Partial Least-Squares Point Matching under Translations

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Abstract

We consider the problem of translating a given *pattern set* B of size m, and matching every point of B to some point of a larger *ground set* A of size n in an injective way, minimizing the sum of the squared distances between matched points. We show that when B can only be translated along a line, there can be at most $m(n-m)+1$ different matchings as B moves along the line, and hence the optimal translation can be found in polynomial time.

1 Introduction

In the partial pattern matching problem we are looking for an occurrence of some pattern B as part of a larger structure A. In this paper, we consider the case when A and B are finite points sets in the plane of size n and m respectively. (The results extend to higher dimensions, but for simplicity, we remain in the plane.)

Thus, we are looking for a subset $A' \subset A$ of size m that is as similar to B as possible. In this paper we measure similarity by the sum of the squared distances between corresponding points in some bijective mapping between B and A'. In other words, we insist that every point of B is matched with a distinct point of A.

In addition we allow B to be translated by some vector t. Thus, we are trying to solve the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\pi,t) := \sum_{i=1}^{m} \|(b_i+t)-a_{\pi(i)}\|^2 \qquad (1)\\ \text{subject to} \quad & \pi\colon B \to A, \text{ injective},\\ & t \in \mathbb{R}^2. \end{aligned}$$

Related Work. This is a rich area of research. See for example [8, 9] for the case of least-squares matching between two equal sets. See [1, 4, 5] for other distance measures.

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2 Basic Observations. The Partial Matching Subdivision

For a fixed assignment π, the objective function f can be rewritten in the form

$$\begin{aligned} f(\pi,t) &= \sum_{i=1}^{m} \|(b_i+t)-a_{\pi(i)}\|^2\\ &= \sum_{i=1}^{m} \|b_i - a_{\pi(i)}\|^2\\ &\qquad + \Big\langle t, \sum_{i=1}^{m}(b_i - a_{\pi(i)})\Big\rangle + m\|t\|^2\\ &= c_\pi + \langle t, d_\pi\rangle + m\|t\|^2, \qquad (2) \end{aligned}$$

for a constant $c_\pi \in \mathbb{R}$ and a vector $d_\pi \in \mathbb{R}^2$.

We can thus rewrite the objective function (1) as

$$\min_t F(t) + m\|t\|^2,$$

where

$$F(t) = \min_{\substack{\pi\colon B\to A\\ \pi \text{ injective}}} (c_\pi + \langle t, d_\pi\rangle)$$

For a given translation t, minimizing $f(\pi,t)$ over all π is equivalent to determining the minimum in the expression for $F(t)$, since the difference is the constant term $m\|t\|^2$. The function $F(t)$ is the minimum of a finite number of linear functions. The regions where the minimum is attained by a particular linear function is hence a convex polygonal region. We call the subdivions of the plane into these regions the *partial matching subdivision* $\mathcal{D}_{B,A}$:

Theorem 1 *This is a basic theorem.*

The space of parameters $t \in \mathbb{R}^2$ is subdivided into finitely many polygonal regions R_π, $\pi \in \Pi_0$. For all values t in one region R_π the same optimum assignment π optimizes (1) *(or the expression in $F(t)$).*

When B consists of a single point, the partial matching subdivision $\mathcal{D}_{B,A}$ is just the Voronoi diagram of A. When A is a large dense point set and B consists of few points that are relatively spread out the subdivision looks like an overlay of several translated copies of the Voronoi diagram of A, since each point of B is just independently matched to its nearest neighbor in A. At least, this is true as long as the points of B lie "within" the set A; when they move far away, several points of B will have the same closest point, and they have to compete for the point to which they are matched. Unfortunately, I could not produce interesting illustrations of partial matching subdivisions so far.

3 Exploring the Parameter Space

Inside each region R_π, the function $f(\pi, t)$ is a convex quadratic function of t, and hence it can be optimized easily. Thus, the straightforward approach to solving (1) is to search all regions R_π and compute the optimum in each region.

For a fixed vector t, the problem (1) is a minimum-cost bipartite matching problem and can be solved in polynomial time, for example by using network flow techniques. In this way, one can find the region R_π to which a parameter t belongs. By parametric linear programming techniques, one can then find the boundaries of this region, and one can also determine the adjacent regions across the boundary edges.

The running time of this approach is, up to a polynomial factor, determined by the number of regions that are to be explored. The crucial question is therefore, how many regions R_π there are.

We know a polynomial bound only for a very restricted case: namely when the translations t are restricted to a line only. In other words, we consider the intersection of the partial matching subdivision with a line.

Theorem 2 *This is the most important theorem.*

A line can intersect the interior of at most $1+m(n-m)$ *different regions of the partial matching subdivision* $\mathcal{D}_{B,A}$*, for* $|A| = n$ *and* $|B| = m$*.* □

For the special case $m = n$, this means that there is only one region, and we get the well-known fact that the least-squares assignment between two sets of equal size is independent of t [8], which is also obvious from the calculation leading to (2).

Proof. It even comes with a proof.

The problem of finding an optimal matching in (1) (for a fixed t) can be formulated as a network flow problem.

We are given an $m \times n$ *cost matrix* (c_{ij}) with $c_{ij} = \|(b_i + t) - a_j\|^2$

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} \quad & \textstyle\sum_{j=1}^{n} x_{ij} = 1, \text{ for } i = 1, \ldots, m \\ & \textstyle\sum_{i=1}^{m} x_{ij} \le 1, \text{ for } j = 1, \ldots, n \\ & 0 \le x_{ij} \le 1 \end{aligned}$$

By network flow theory, there is an optimal solution with $x_{ij} \in \{0, 1\}$, and it represents an assignment where each row i is assigned to exactly one column j and each column j is assigned to at most one row i. (The special case where $m = n$ is the usual assignment problem.) Among the n points of A, there will be m matched and $n - m$ unmatched vertices. We denote the set of matched vertices by $M(x)$.

Now, if we change t continuously, the solution (x_{ij}) will at some point change to a different solution $(\bar{x}_{ij})$. Some vertices will become matched and others will become unmatched.

Lemma 3 *And then we also found this lemma. Let* (x_{ij}) *and* $(\bar{x}_{ij})$ *be optimal solutions for parameter values* t *and* $\bar{t}$*, respectively. Then there is a one-to-one matching* σ *between the points in* $M(x) \setminus M(\bar{x})$ *and the points in* $M(\bar{x}) \setminus M(x)$ *such that*

$$\langle a_{\sigma(j)} - a_j, \bar{t} - t \rangle \ge 0,$$

for all $j \in M(x) \setminus M(\bar{x})$ *As a consequence, we have*

$$\Big\langle \sum_{j \in M(\bar{x})} a_j - \sum_{j \in M(x)} a_j,\ \bar{t} - t \Big\rangle \ge 0 \qquad (3)$$

Proof. The difference $\bar{x} - x$ between two assignments can be decomposed into an edge-disjoint union of (a) alternating even-length cycles and (b) alternating paths of even length starting at a matched vertex a_- of (x_{ij}) and ending at an unmatched vertex a_+ of (x_{ij}). For each such path of type (b), the vertex a_+ will be matched in the new assignment, and the vertex a_- will become unmatched.

Now let $a_- = a_0, b_1, a_1, b_2, \ldots, a_{k-1}, b_k, a_k = a_+$ be such an alternating path or cycle (for $a_- = a_+$).

The cost difference $\Delta c = c(\bar{x}) - c(x)$ between the old matching x and the new matching $\bar{x}$ can be expressed as follows. In order to simplify notation, we have first written the formulas without translation ($t = 0$).

$$\begin{aligned} \Delta c &= \sum_{i=1}^{k} \|b_i - a_i\|^2 - \sum_{i=1}^{k} \|b_i - a_{i-1}\|^2 \\ &= \sum_{i=1}^{k} (\|b_i\| - 2\langle b_i, a_i \rangle + \|a_i\|^2) \\ &\quad - \sum_{i=1}^{k} (\|b_i\| - 2\langle b_i, a_{i-1} \rangle + \|a_{i-1}\|^2) \\ &= \|a_+\|^2 - \|a_-\|^2 - 2\sum_{i=1}^{k} \langle b_i, a_i \rangle + 2\sum_{i=1}^{k} \langle b_i, a_{i-1} \rangle \\ &= \|a_+\|^2 - \|a_-\|^2 - 2\sum_{i=1}^{k} \langle b_i, a_i - a_{i-1} \rangle \end{aligned}$$

Now let us bring in the dependence on t and replace

b_i by $b_i + t$:

$$\begin{aligned}\Delta c(t) &= \|a_+\|^2 - \|a_-\|^2 - 2\sum_{i=1}^{k}\langle b_i + t, a_i - a_{i-1}\rangle \\ &= \|a_+\|^2 - \|a_-\|^2 - 2\sum_{i=1}^{k}\langle b_i, a_i - a_{i-1}\rangle \\ &\quad - 2\sum_{i=1}^{k}\langle t, a_i - a_{i-1}\rangle \\ &= \|a_+\|^2 - \|a_-\|^2 - 2\sum_{i=1}^{k}\langle b_i, a_i - a_{i-1}\rangle \\ &\quad - 2\langle t, a_+ - a_-\rangle\end{aligned}$$

The only term that depends on t is the last term $-2\langle t, a_+ - a_-\rangle$. Now if x is optimal at t, then $\Delta c(t)$ must be nonnegative; otherwise we could use the alternating path or cycle to obtain a better solution. Similarly, since $\bar{x}$ is optimal at $\bar{t}$, we must have $\Delta c(\bar{t}) \leq 0$. Thus we get $\Delta c(t) - \Delta c(\bar{t}) \geq 0$, or

$$\langle \bar{t} - t,\, a_+ - a_-\rangle \geq 0$$

If we add this relation for all alternating paths and cycles the form the difference $\bar{x} - x$, we obtain (3). (The alternating cycles give no contribution.) □

Now we can conclude the proof of the theorem. Let us vary t along a line in direction s. Lemma 3 tells us that, whenever the assignment changes, a matched point a_- can only be replaced by a new matched point a_+ with $\langle \bar{t} - t,\, a_+ - a_-\rangle > 0$, or in other words, $\langle a_+, s\rangle > \langle a_-, s\rangle$. If we sort the points a by $\langle a, s\rangle$, and classify the subsets $M(x)$ of matched points of A by the sum of the ranks in this order, this means that the sum of the ranks can only go up. The minimum sum of ranks is $\sum_{i=1}^{m} i = m(m+1)/2$, and the maximum sum of ranks is $\sum_{i=1}^{m}(n+1-i) = (n+1)m - m(m+1)/2$. Between these two extreme values, there can be only $(n-m)m$ changes. □

An example showing that the bound is tight can be easily constructed in one dimension already: the set A consists of n uniformly spaced points, and B consists of m points very close together (much closer than the spacing between the points of A).

4 Conclusion

What we did is amazing and improves everything that was there before.

Still, the most important question is open: is the complexity of the partial matching subdivision $\mathcal{D}_{B,A}$ bounded by a polynomial? It is possible that a bound can already be derived from Theorem 2.

Another question arises if we allow rotations. Even if A and B have the same size and we consider only the one-parameter family of rotations about a fixed point, there can be many different optimal assignments, No polynomial bound is known. This problem can be formulated as a special parametric assignment problem where the costs depend linearly on a parameter x. For this more general problem, a superpolynomial lower bound of the form $2^{\sqrt{n}}$ on the number of optimal assignments has been proved by Patricia Carstensen [3, 2], based on a construction of Zadeh [7].

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