The algebraic conspiracy

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(joint work with Mikkel Abrahamsen)

1. Problem Statement and Motivation

We consider the problem of sandwiching a polytope Δ with a given number k of vertices between two nested polytopes $P \subset Q \subset \mathbf{R}^d$: Find Δ such that $P \subseteq \Delta \subseteq Q$. The polytope P is not necessarily full-dimensional.

Besides the problem of computing Δ, we study the following question: Assuming that the given polytopes P and Q are rational polytopes (they have rational vertex coordinates), does it suffice to look for Δ among the rational polytopes?

This problem has several applications: (1) When Q is a dilation of P (or an offset of P), Δ can serve as a thrifty approximation of P. (2) The polytope nesting problem can model the nonnegative rank of a matrix, and thereby the extension complexity of polytopes, as well as other problems in statistics and communication complexity. It was in this context that question (b) was first asked [3].

2. Nested Polygons in the Plane

In the plane ($d = 2$), it has been shown in 1989 by Aggarwal, Booth, ORourke, Suri & Yap [2] that Δ can be computed in $O(n \log k)$ time, assuming unit-cost arithmetic operations. This algorithm computes in fact the smallest possible k for which Δ exists, while for $d \geq 3$, minimizing k is NP-hard [4, 5].

The approach of [2] is as follows: Choose a starting point x_0 on the boundary of Q and wind a polygonal path $x_1 = f_1(x_0)$, $x_2 = f_2(x_1)$, $\ldots$, $x_k = f_k(x_{k-1})$, around P by putting a sequence of tangents to P and intersecting them with the boundary of Q, see Figure 1a. If $x_k \geq x_0$, then a k-gon Δ can be found. We

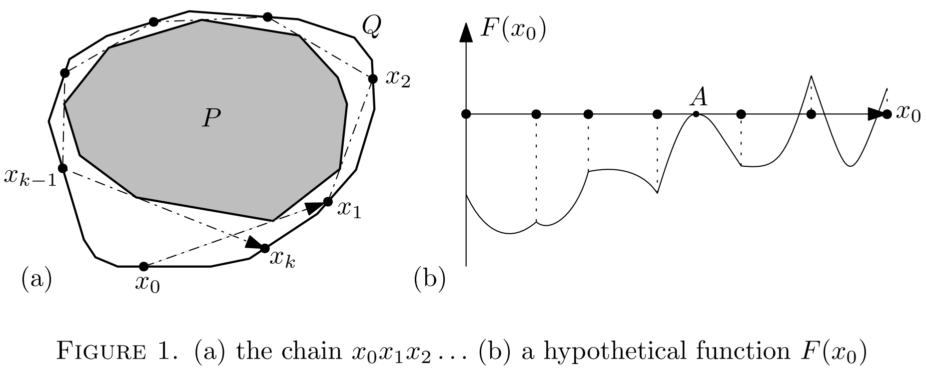

Figure 1. (a) the chain $x_0x_1x_2\ldots$ (b) a hypothetical function $F(x_0)$

parameterize the points x_0 by arc length along the boundary of Q from some fixed starting point. Now vary x_0 and follow the other points. As long as each point x_i moves on a fixed edge of Q and each segment $x_{i-1}x_i$ touches a fixed vertex of P, the function f_i is a rational linear function of the form $f_i(x) = \frac{ax+b}{cx+d}$. The composition of such functions is also of the same form. The function changes at the *breakpoints*, when an edge $x_{i-1}x_i$ of Δ lies flush with an edge P or a vertex x_i coincides with a vertex of Q. It follows that the function

$$F(x_0) := f_k(f_{k-1}(\cdots f_2(f_1(x_0))\cdots)) - x_0 \tag{1}$$

is piecewise rational, see Figure 1b. A solution of $F(x_0) \geq 0$ can be found by looking at the pieces and solving a quadratic equation for each piece.

Now, for some interval where the function f_i is smooth, the graph of the function is a hyperbola. It is easy to see that, for the range of the variable x_{i-1} that is of interest, the graph of $f_i(x_{i-1})$ lies on that branch of the hyperbola which is increasing and convex. The property of being increasing and convex is preserved under composition. Therefore, the function F in (1) is piecewise convex, unlike the function in Figure 1b. We obtain the following simplification of the algorithm.

Proposition 1. *To find the solutions of $F(x_0) \geq 0$, it is sufficient to look at the breakpoints of F.*

(For $k = 3$, this has been established before by Kubjas, Robeva, and Sturmfels [7], based on results from [8].) This implies in particular that the solution Δ can be found among the rational polygons. The existence of a rational solution has also been established in [9, Theorem 8] by observing that an isolated solution x_0 of $F(x_0) \geq 0$, like the point A in Figure 1b, would have to be rational for algebraic reasons, being a double zero of a quadratic equation. Our proof of Proposition 1 shows that such a situation cannot arise.

3. The Quest for an Irrational Solution in Higher Dimensions

A 3-dimensional example, in which the only polytope Δ with $k = 5$ vertices has irrational coordinates, has been constructed in [9], and it has been lifted to 4-dimensions (with a 3-dimensional polytope P) [10]. The case of a tetrahedron ($k = 4$) in 3 dimensions is open. It would also be interesting to have a 4-dimensional example where P is full-dimensional. (This corresponds to the *restricted* nonnegative rank [6].)

Figure 2 shows an attempt to construct a 3-dimensional instance which only has an irrational tetrahedron as a solution. Q has a horizontal bottom face Q_{bottom} and a horizontal top face Q_{top}. (The edges of Q are not fully shown.) P has six vertices and sits on Q_{bottom} with three vertices $P_1P_2P_3$. The tetrahedron Δ has an irrational vertex Δ_4 in the interior of Q_{top}. Figure 2b shows Q_{bottom} together with the projection $P_4'P_5'P_6'$ of the remaining vertices as seen from Δ_4, and it shows how the bottom face $\Delta_1\Delta_2\Delta_3$ of Δ is squeezed between $P_1P_2P_3 \cup P_4'P_5'P_6'$ and Q_{bottom}.

We have tried to construct such an example in reverse by building Q around Δ: After choosing a rational polytope $P = P_1P_2P_3P_4P_5P_6$ with $P_1P_2P_3$ on the horizontal plane of Q_{bottom}, we choose Δ_4 as an irrational point with coordinates in some quadratic extension field $\mathbb{Q}[\sqrt{r}]$. This leads to irrational projected points $P_4'P_5'P_6'$, and from this, the irrational points $\Delta_1\Delta_2\Delta_3$ can be constructed. Through each of these points, there is a unique rational line q_1, q_2, q_3, and these lines can be combined to form the boundary of Q_{bottom}. However, no matter how we try

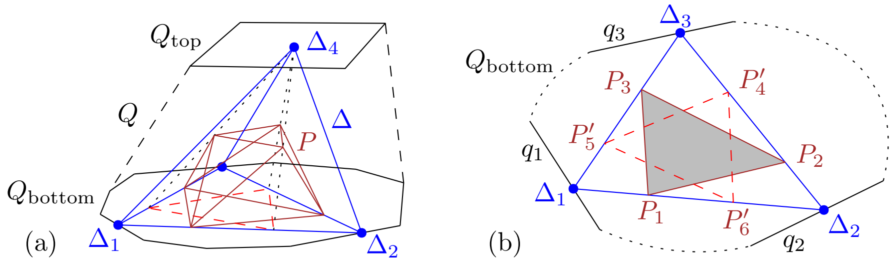

FIGURE 2. (a) $P \subset \Delta \subset Q$; (b) the situation on the bottom face Q_{bottom}

to choose the data, as if by some conspiracy, one of the lines q_1, q_2, q_3 always cuts into the triangle $\Delta_1\Delta_2\Delta_3$, making the completion of the construction impossible. Some experiments with dynamic geometry software suggest that this might be a systematic phenomenon: When we adjust the data so that one of the lines q_1, q_2, q_3 moves out of the triangle $\Delta_1\Delta_2\Delta_3$, another lines moves in precisely at the same time. If such an irrational example is indeed impossible, and examples of a different combinatorial type can also be excluded, it is conceivable that the solution for $k = 4$ is always rational if it exists. But this would so be for some deeper reason.

A similar "conspiracy" phenomenon has been observed in the construction of art gallery problems which require irrational guards [1]. The problem could be circumvented by modifying the construction and using more guards.

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