How Difficult is it to Walk the Dog?

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Abstract

We study the complexity of computing the Fréchet distance (also called dog-leash distance) between two polygonal curves with a total number of n vertices. For two polygonal curves in the plane we prove an $\Omega(n \log n)$ lower bound for the decision problem in the algebraic computation tree model allowing arithmetic operations and tests. Up to now only a $O(n^2)$ upper bound for the decision problem was known.

The $\Omega(n \log n)$ lower bound extends to variants of the Fréchet distance such as the weak as well as the discrete Fréchet distance. For the one-dimensional case we give a linear-time algorithm to solve the decision problem for the weak Fréchet distance between one-dimensional polygonal curves.

1 Introduction

The Fréchet distance is a metric for comparing parameterized shapes. In this paper we consider the Fréchet distance between polygonal curves. We also study variants of the Fréchet distance, namely the weak Fréchet distance, the discrete Fréchet distance, and the weak discrete Fréchet distance. There is a quadratic upper bound for solving the decision problem for the Fréchet distance of polygonal curves [2] and its variants, but so far no non-trivial lower bound was known.

In this paper we prove the following lower bound:

Theorem 1 *Determining whether or not the Fréchet distance between two polygonal curves in the plane of total complexity (i.e., number of vertices) n is less than a value ε takes $\Omega(n \log n)$ time in the algebraic computation tree model allowing arithmetic operations $(+, -, \times, /)$ and tests $(>, \geq, =)$.*

The same holds for the weak Fréchet distance with and without the restriction that endpoints are mapped to endpoints, the discrete Fréchet distance, and the weak, discrete Fréchet distance with and without endpoint restriction.

We prove Theorem 1 by reducing a problem with an $\Omega(n \log n)$ lower bound in linear time to the decision problem for the Fréchet distance. The problem we reduce from is *set inclusion* for which the lower bound in the above model has been proved by Ben-Or [3].

The lower bound in the theorem holds for polygonal curves in the plane. For the one-dimensional case we show that the lower bound for the weak Fréchet distance between one-dimensional polygonal curves does not hold. We give a linear-time algorithm for this case.

Note that the definition of the *weak Fréchet distance* does not require endpoints to be mapped to endpoints as does the definition of the *non-monotone Fréchet distance* in [2] which coincides with the *weak Fréchet distance with endpoint restriction*.

Theorem 2 *The weak Fréchet distance of one-dimensional polygonal curves can be computed in linear time.*

For the weak Fréchet distance with endpoint restriction Theorem 2 holds if the polygonal curves lie between their endpoints. It remains open whether the lower bound holds if the endpoints lie inside and whether the lower bound holds for the Fréchet distance of one-dimensional curves.

2 Fréchet Distance

In this section we recall the definitions of the Fréchet distance and its variants. For two parameterized curves $f_1, f_2 : [0,1] \to \mathbb{R}^d$ their Fréchet distance is defined as

$$\inf_{\substack{\alpha:[0,1]\to[0,1] \\ \beta:[0,1]\to[0,1]}} \max_{t\in[0,1]} |f_1(\alpha(t)) - f_2(\beta(t))|$$

where $|\cdot|$ denotes the Euclidean metric in $\mathbb{R}^d$ and the reparametrizations α, β range over all orientation-preserving homeomorphisms.

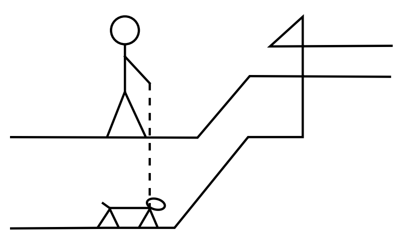

Figure 1: Fréchet Distance: length of shortest leash.

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The Fréchet distance can be illustrated by a man and a dog walking on the two curves as in Figure 1. The man has the dog on a leash. Both may choose their speed and may stop but not walk backwards. Then the Fréchet distance corresponds to the length of the shortest leash that allows them to walk on their respective curves from beginning to end. The Fréchet distance is therefore also called the *dog-leash distance*.

We focus on polygonal curves. Since the Fréchet distance is invariant under reparametrization we can assume a polygonal curve P to be given by the ordered list of its vertices, i.e., $P = (p_1, \ldots, p_l)$.

The *weak Fréchet distance* of polygonal curves is defined in the same way except that the reparametrizations α, β range over all surjective continuous functions. For the *weak Fréchet distance with endpoint restriction* the reparametrizations are further required to map 0 to 0 and 1 to 1, respectively. In the man-dog illustration the weak Fréchet distance with endpoint restriction allows the man and dog to walk also backwards. In the weak Fréchet distance they may also choose their starting and ending point, but must cover both curves.

Since the weak versions of the Fréchet distance are defined as the Fréchet distance but with less constraints, the weak Fréchet distance with endpoint restriction is less or equal to the Fréchet distance, and the weak Fréchet distance is less or equal to the weak Fréchet distance with endpoint restriction.

The *discrete Fréchet distance* is defined using discrete maps on the vertices instead of homeomorphisms on the parameter spaces. Let $P = (p_1, \ldots, p_l)$ and $Q = (q_1, \ldots, q_m)$ be two polygonal curves given by their ordered lists of vertices. A *coupling* of the vertices is an ordered sequence of pairs of vertices in P, Q, i.e., $C = (c_1, \ldots, c_k)$ with

$$c_r = (a, b),\ a \in P,\ b \in Q \quad \text{for } 1 \leq r \leq k,$$

fulfilling $(0, 0)$, $(l, m) \in C$ and for $1 \leq r < k$

$$\begin{aligned} &c_r = (a_i, b_j) \ \Rightarrow \\ &c_{r+1} \in \big\{(a_i + 1, b_j),\ (a_i, b_j + 1),\ (a_i + 1, b_j + 1)\big\}. \end{aligned}$$

The discrete Fréchet distance between polygonal curves is defined by taking the minimum over all couplings and the maximum over all distances between coupled vertices, i.e.,

$$\min_{C \text{ coupling}} \ \max_{(a_i, b_j) \in C} \ |a_i - b_j|.$$

A coupling of the vertices can be extended to a limit of homeomorphisms on the parameter spaces of the curves. This implies that the Fréchet distance is less than or equal to the discrete Fréchet distance. Furthermore, for any homeomorphism there exists a coupling which yields a distance that is not more than the distance of the homeomorphism plus half the length of the longest edge of either curve. Thus, if we add vertices to the curves P, Q so that their edge lengths tend to zero, their discrete Fréchet distance will tend to the Fréchet distance. The weak versions of the discrete Fréchet distance are defined analogously to the continuous case, i.e., a coupling can also make backward steps in the sense that from a_i it can step to $a_i - 1, a_i$, or $a_i + 1$.

3 Lower Bound

We reduce the problem of set inclusion to the decision problem for the Fréchet Distance.

Fréchet Distance Given two polygonal curves in $\mathbb{R}^d$ with vertices $P = (p_1, \ldots, p_l)$, $Q = (q_1, \ldots, q_m)$, $l + m \leq n$ and $\varepsilon > 0$, determine whether or not the Fréchet distance of the curves is less than ε.

Set Inclusion Given two sets $A = a_1, \ldots, a_n \subset \mathbb{R}$, $B = b_1, \ldots, b_n \subset \mathbb{R}$, determine whether or not $A \subseteq B$.

In terms of distance measures the problem of set inclusion corresponds to deciding whether the directed Hausdorff distance between the point sets is 0, i.e., deciding whether

$$\max_{a \in A} \min_{b \in B} |a - b| = 0.$$

Given sets A and B for which we want to determine whether or not $A \subseteq B$, we first scale A and B such that $A \cup B \subset [0, 1]$ holds. This can be done in linear time. In the following we assume $A \cup B \subset [0, 1]$. For $a_i \in A$ we define

$$p_i := \big(2a_i/(1 + a_i^2),\ (1 - a_i^2)/(1 + a_i^2)\big) \in \mathbb{R}^2$$

and for $b_i \in B$ we define

$$q_i := \big(-2b_i/(1 + b_i^2),\ -(1 - b_i^2)/(1 + b_i^2)\big) \in \mathbb{R}^2.$$

The coordinates of all p_i and q_i can be determined in linear time in total. We define $p_0 := (1, 1)$ and $q_0 := (0, 0)$. Let C_A be the polygonal curve with vertices $(p_0, p_1, p_0, p_2, \ldots, p_0, p_n, p_0)$ and C_B be the polygonal curve with vertices $(q_0, q_1, q_2, \ldots, q_n, q_0)$. The construction is illustrated in Figure 2.

Theorem 1 directly follows from the following lemma:

Lemma 3 *Let the curves C_A, C_B be constructed as above from two finite sets $A, B \subset \mathbb{R}$. Then $A \not\subseteq B$ holds if and only if the Fréchet distance between C_A and C_B is less than 2.*

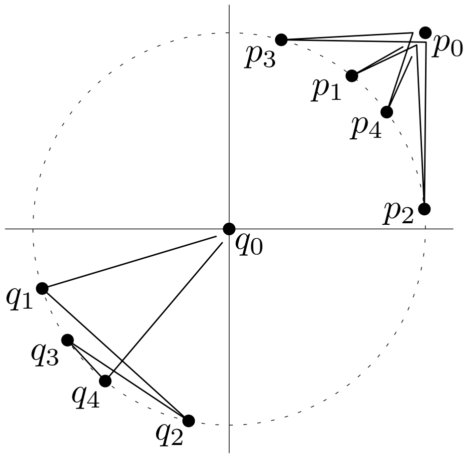

Figure 2: Polygonal curves C_A and C_B. The curves go through p_0 and q_0 but are drawn slightly perturbed for illustration purposes.

The same holds for the discrete Fréchet distance, the weak Fréchet distance with and without endpoint restriction, and the weak variants of the discrete Fréchet distance.

Proof. We prove the lemma first for the Fréchet distance, and then generalize it to the weak and discrete variants of the Fréchet distance. An important property of our construction is that the Euclidean distance between p_i and q_j equals 2 if and only if $a_i = b_j$, otherwise it is strictly less than 2.

Now assume $A \not\subseteq B$, i.e., there is an $a_k \notin \{b_1, \ldots, b_n\}$. Consider the following parametrizations of C_A and C_B: First traverse C_A until p_k is reached. So far the distance between pairs of points on the two curves is clearly less than 2 (actually at most $\sqrt{2}$ since on C_B we stay in q_0). Then C_B is traversed completely. Since no b_i equals a_k, all pairwise distances are less than 2. Now the rest of C_A is traversed but since on C_B we are again in q_0 the distance stays less than 2. In total these parametrizations yield a distance less than 2, therefore the Fréchet distance is less than 2.

For the other direction assume the Fréchet distance between the two curves is less than 2. Then there are parametrizations yielding a distance less than 2. Consider such parametrizations. At the point when the parametrization of C_B reaches q_1, the parametrization of C_A must be in the *neighborhood* of some p_k. The *neighborhood* of p_k is the subcurve of C_A with the vertices p_0, p_k, p_0 excluding the two endpoints p_0. Now, until the parametrization of C_B reaches q_n, the parametrization of C_A cannot leave the neighborhood of p_k because the closest possible point on C_B to p_0 is the point $(-1/2, -1/2)$, which still has distance $3/2 \cdot \sqrt{2} > 2$ to p_0.

It follows that all points in the neighborhood of p_k have distance less than 2 to $q_1, \ldots, q_n$. Since p_k is the closest point in its neighborhood to all of the $q_i, 1 \leq i \leq n$, the distance from p_k to all of them is less than 2. From this we get that $a_k \neq b_i$ for all $1 \leq i \leq n$, thus $A \not\subseteq B$.

This proves the lemma for the Fréchet distance. Since we did not use the monotonicity of the parametrizations the proof directly transfers to the weak Fréchet distance with and without endpoint restriction.

For the discrete Fréchet distance, consider again the two directions of the proof. For $A \not\subseteq B$ we constructed parametrizations realizing a Fréchet distance less than two. But these parametrizations also give a discrete Fréchet distance less than two since they always map vertices to vertices. For the other direction, we need to show that a discrete Fréchet distance less than two implies that $A \not\subseteq B$. This is equivalent to showing that $A \subseteq B$ implies a discrete Fréchet distance greater than or equal to two. This follows from the fact that the discrete Fréchet distance is always greater than or equal to the Fréchet distance. Combining the arguments for the weak Fréchet distance and the discrete Fréchet distance yields the result for the weak discrete Fréchet distance with and without endpoint restriction. ☐

4 Curves on a Line

In the previous section we showed an $\Omega(n \log n)$ lower bound for the decision problem for various variants of the Fréchet distance between polygonal curves in 2D. A natural question is whether these bounds still hold in 1D, i.e., in the case that the curves are restricted to lie on a line.

For the weak Fréchet distance we show that the lower bound does not hold. We show instead (Proposition 5) that the weak Fréchet distance can be computed in linear time by simply considering the differences of the extremal vertices. If the extremal vertices are the endpoints of the curves then this equals also the weak Fréchet distance with endpoint restriction.

Thus, the distance between polygonal curves is simpler to compute if we weaken the constraints of the Fréchet distance and restrict the dimension of the curves. Interestingly, there are similar results for the Fréchet distance between surfaces. While computing the Fréchet distance between simplicial surfaces is NP-hard [5], the weak Fréchet distance between simplicial surfaces in 3D can be computed in polynomial time [1]. If the surfaces are restricted to lie in a plane and to not self-intersect, i.e., to be simple polygons, then even the Fréchet distance can be computed in polynomial time [4].

The weak Fréchet distance between curves in 1D is closely related to the *Mountain Climbing Problem.*

The Mountain Climbing Problem
Two climbers start at sea-level on opposite sides of a mountain range and want to meet at the highest peak without resting on the way. Can they travel in a way that they stay on equal altitude at all times?

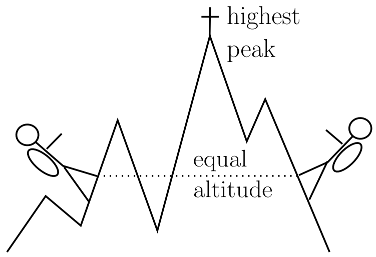

Figure 3: Mountain Climbing.

This problem is illustrated in Figure 3. It has been answered many times (see [8]). Homma [6] proved in 1952 that the climbers can stay at equal altitude if the mountains are locally non-constant. He also gave an example where it is not possible for the climbers, where one mountain range has a plateau while the other mountain range oscillates infinitely often.

In mathematical terms the problem asks for a characterization of the continuous functions $f_1, f_2 : [0,1] \to [0,1]$ with $0 = f_1(0) = f_2(0)$ and $1 = f_1(1) = f_2(1)$ for which there are continuous functions $g_1, g_2 : [0,1] \to [0,1]$ with $0 = g_1(0) = g_2(0)$ and $1 = g_1(1) = g_2(1)$ and

$$f_1 \circ g_1 = f_2 \circ g_2.$$

Weak Fréchet Distance in 1D The mountain climbing problem can be interpreted in terms of the weak Fréchet distance: Are there reparametrizations of f_1 and f_2 mapping endpoints to endpoints that realize a distance of 0?

The answer above directly yields that the weak Fréchet distance with endpoint restriction is 0 for curves f_1 and f_2 which take values in $[0,1]$, are locally non-constant, and satisfy $0 = f_1(0) = f_2(0)$ and $1 = f_1(1) = f_2(1)$. For curves $f_i,\ i = 1,2$, which take values in $[a_i, b_i]$, are locally non-constant, and satisfy $a_i = f_i(0)$ and $b_i = f_i(1),\ i = 1,2$, this implies that the weak Fréchet distance with endpoint restriction of f_1 and f_2 is $\max(|a_2 - a_1|, |b_2 - b_1|)$.

A characterization of the functions f_1, f_2 [7] for which such reparametrizations exist implies that the weak Fréchet distance between continuous functions with the same image is 0 even in the general case, i.e., where f_1, f_2 may be locally constant. In this case the "climbers" can maintain almost the same altitude.

Corollary 4 (Huneke [7]) *For any two continuous, surjective functions $f_1, f_2 : [0,1] \to [0,1]$ and for any $\varepsilon > 0$, there exist continuous, surjective functions $g_1, g_2 : [0,1] \to [0,1]$ such that for all $x \in [0,1]$*

$$|f_1 \circ g_1(x) - f_2 \circ g_2(x)| < \varepsilon.$$

Note that f_1 and f_2 no longer need to start at 0 and end at 1. For the weak Fréchet distance this implies:

Proposition 5 *Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be continuous functions with $f_i([0,1]) = [a_i, b_i]$ for $i = 1,2$. The weak Fréchet distance between f_1 and f_2 is*

$$\max(|a_2 - a_1|, |b_2 - b_1|).$$

Theorem 2 directly follows from Proposition 5. If a_1, a_2, b_1, and b_2 are known, the weak Fréchet distance can even be computed in constant time.

5 Discussion

We presented an $\Omega(n \log n)$ lower bound for the decision problem for the Fréchet distance between polygonal curves in the plane. An open problem is to close the gap to the known quadratic upper bound. Furthermore, it is open whether the lower bound holds for underlying metrics other than the Euclidean metric.

We showed that the lower bound does not hold for the weak Fréchet distance between curves on a line. It remains to investigate the complexity of the Fréchet distance for curves on a line for which we only know the quadratic upper and trivial linear lower bound.

Acknowledgment

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