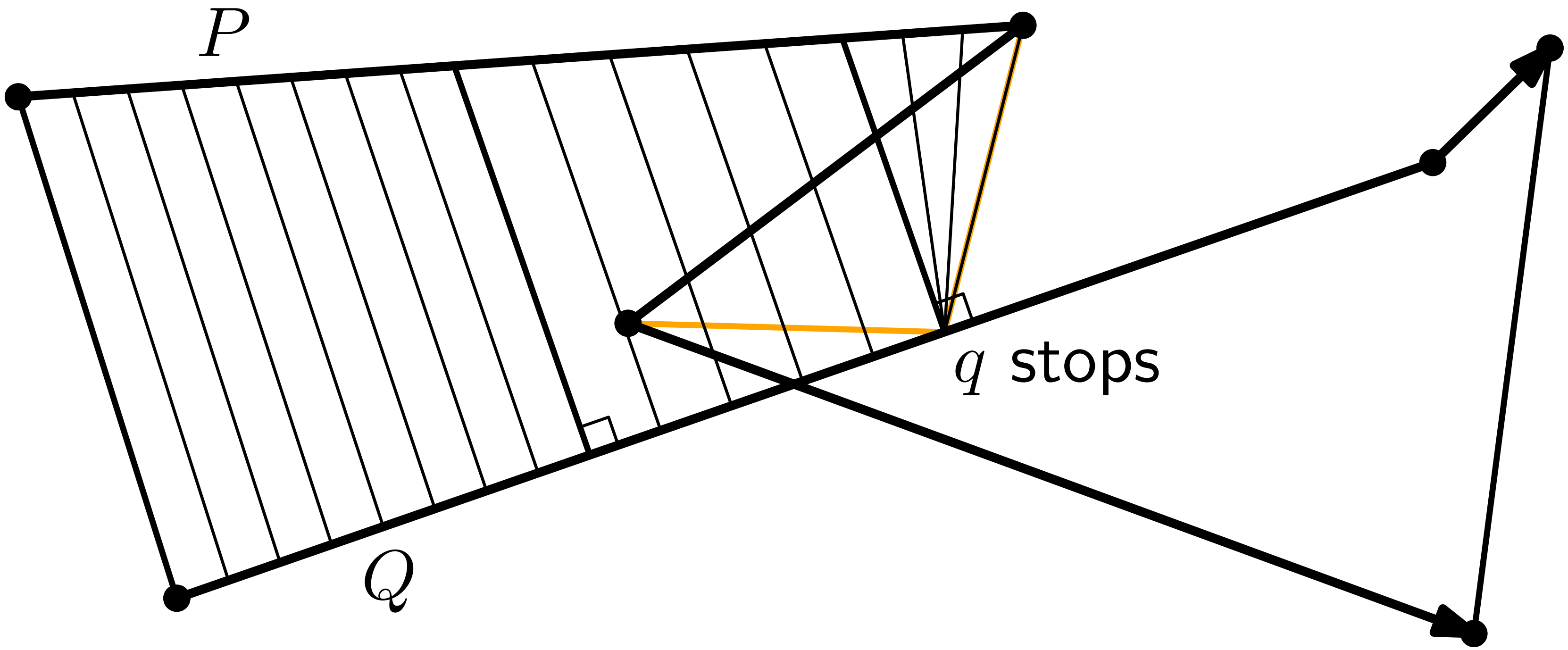
Example



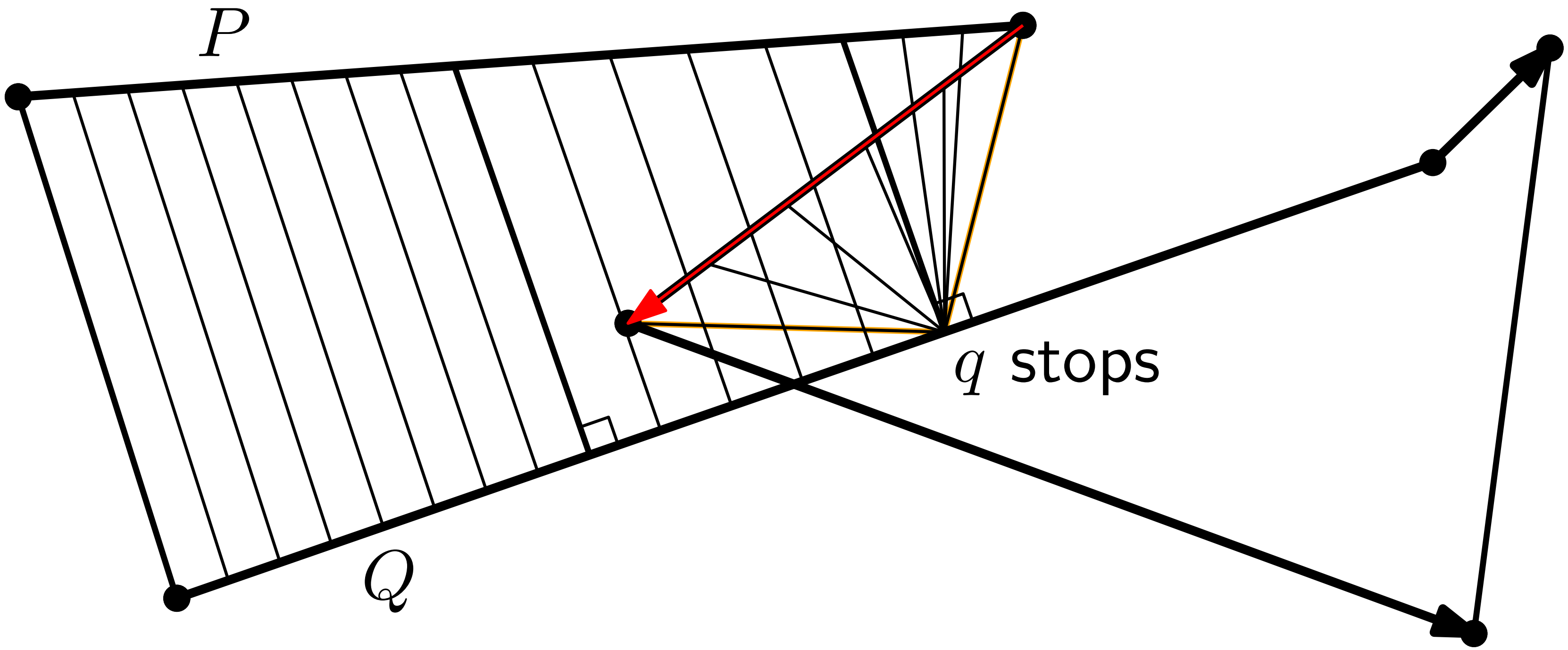
Example



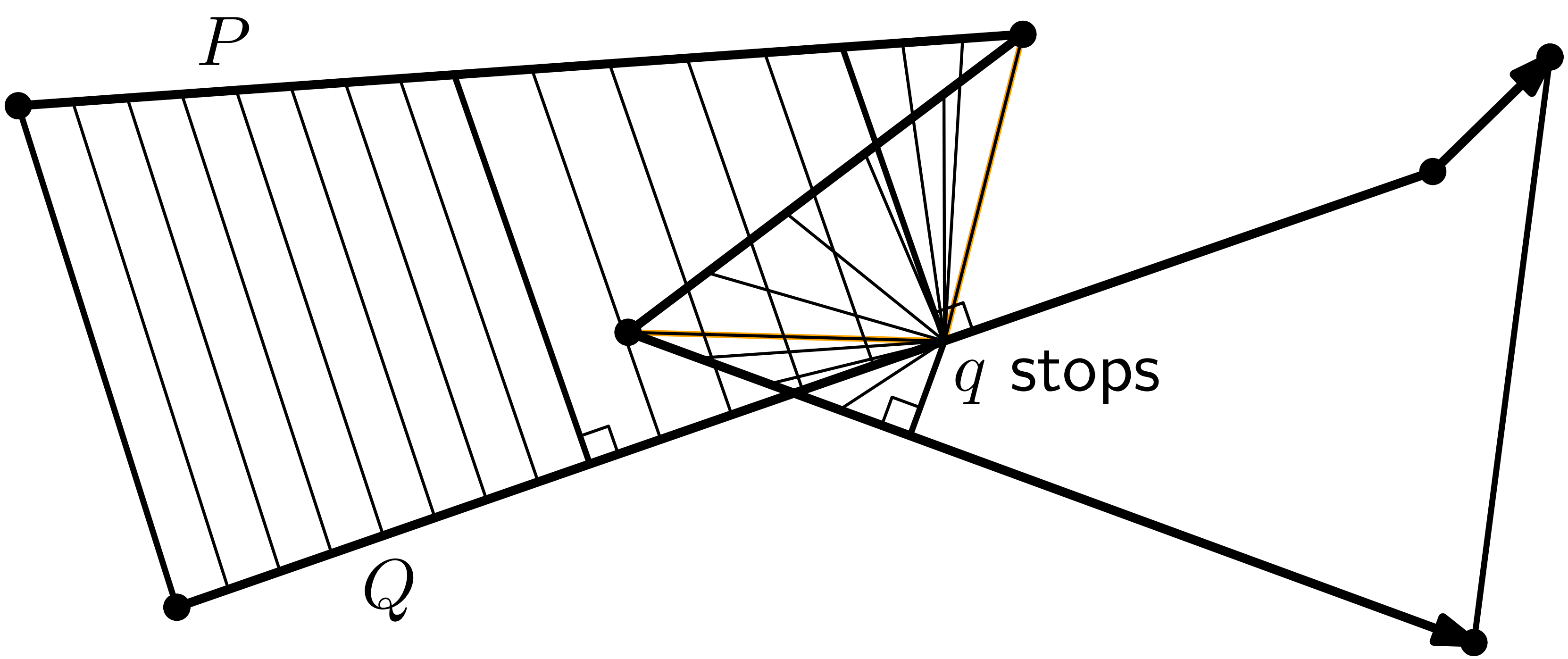
Example



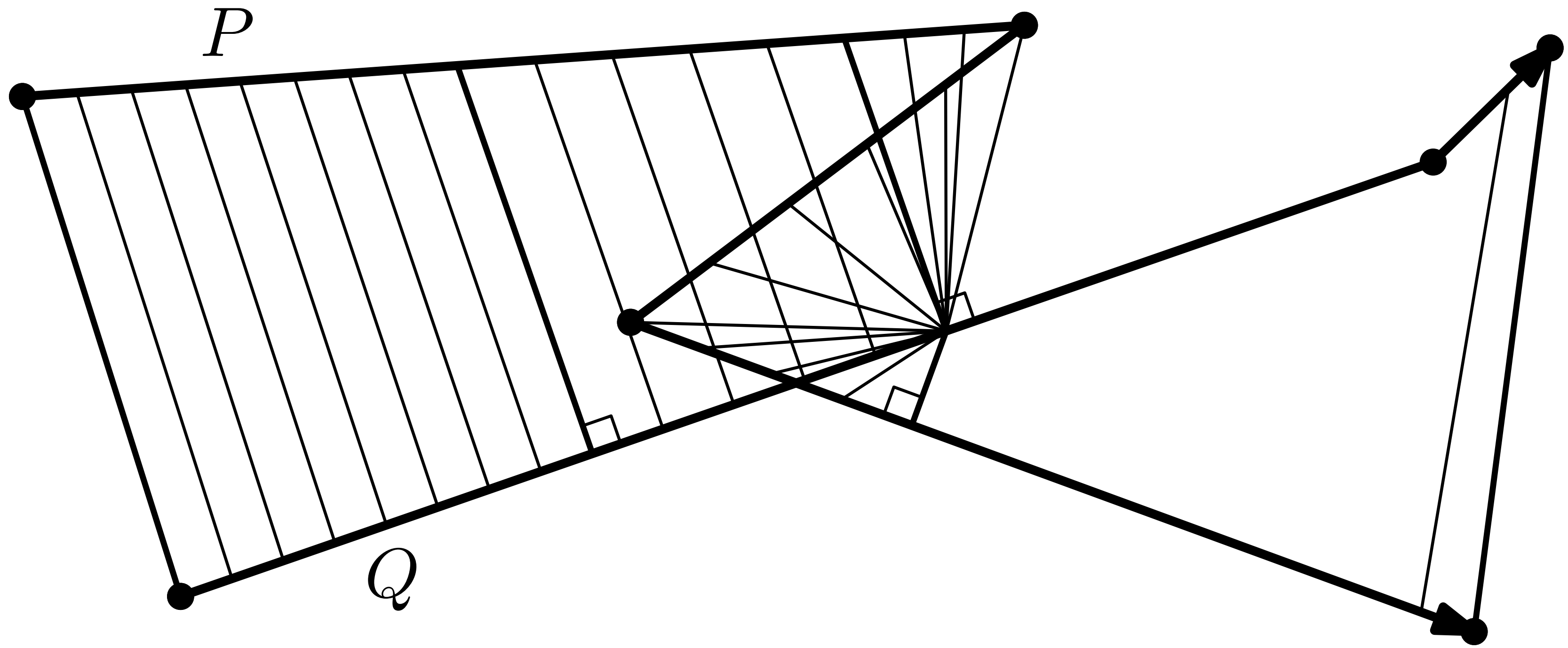
Example



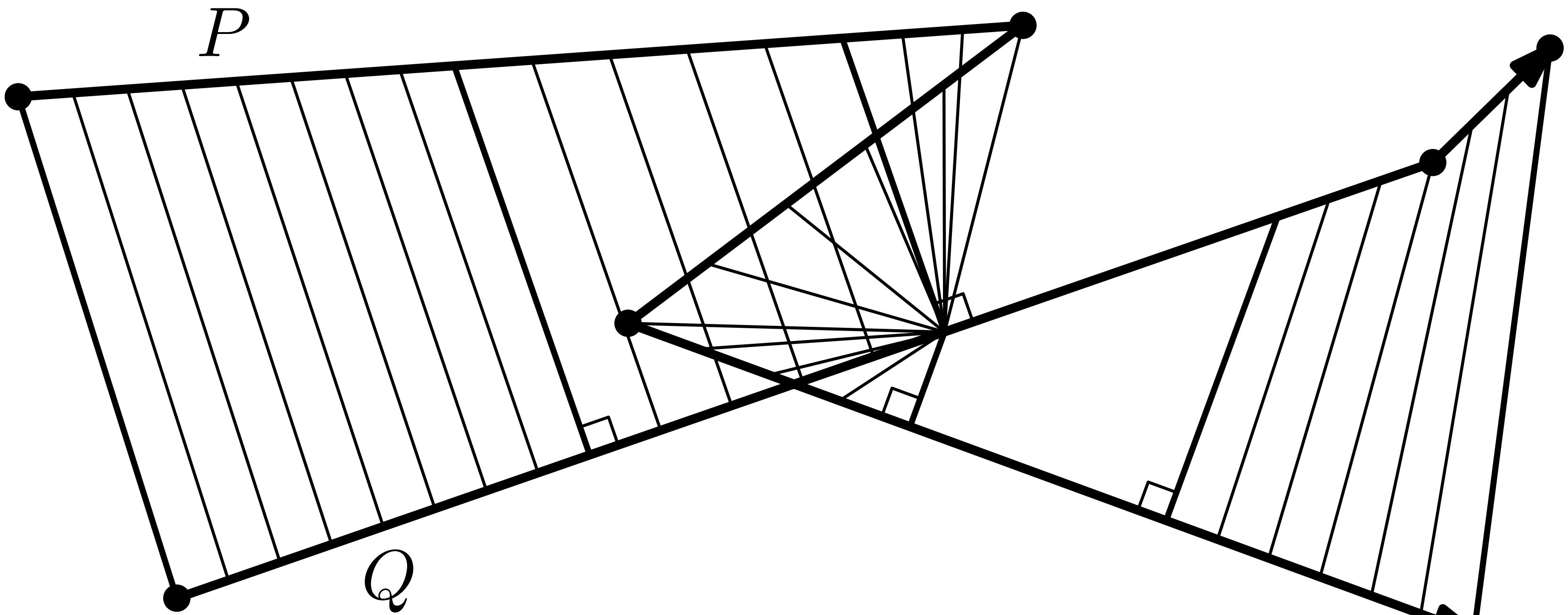
Example



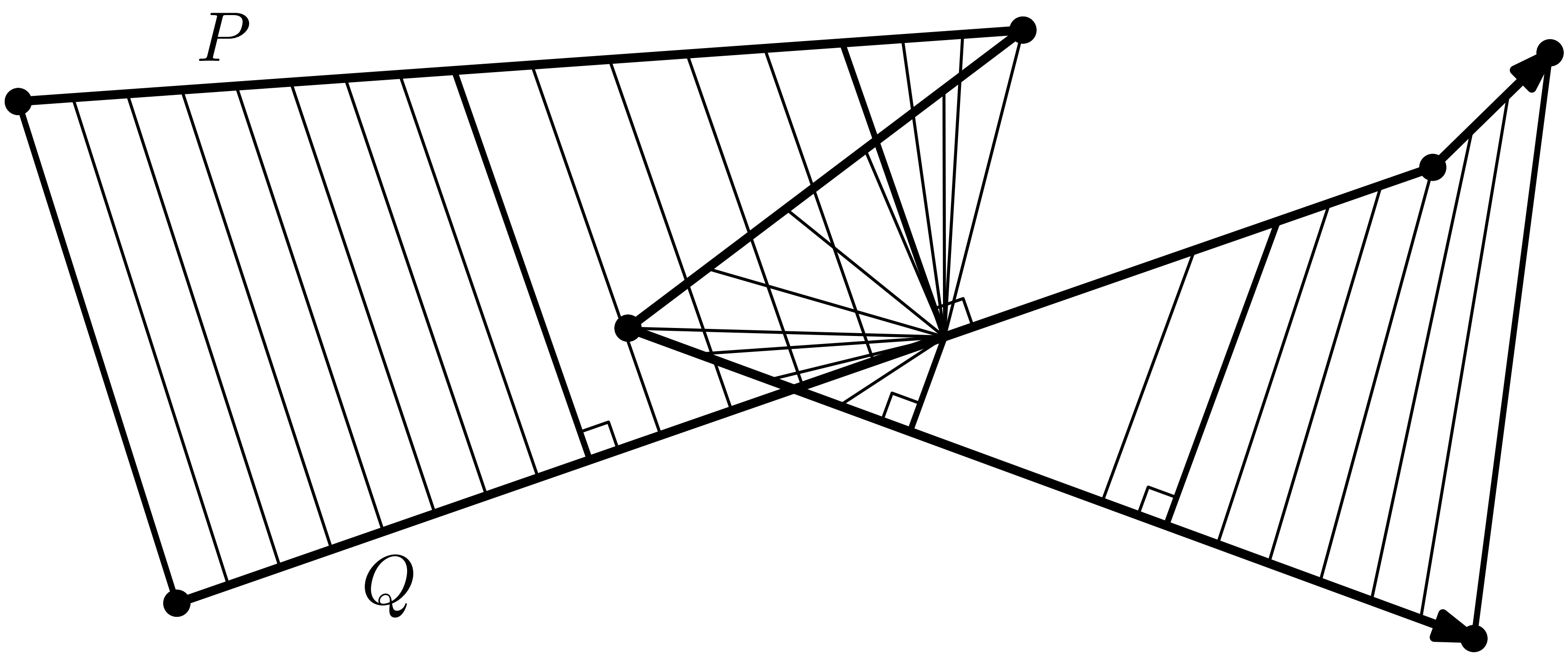
Example



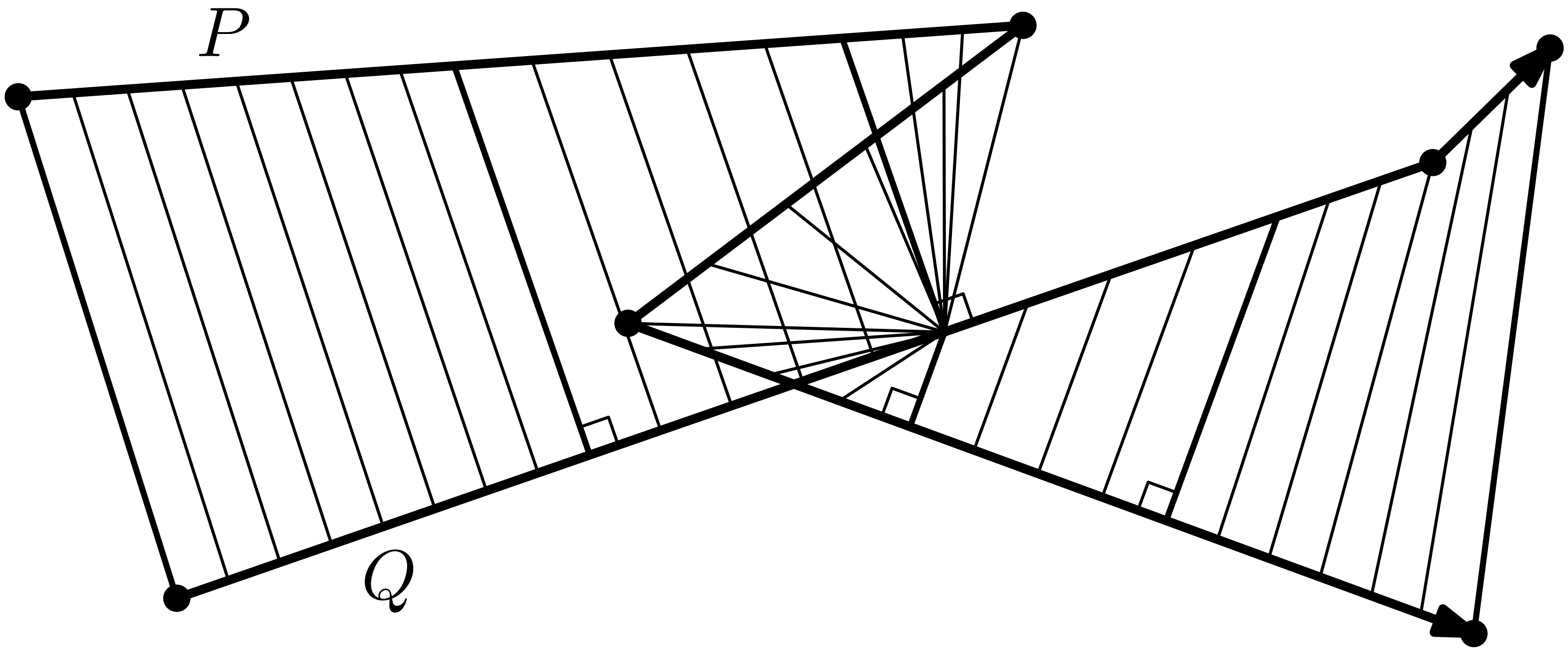
Example



Example



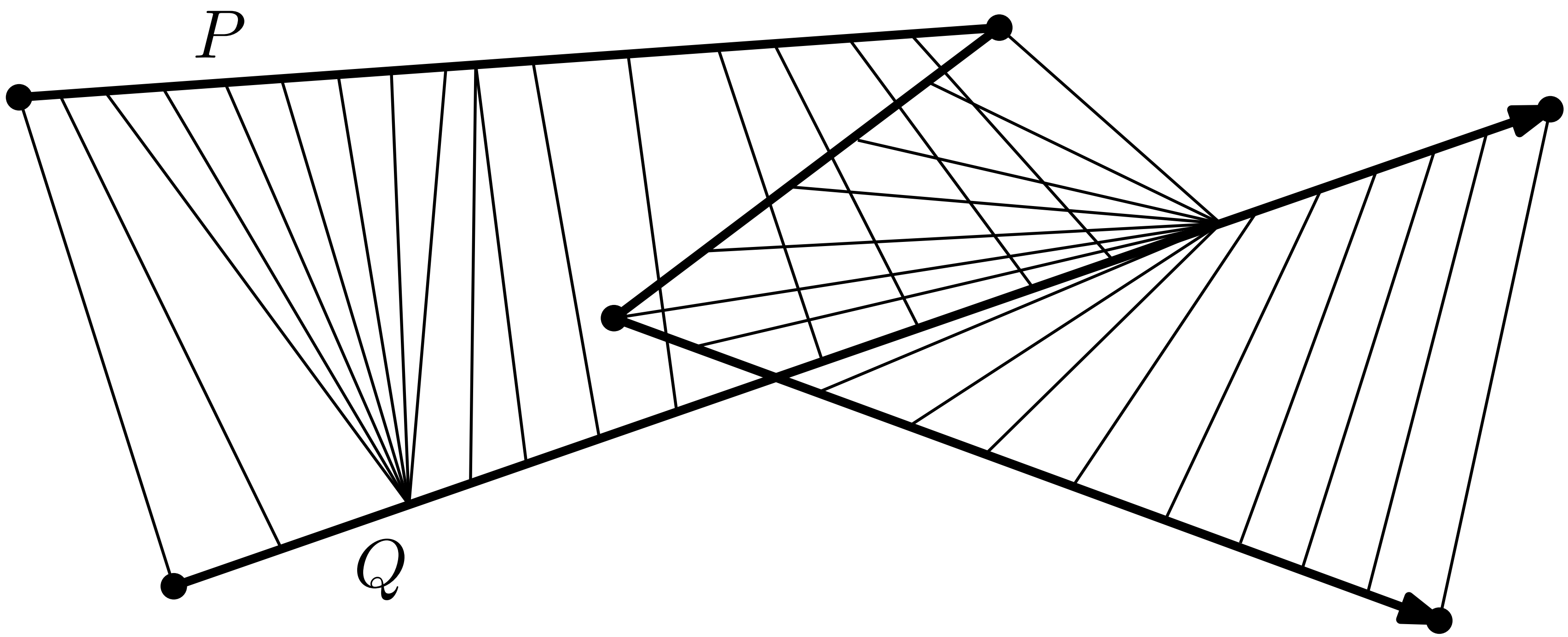
Example



Locally Correct Fréchet Matching

Related Work: [Buchin, Buchin, Meulemans, Speckmann 2012]

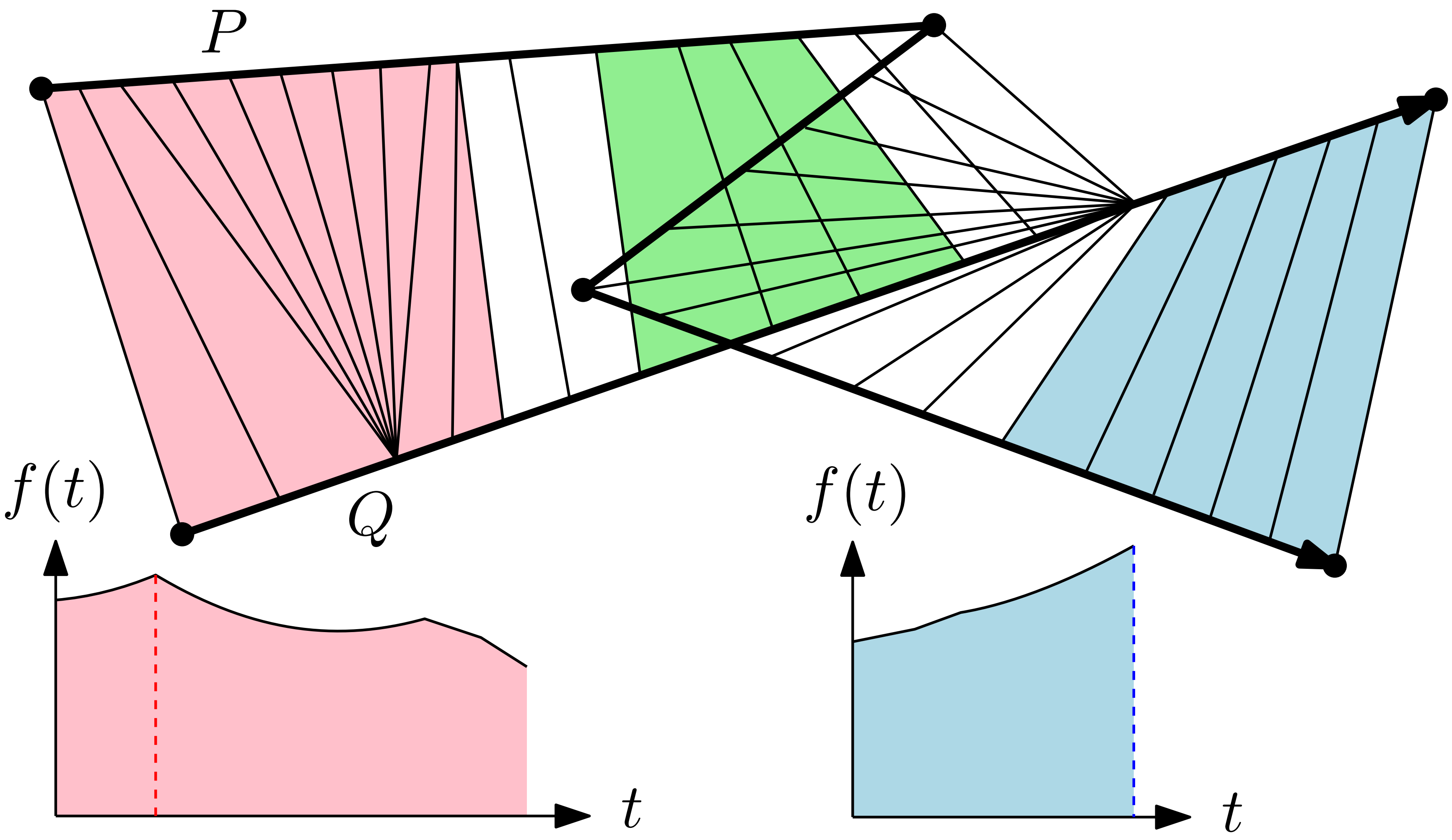
The maximum distance between any two matched subcurves must be the Fréchet distance between these two curves.



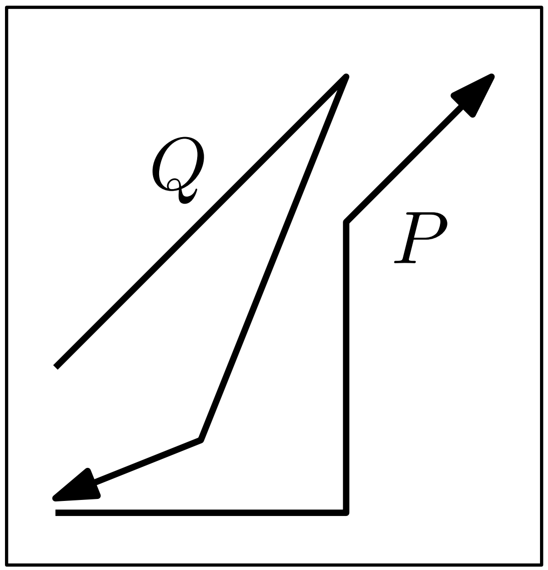
Locally Correct Fréchet Matching

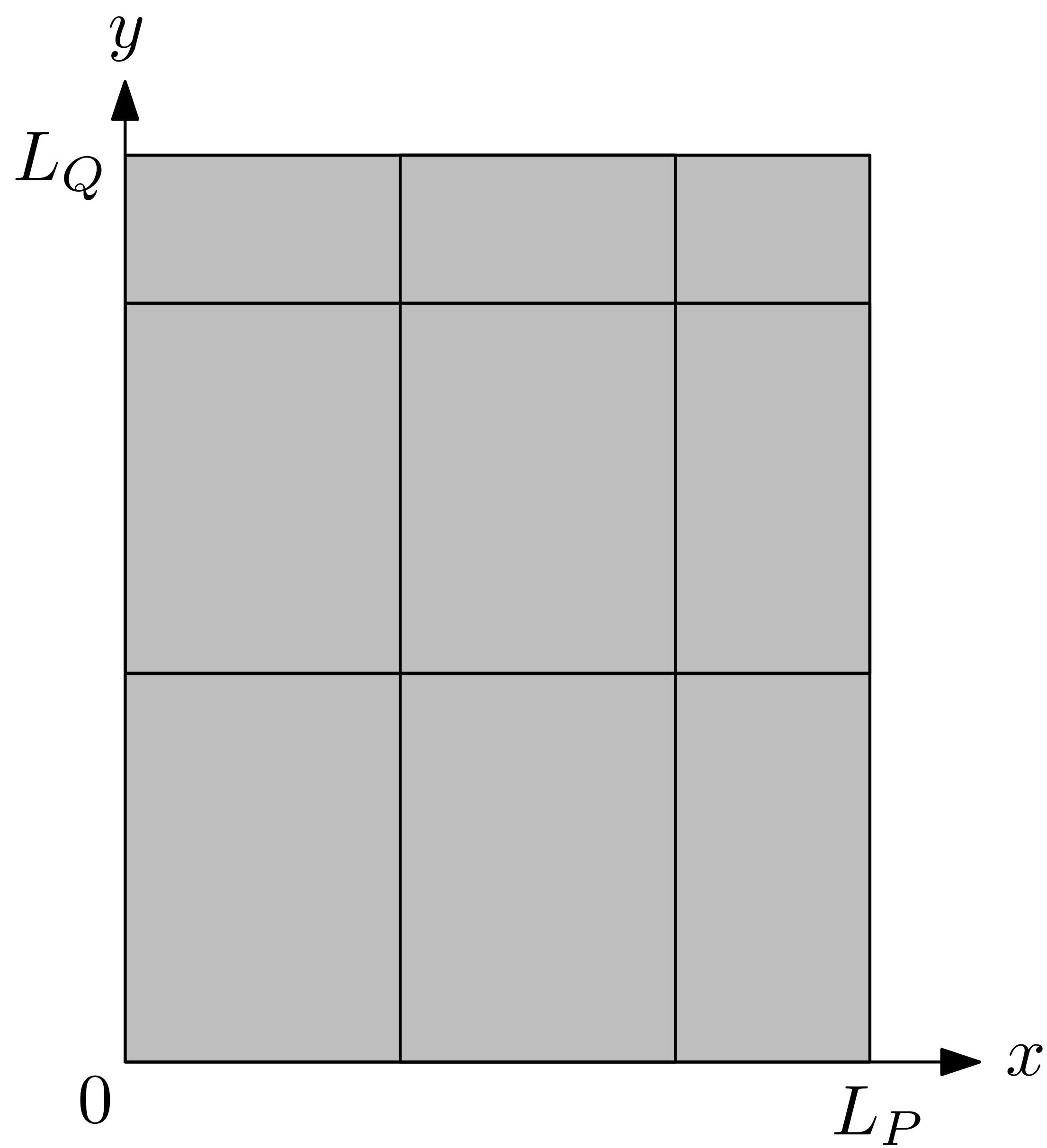
Related Work: [Buchin, Buchin, Meulemans, Speckmann 2012]

The maximum distance between any two matched subcurves must be the Fréchet distance between these two curves.

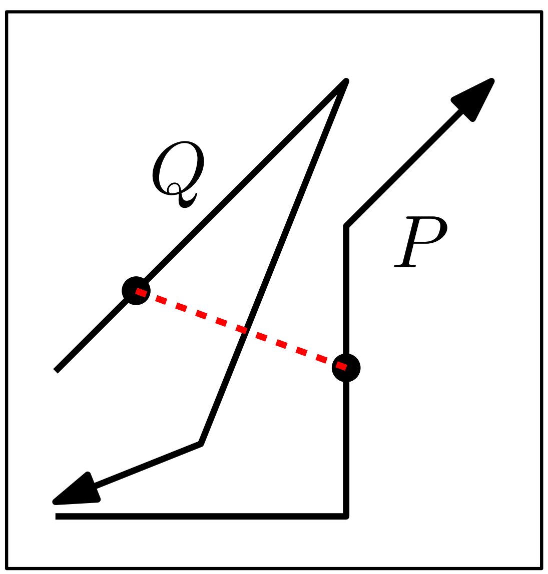


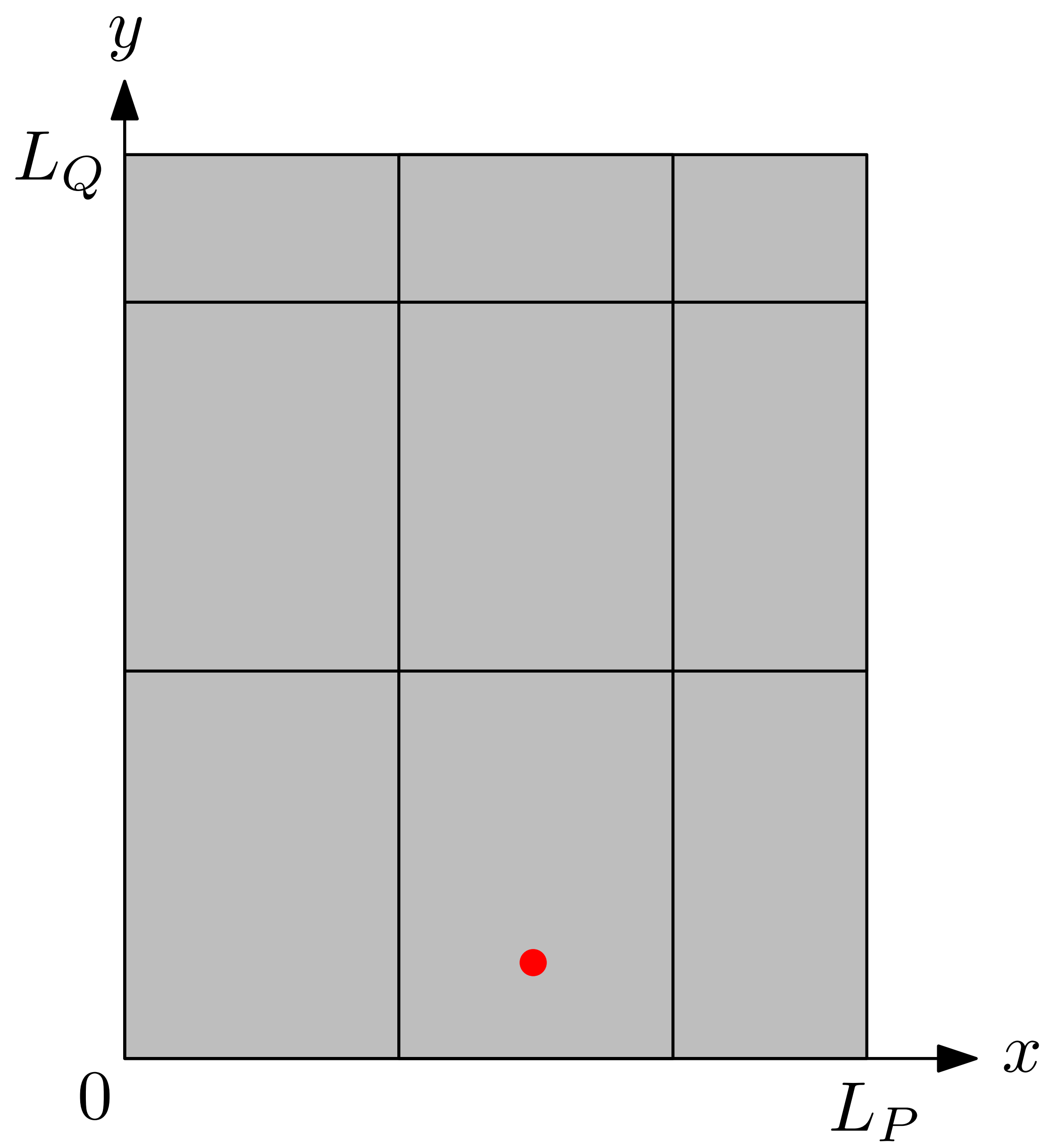
The Parameter Rectangle R



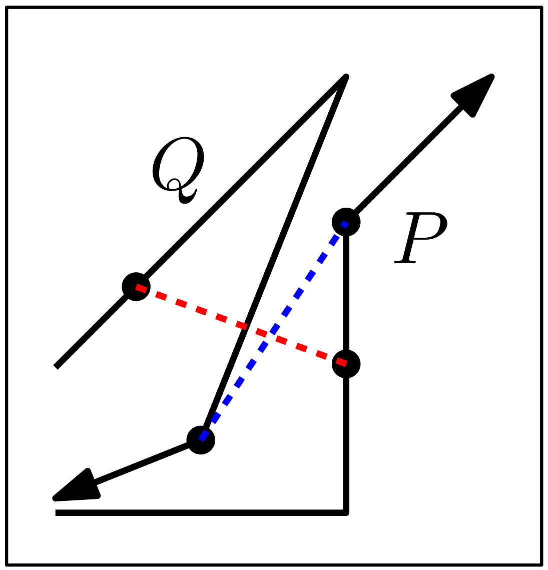


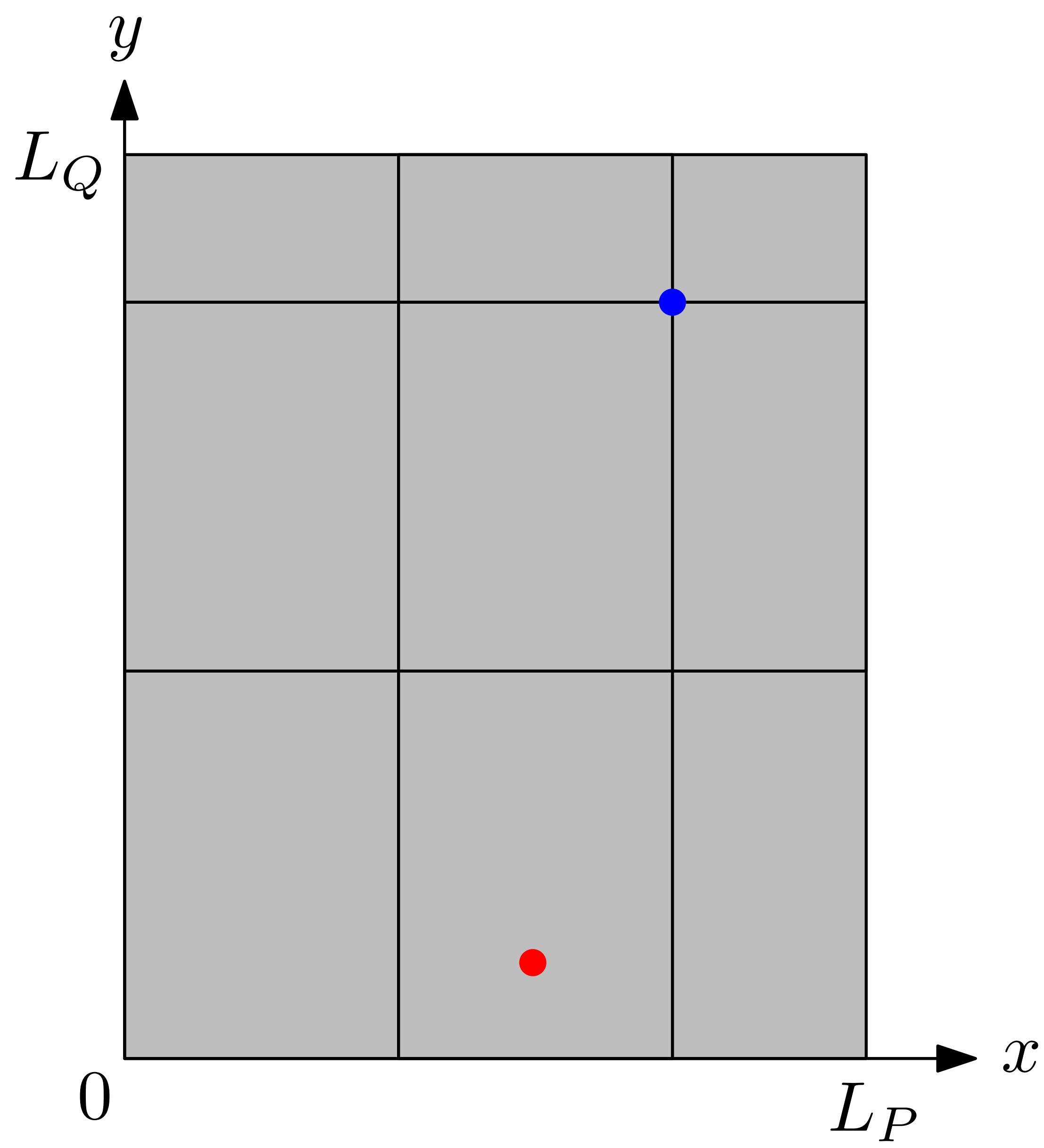
The Parameter Rectangle R



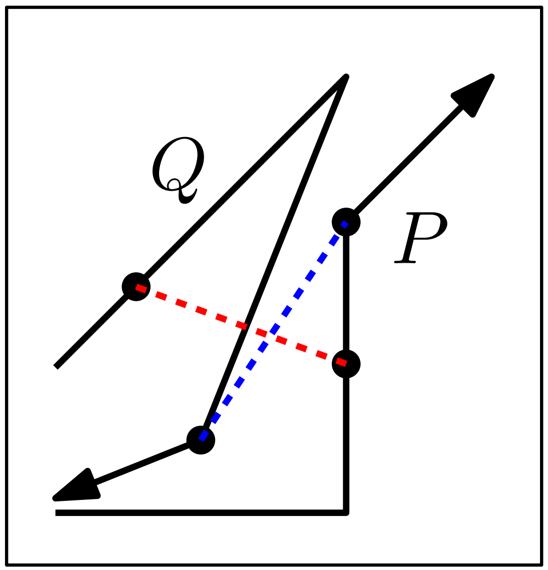


The Parameter Rectangle R

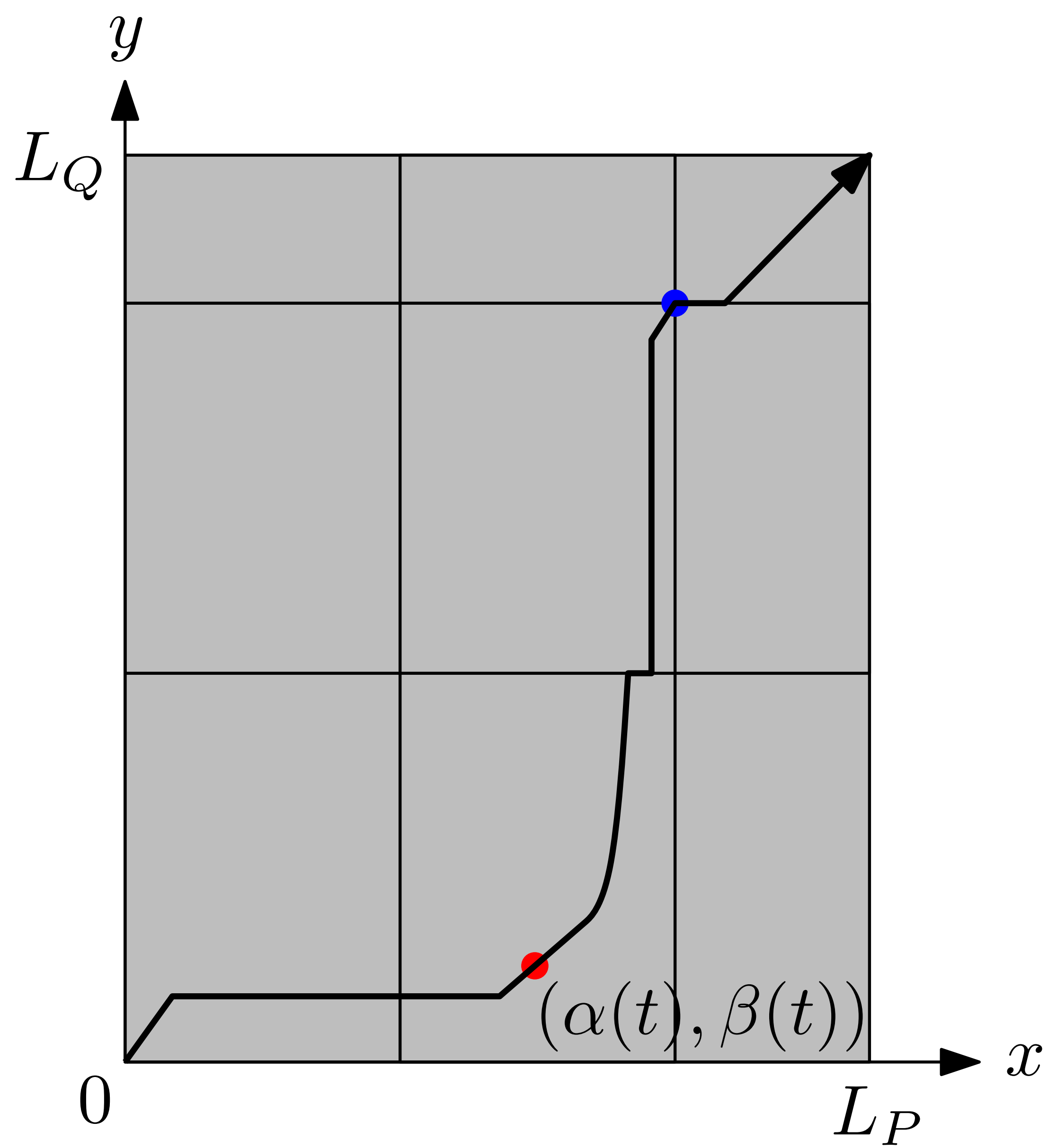




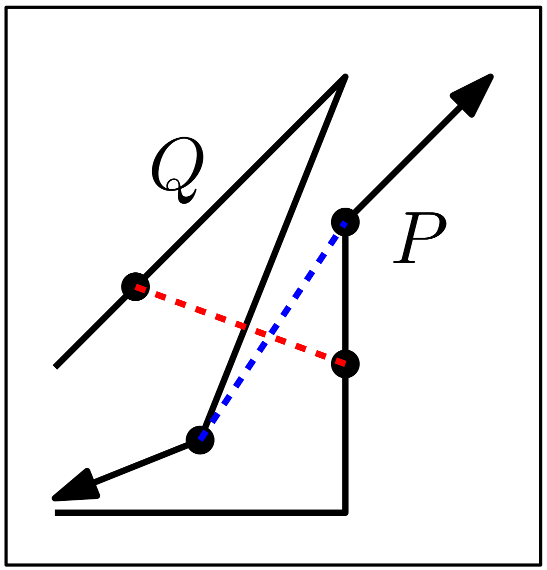
The Parameter Rectangle R



wanted: monotone path

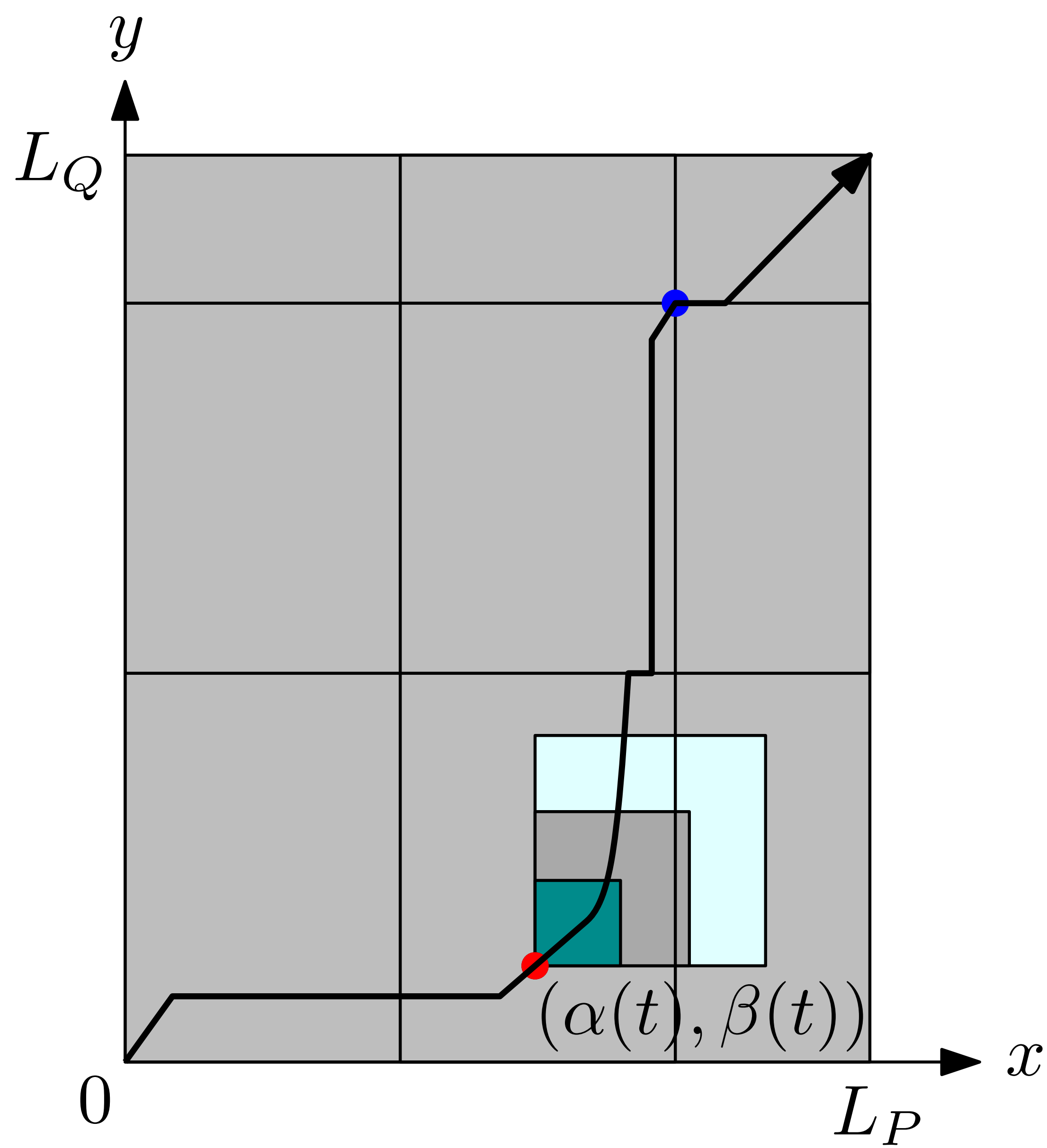


The Parameter Rectangle R



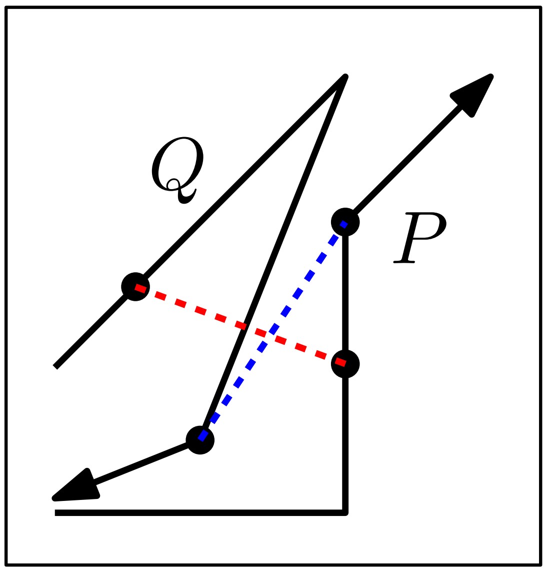
wanted: monotone path

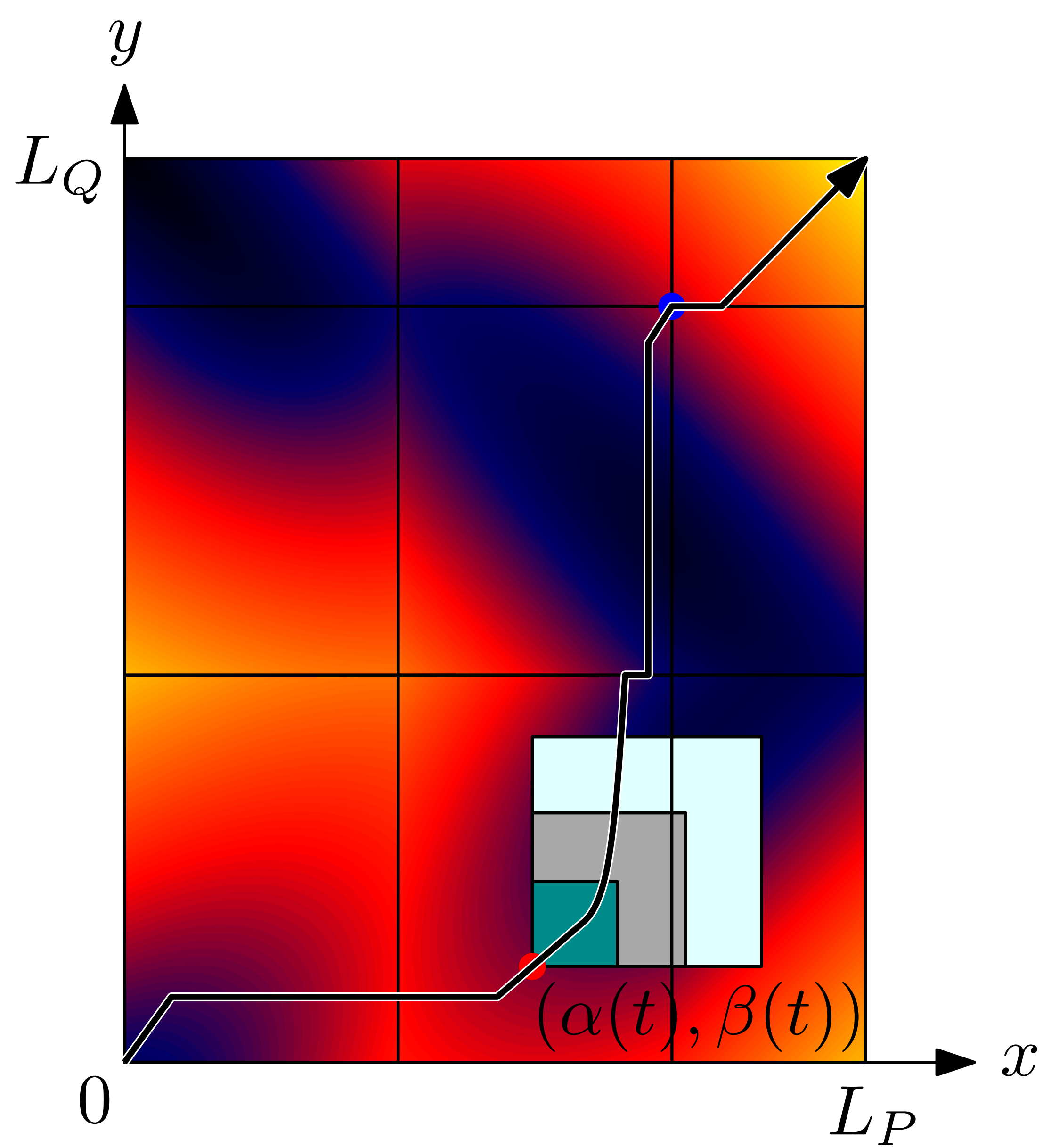
speed limit in $L_{\max}$-norm



The Parameter Rectangle R

Freie Universität Berlin



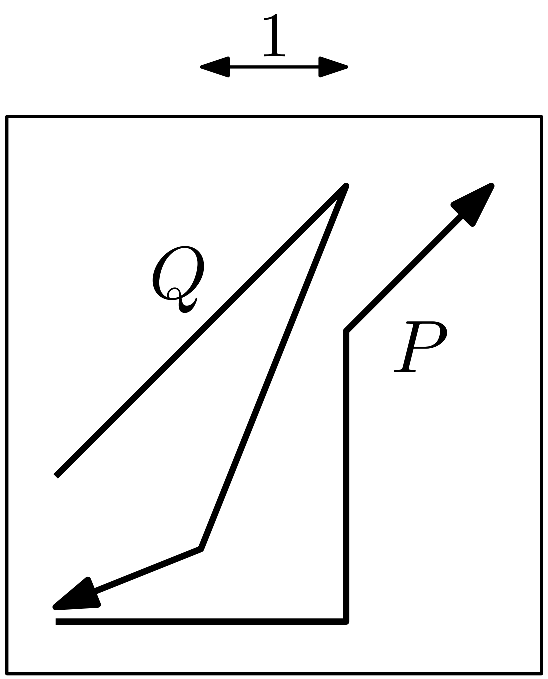


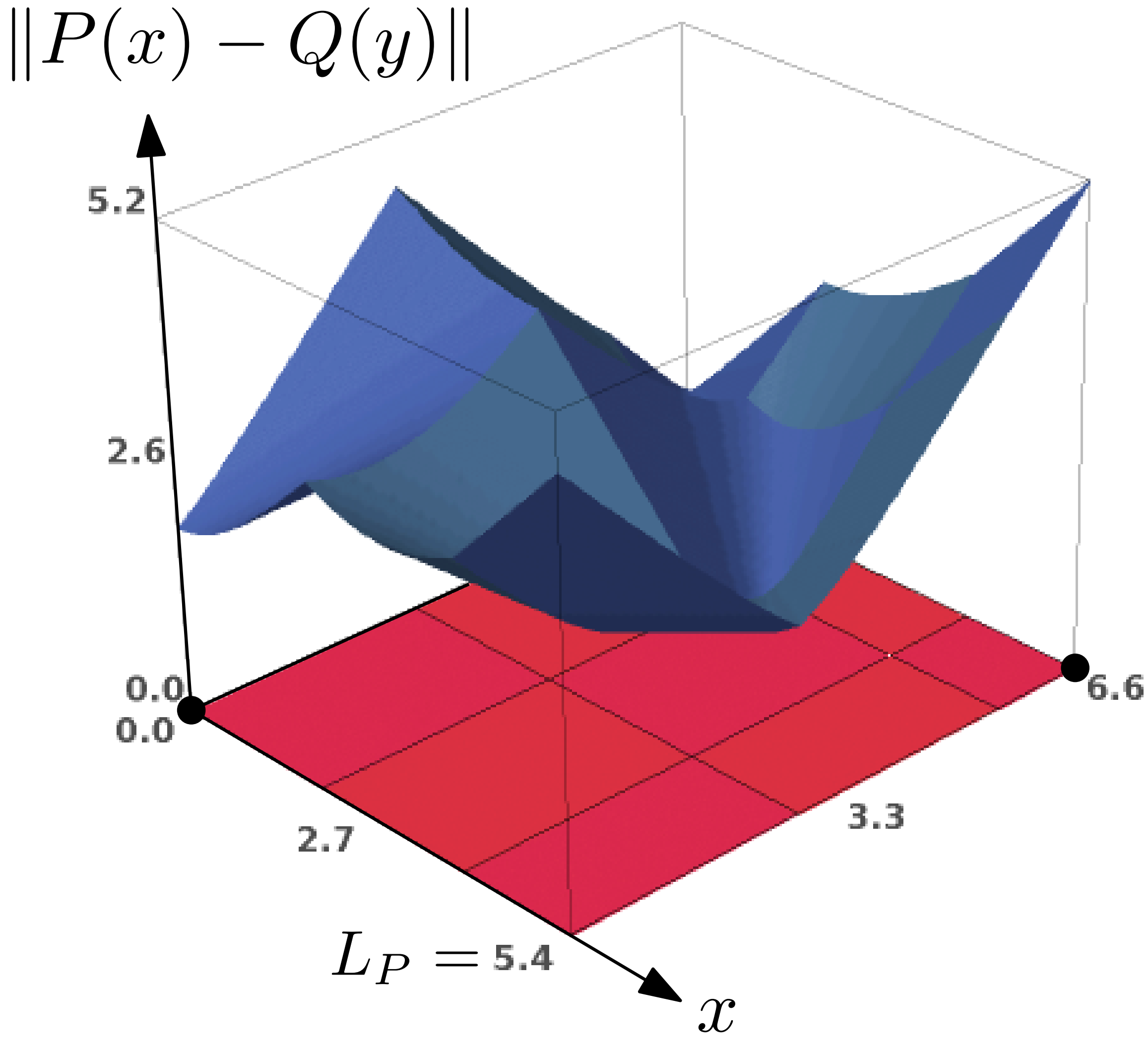
wanted: monotone path

speed limit in $L_{\max}$-norm

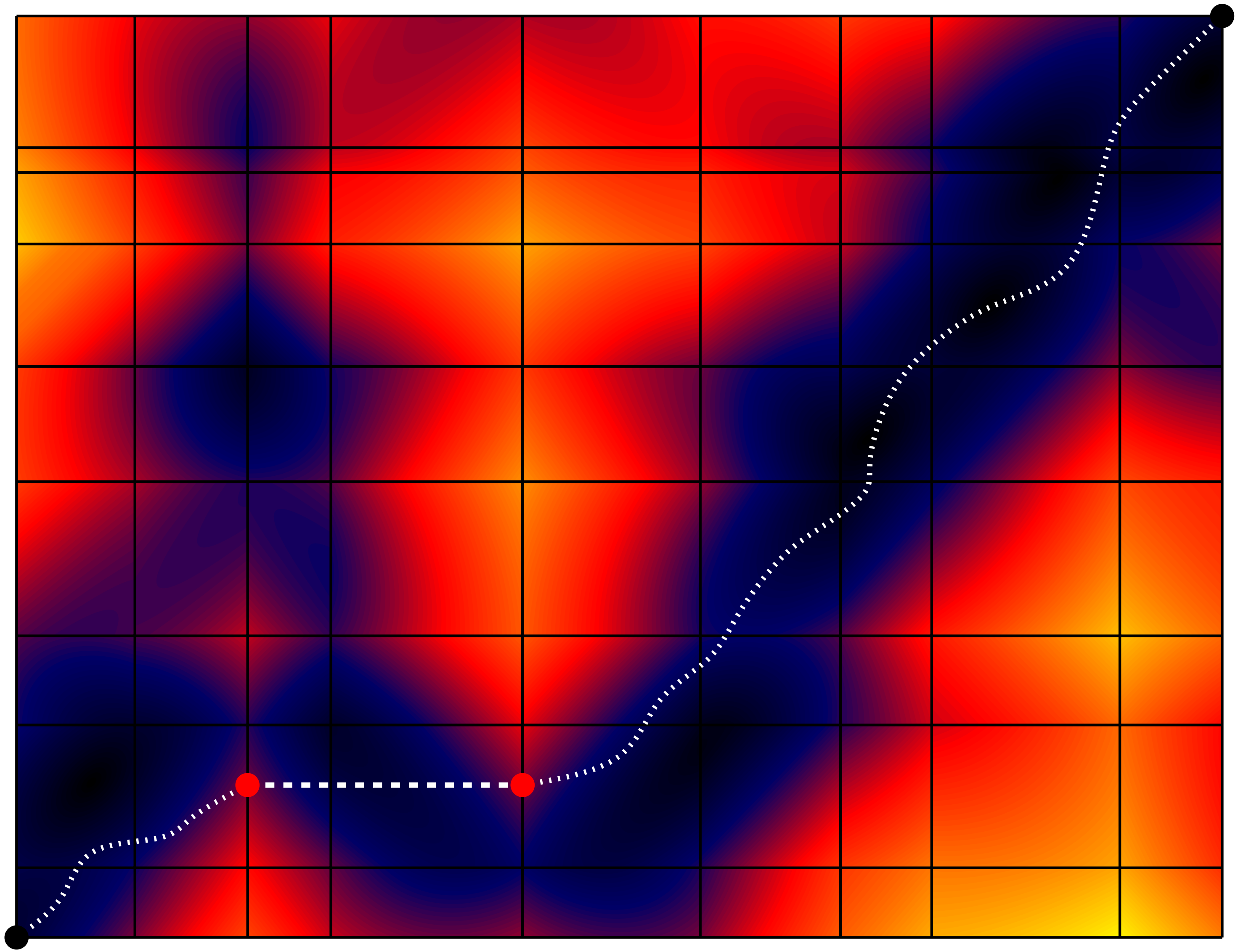
Avoid high values of $\|P(x) - Q(y)\|$

The Distance Landscape





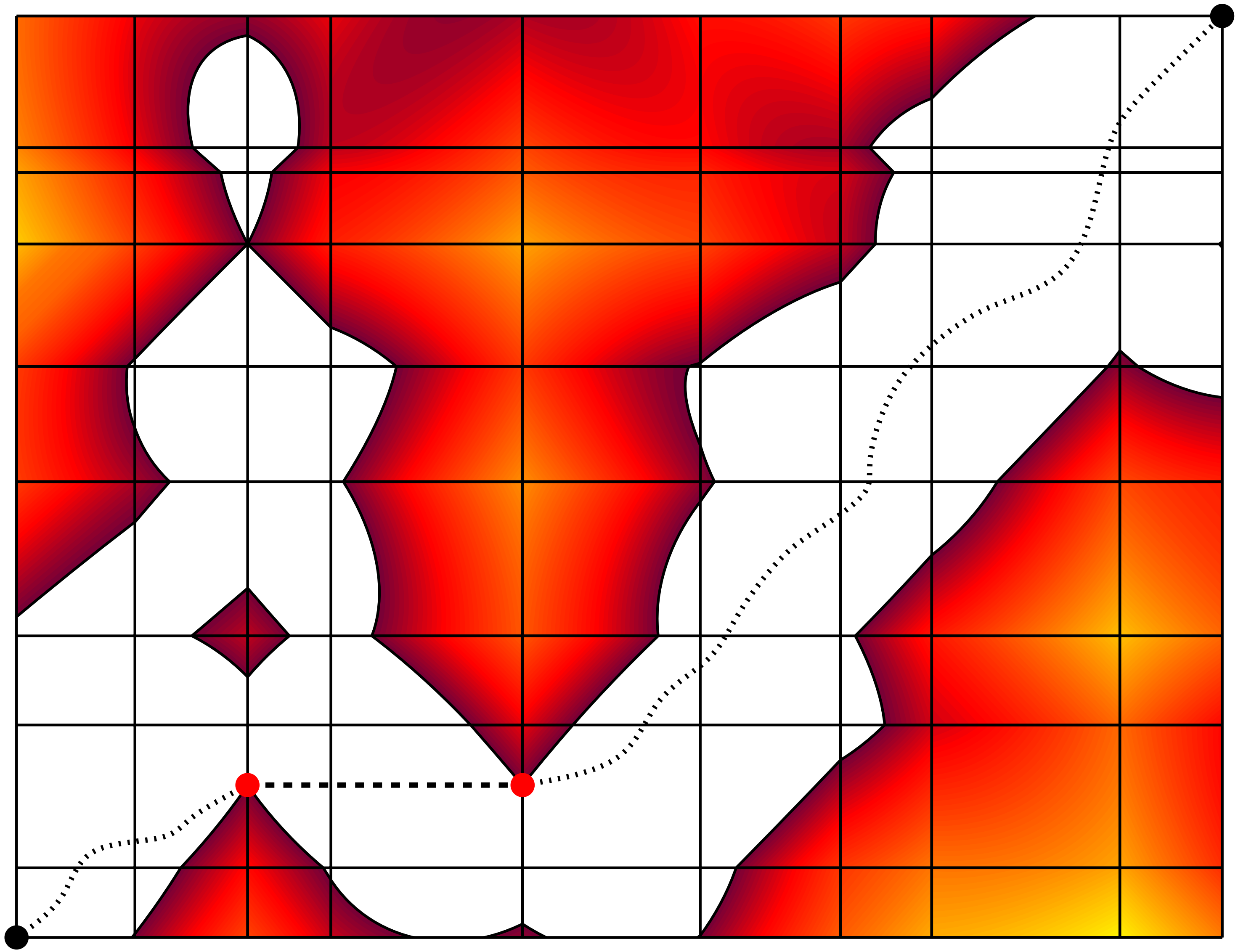
A Critical Passage



ASSUME: There is a UNIQUE critical passage.

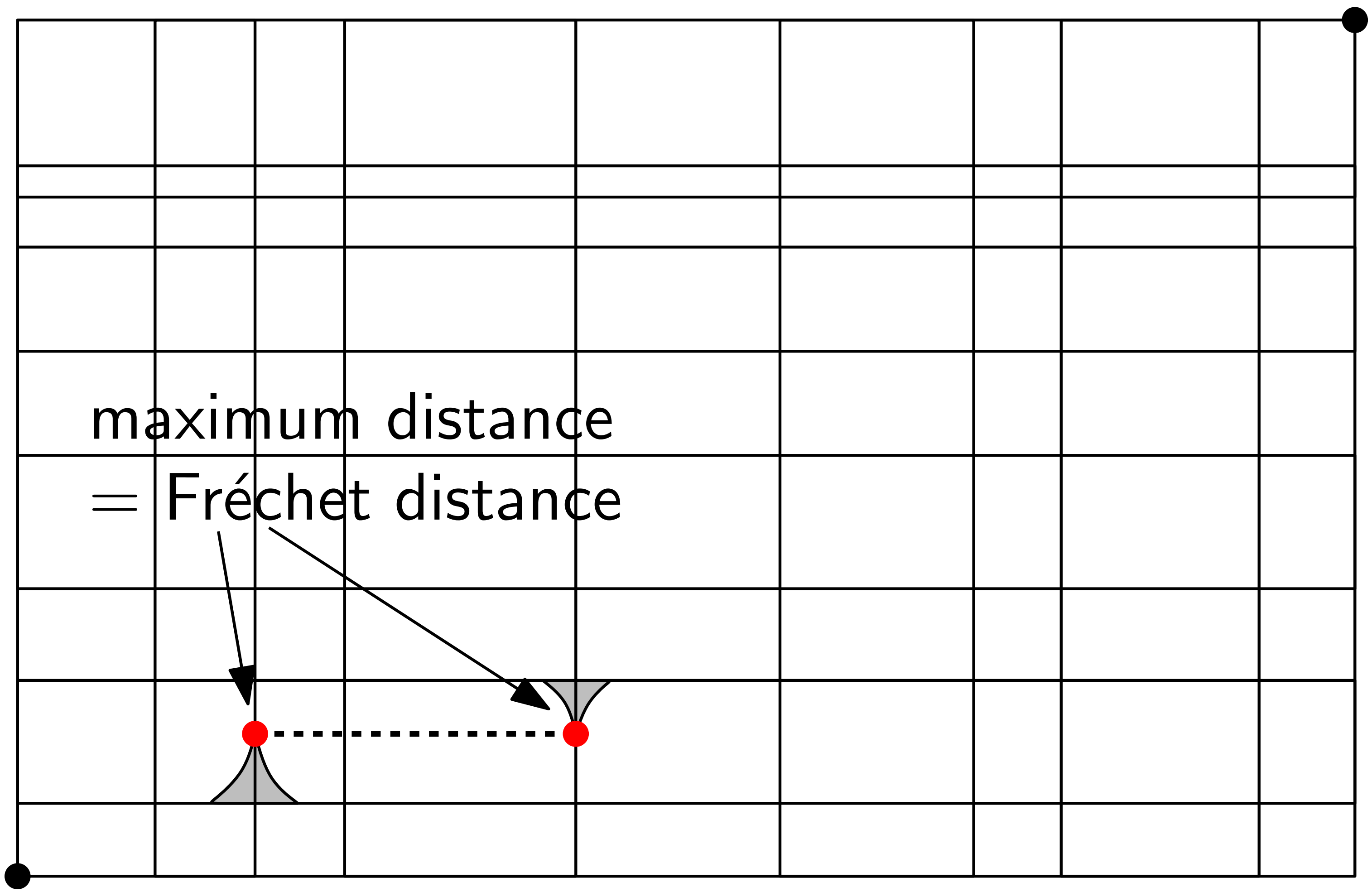
A Critical Passage

Freie Universität Berlin



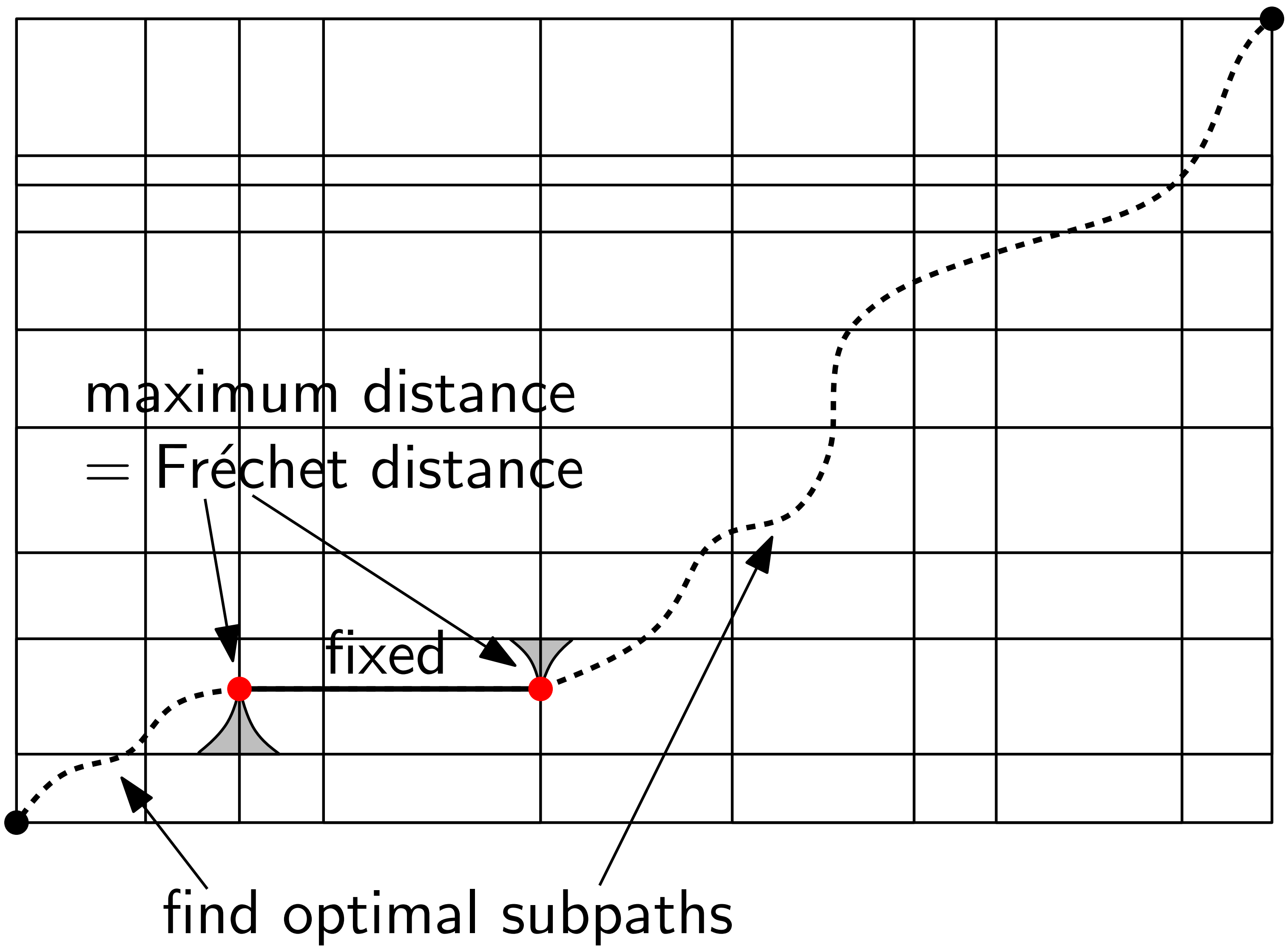
ASSUME: There is a UNIQUE critical passage.

A Critical Passage



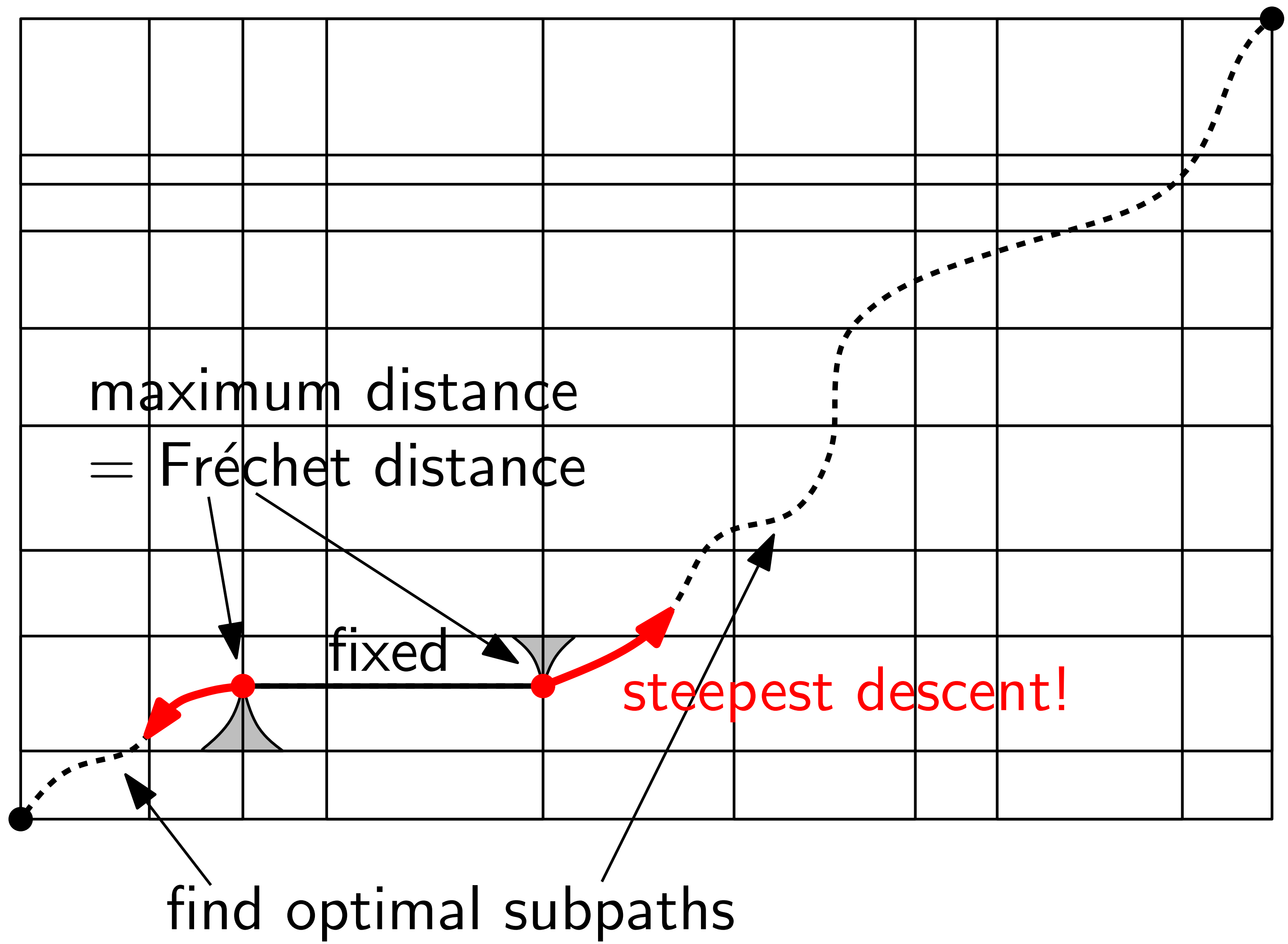
ASSUME: There is a UNIQUE critical passage.

A Critical Passage



ASSUME: There is a UNIQUE critical passage.

A Critical Passage



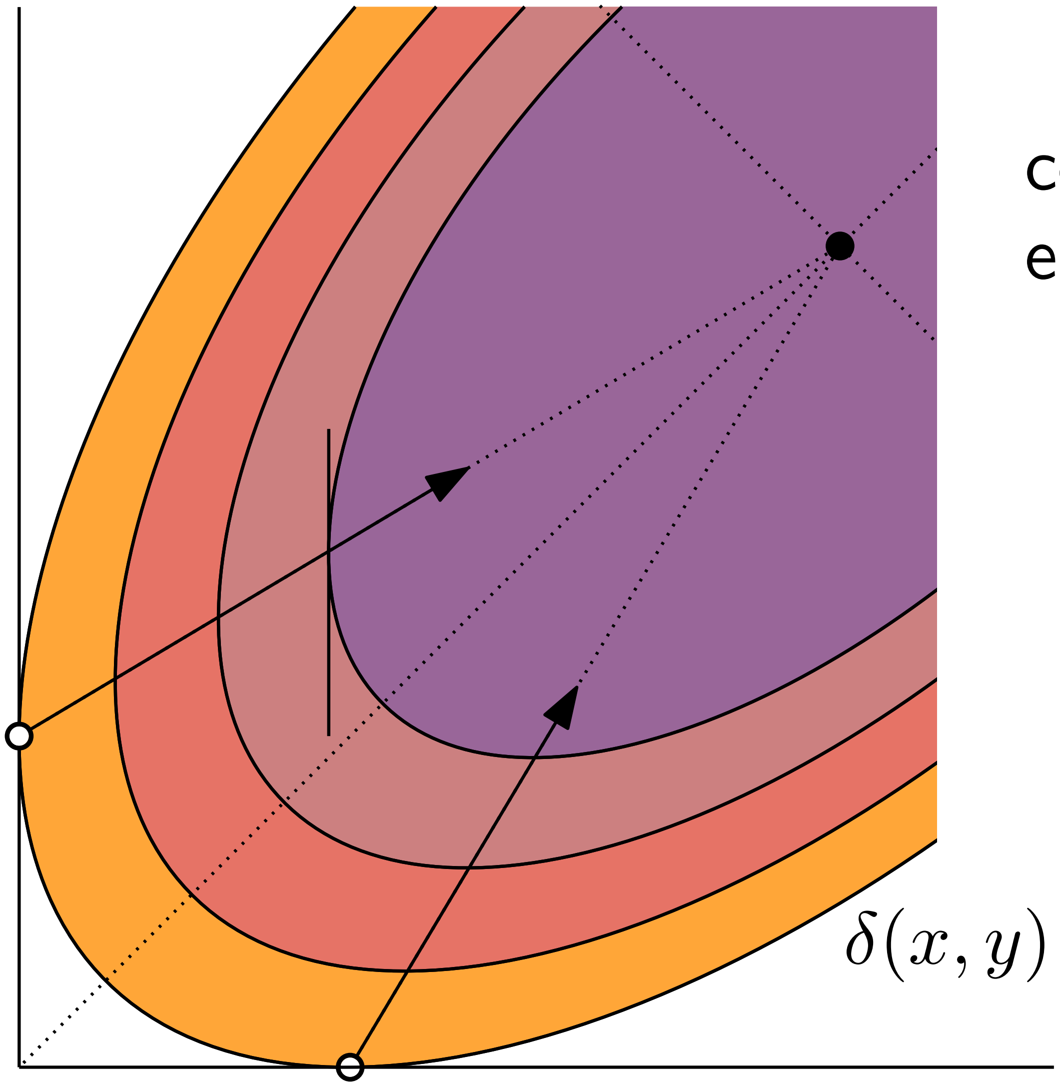
ASSUME: There is a UNIQUE critical passage.

Steepest Descent

Inside one cell: $\delta(x,y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$

$$(-2 \le \lambda \le 2,\ c \ge 0)$$



concentric homothetic ellipses with 45° axes

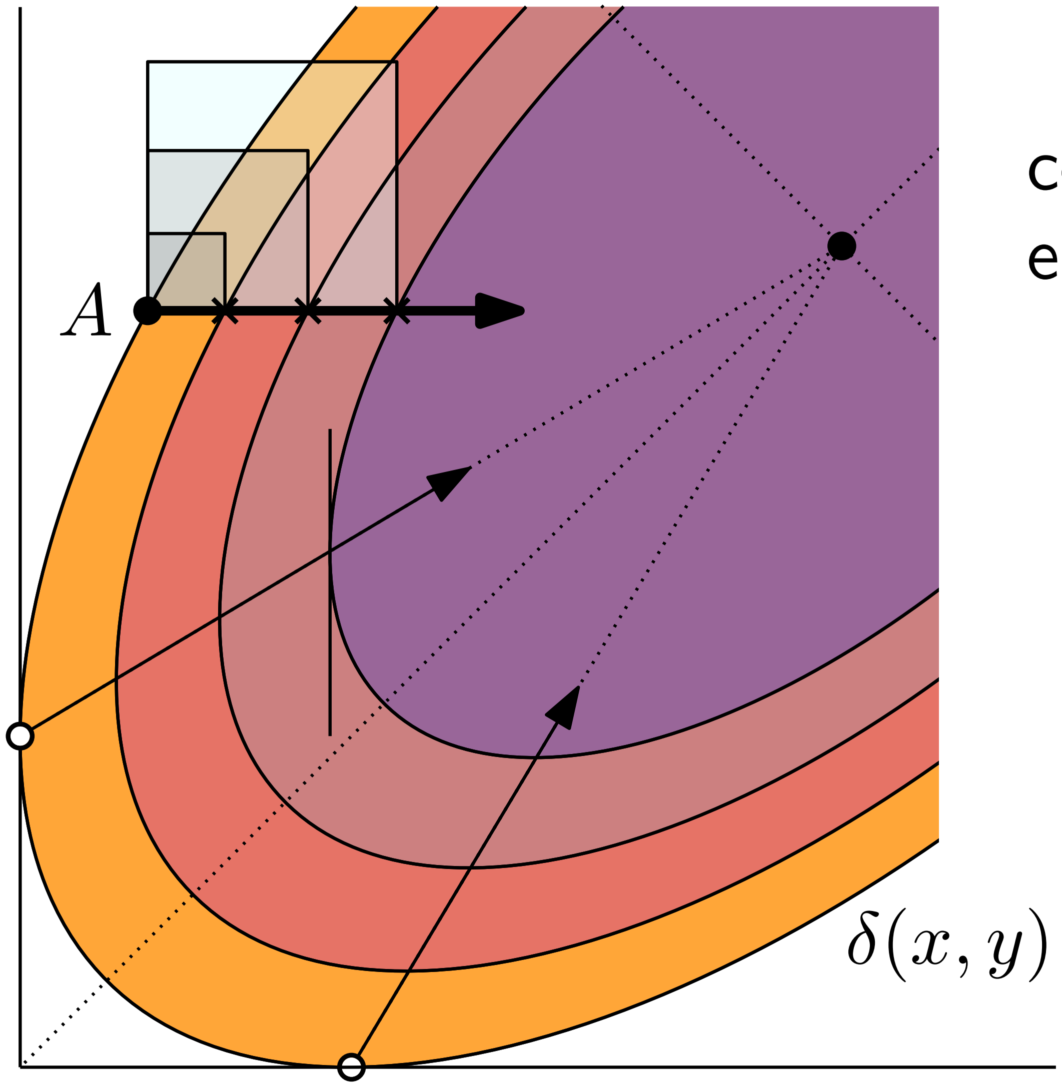
$\delta(x,y) = \text{const}$

Steepest Descent

Inside one cell: $\delta(x, y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$

$$(-2 \le \lambda \le 2,\ c \ge 0)$$



concentric homothetic ellipses with 45° axes

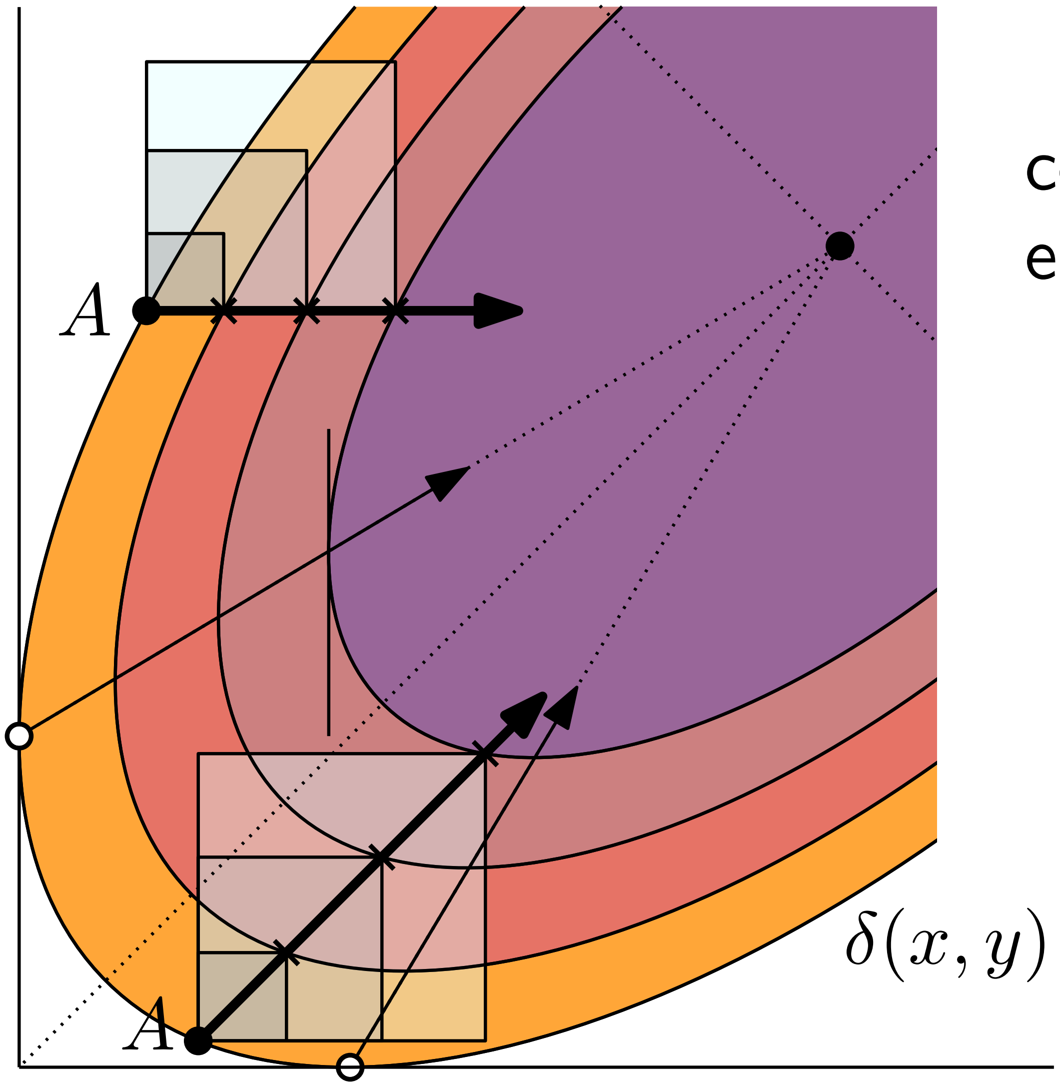
$\delta(x, y) =$ const

Steepest Descent

Inside one cell: $\delta(x,y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$

$$(-2 \le \lambda \le 2,\ c \ge 0)$$



concentric homothetic ellipses with 45° axes

$\delta(x,y) =$ const

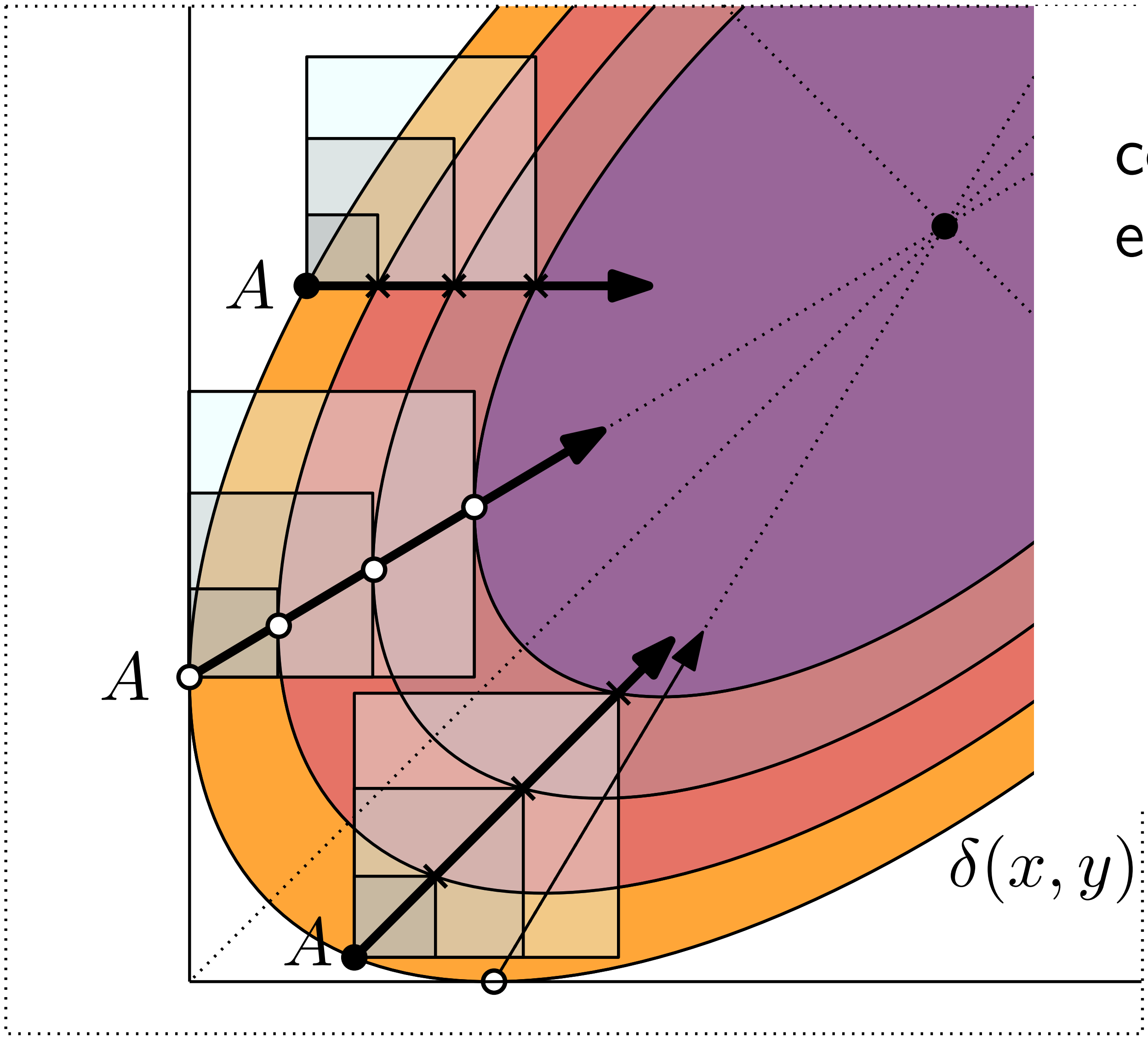
Steepest Descent

Inside one cell: $\delta(x,y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$

$$(-2 \le \lambda \le 2,\ c \ge 0)$$

concentric homothetic ellipses with 45° axes

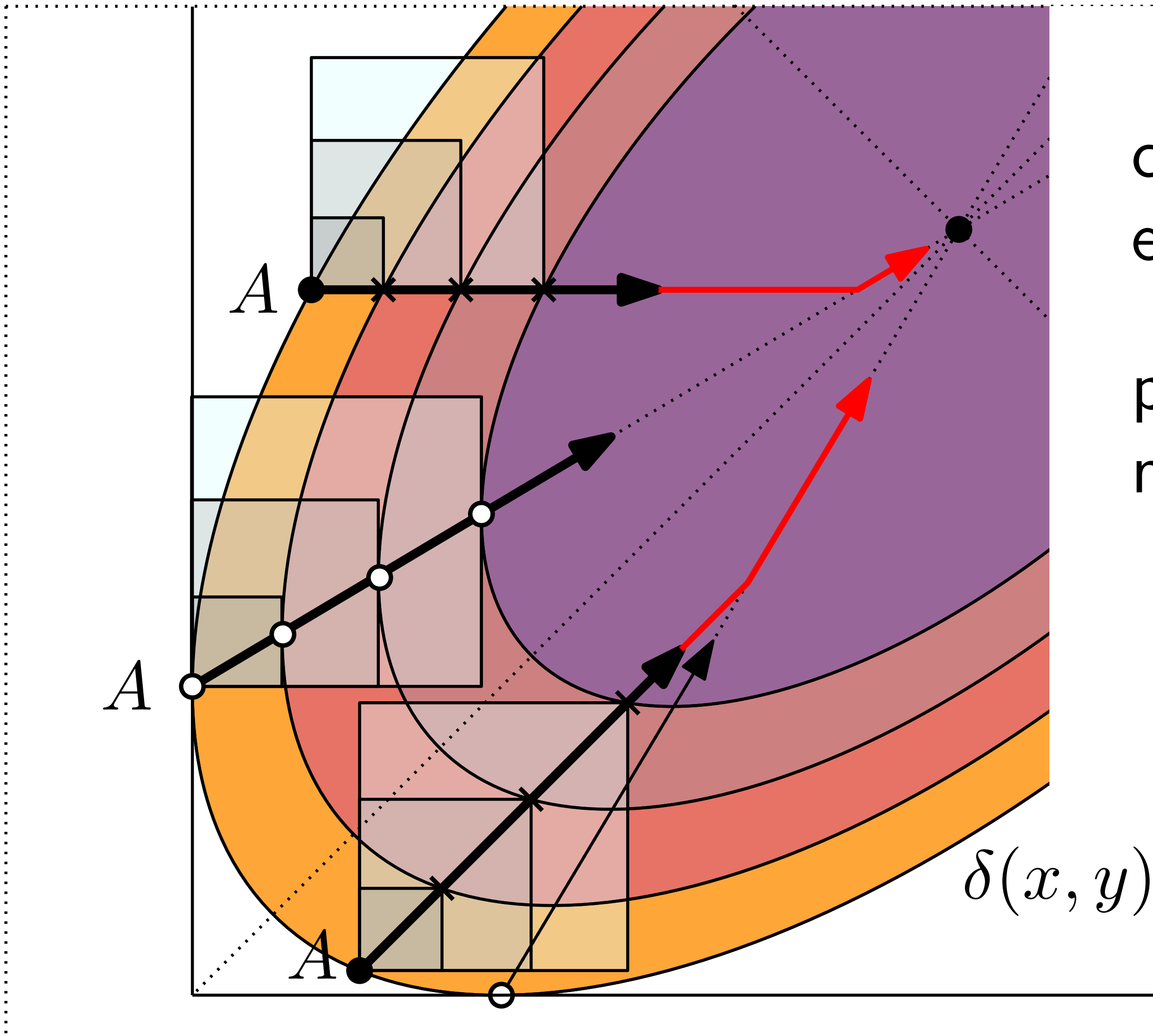


$\delta(x,y) = \text{const}$

Steepest Descent

Inside one cell: $\delta(x,y) := \|P(x) - Q(y)\|$

$$= \sqrt{(x-a)^2 + (y-b)^2 + \lambda(x-a)(y-b) + c}$$



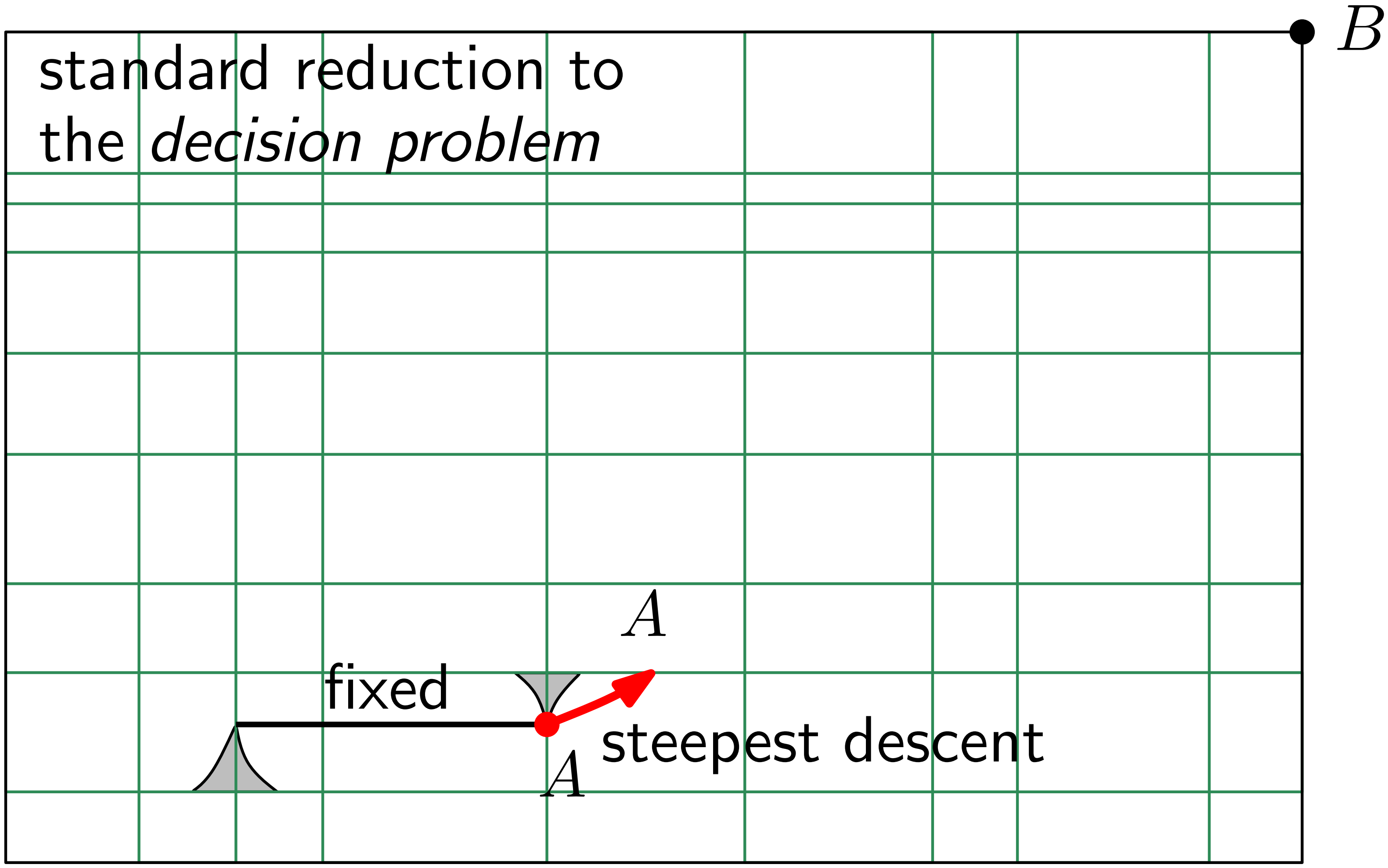
$(-2 \le \lambda \le 2,\ c \ge 0)$

concentric homothetic ellipses with 45° axes

piecewise linear with at most two pieces

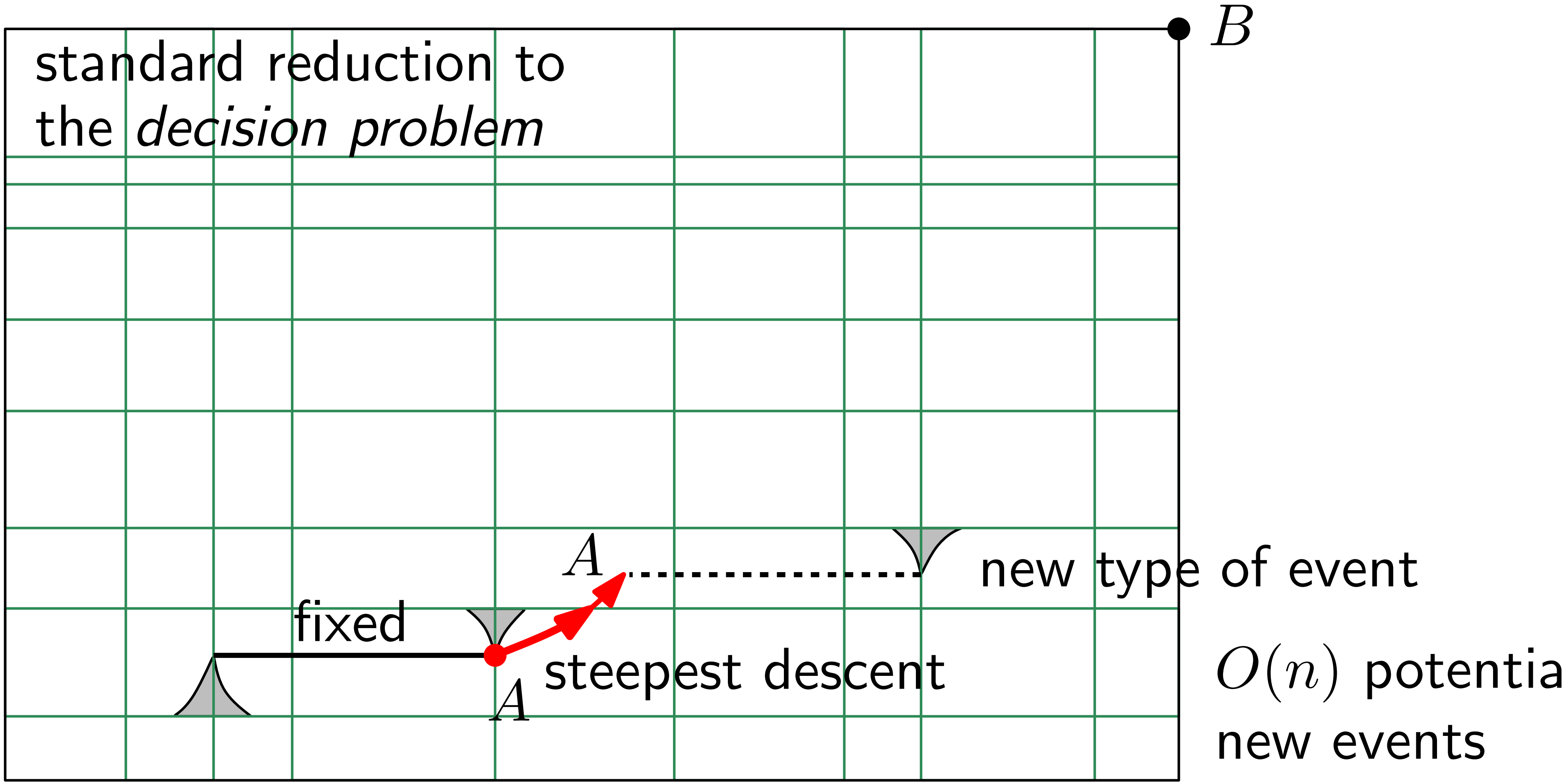
$\delta(x,y) = \text{const}$

Searching among Events



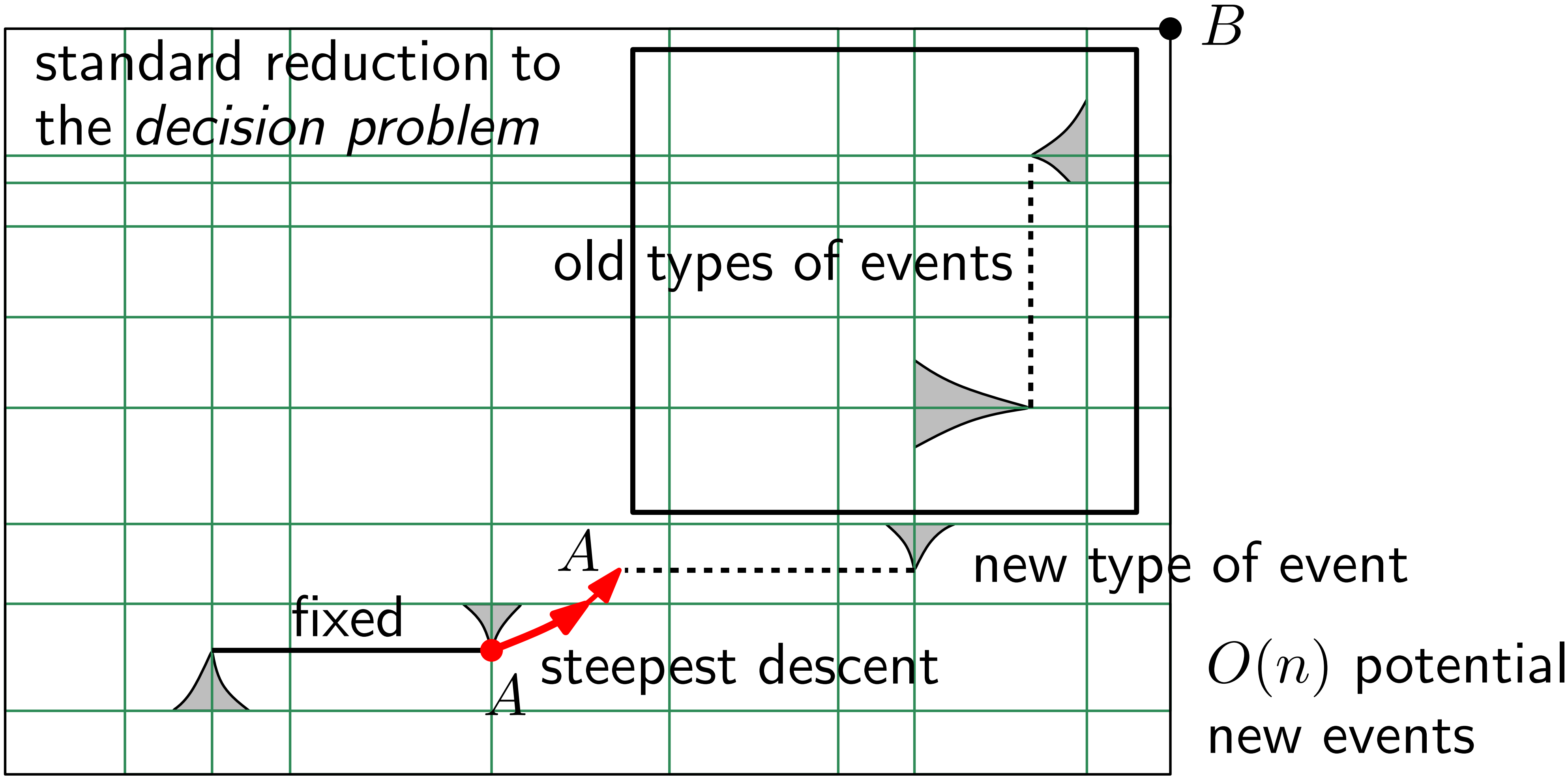
A follows steepest descent path: decrease height ε
while $\exists$ monotone path from A to B with height $\leq \varepsilon$

Searching among Events



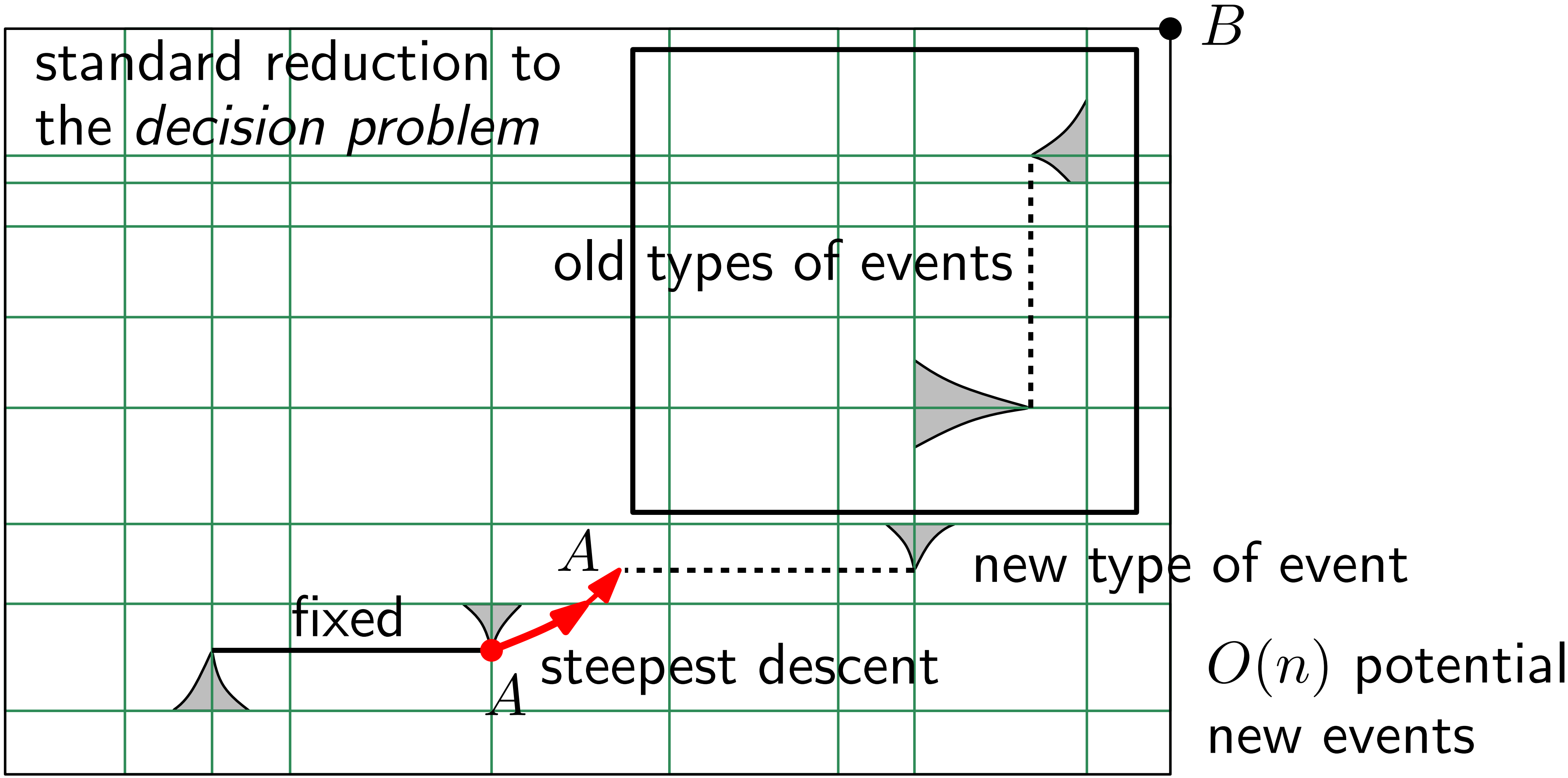
A follows steepest descent path: decrease height ε
while $\exists$ monotone path from A to B with height $\leq \varepsilon$

Searching among Events



A follows steepest descent path: decrease height ε
while $\exists$ monotone path from A to B with height $\le \varepsilon$

Searching among Events



While A is in one cell (i.e., $O(n)$ times):

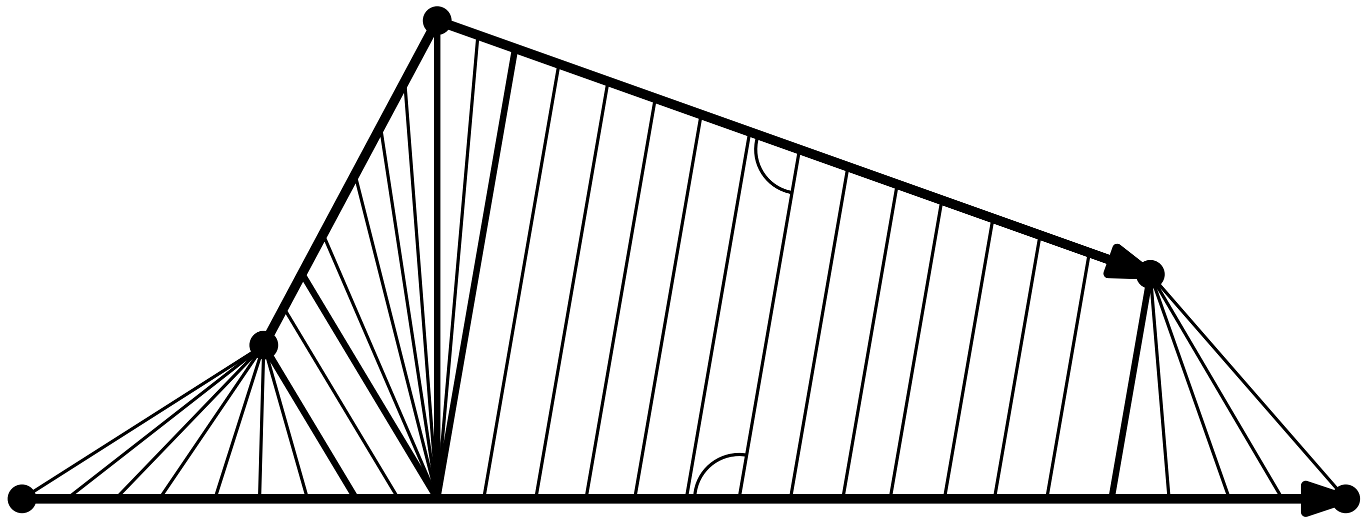
Search among *new* events: $O(\log n) \times$feasibility $= O(n^2 \log n)$

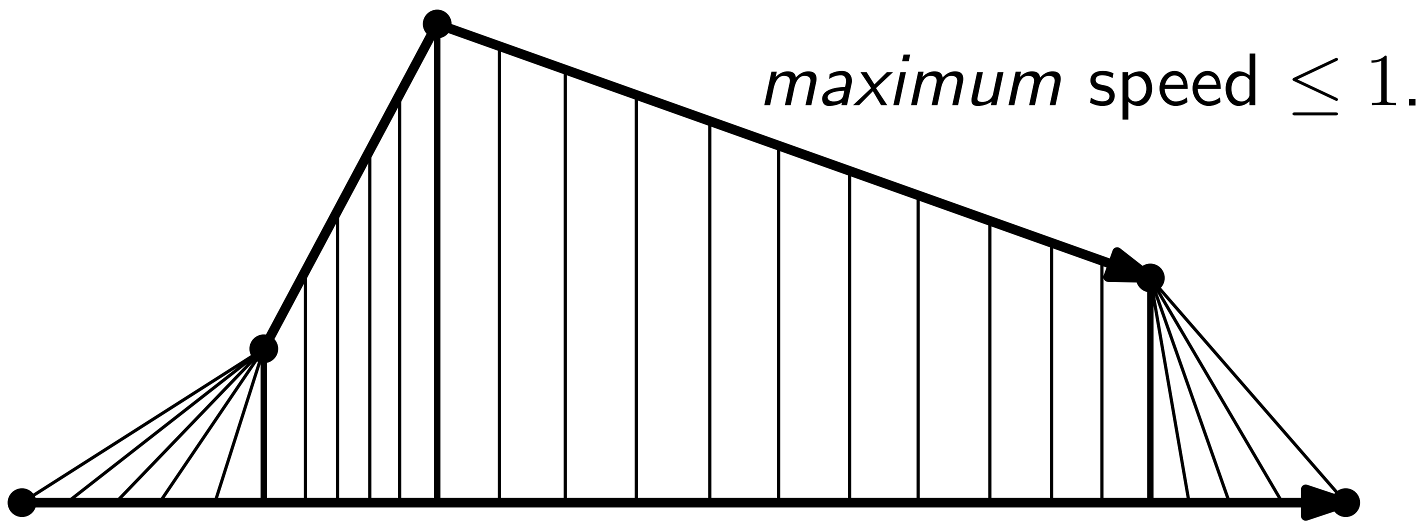
Search among *old* events: classical Fréchet $= O(n^2 \log n)$

$\rightarrow$ Overall time $= O(n^3 \log n)$

Other Normalizations

The *sum* of the speeds is ≤ 1. (L_1-norm)





An Unresolved Issue

Freie Universität Berlin

several critical passages