Convex Equipartitions of Colored Point Sets

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Abstract

We show that any d-colored set of points in general position in $\mathbb{R}^d$ can be partitioned into n subsets with disjoint convex hulls such that the set of points and all color classes are partitioned as evenly as possible. This extends results by Holmsen, Kynčl & Valculescu (2017) and establishes a central case of their general conjecture. Our proof utilizes a result of Soberón (2012) on simultaneous equipartitions of d continuous measures in $\mathbb{R}^d$ by n convex regions, which gives a convex partition of $\mathbb{R}^d$ with the desired properties, except that points may lie on the boundaries of the regions. In order to resolve the ambiguous assignment of these points, we set up a network flow problem. The equipartition of the continuous measures gives a fractional flow. The existence of an integer flow then yields the desired partition of the point set.

1 Introduction

A (finite) set X of points in $\mathbb{R}^d$ is in *general position* if every subset of size at most $d+1$ is affinely independent. A partition $X = X_1 \sqcup \cdots \sqcup X_m$ of X into m disjoint subsets is an *m-coloring* of X. The sets $X_1, \dots, X_m$ are called *color classes* and we say that the set X is *m-colored*. A subset $Y \subseteq X$ containing points from at least j distinct color classes is said to be *j-colorful*.

In this language, the classical partition result of Akiyama and Alon reads as follows.

Theorem 1 (Akiyama–Alon [2]). *Let n, d be positive integers, and let X be a d-colored set of points in general position in $\mathbb{R}^d$, with each color class containing n points. Then there is a partition of X into n d-colorful sets of size d whose convex hulls are pairwise disjoint.*

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Akiyama and Alon gave a beautifully simple proof using a discrete version of the ham sandwich theorem, which is a well known consequence of the Borsuk–Ulam theorem. The use of such topological methods created a lot of progress in solving discrete partitioning problems. In fact, many related partition results have both a continuous mass partition as well as a discrete colored version—often equivalent.

In this paper, we consider the following conjecture of Holmsen, Kynčl and Valculescu [5, Conjecture 3].

Conjecture 2 (Holmsen–Kynčl–Valculescu, 2016)**.** *Let m, k, n, and d be positive integers, and let X be an m-colored set of kn points in general position in $\mathbb{R}^d$. Suppose there is a partition of X into n d-colorful sets of size k. Then there is also such a partition with the additional property that the convex hulls of the n sets are pairwise disjoint.*

Here, the assumption involves no geometry, and it depends only on the number of color classes and their sizes. It is obviously a necessary condition. In particular, it implies that $m \geq d$ and $k \geq d$.

Theorem 1 answers the case when $k = m = d$. The case $m \geq k = d = 2$ was settled by Aichholzer et al. [1] and by Kano, Suzuki and Uno [8]. Further developments on the planar case were made independently by Bespamyatnikh, Kirkpatrick and Snoeyink [3], Ito, Uehara and Yokoyama [6] as well as Sakai [10], who confirmed the conjecture for two colors ($m = d = 2$) where the sizes of the color classes are divisible by n. Holmsen, Kynčl and Valculescu resolved the conjecture for the remaining cases in the plane, as well as for the case when $k - 1 = m = d \geq 2$, the latter by giving a particular discretization of the ham sandwich theorem [5]. Their method is similar to the one used by Kano and Kynčl in [7] to establish the case $m - 1 = d = k$, who proved a generalization of the ham sandwich theorem for $d + 1$ measures in $\mathbb{R}^d$ and called it the hamburger theorem.

Holmsen et al. emphasized the connection of the conjecture with a continuous analogue for the case $m = d$, proved in the plane by Sakai [10] and extended to arbitrary Euclidean space by Soberón [12]. (A more general version, for functions that are not necessarily measures, was obtained soon after by Karasev, Hubard and Aronov [9] and by Blagojević and Ziegler [4].)

Theorem 3 (Soberón [12])**.** *Let n, d be positive integers, and let $\mu_1, \dots, \mu_d$ be absolutely continuous finite measures on $\mathbb{R}^d$ with respect to the Lebesgue measure. Then there exists a partition of $\mathbb{R}^d$ into n convex regions $C_1, \dots, C_n$ that simultaneously equipartitions all d measures, that is,*

$$\mu_i(C_j) = \frac{1}{n}\, \mu_i(\mathbb{R}^d)$$

for all $i \in \{1, \dots, d\}$ and all $j \in \{1, \dots, n\}$.

Holmsen, Kynčl and Valculescu state:

> "However, going from the continuous version to the discrete version seems to require, in many cases, a non-trivial approximation argument, and we do not see how the continuous results [...] could be used to settle our Conjecture 3 for the case $m = d$."

Indeed, this is not straightforward. However, in this paper we show how this can be done: We confirm Conjecture 2 when $m = d$, as a direct corollary of the following main result. For this we say that a partition of a finite set A into n parts is an *equipartition* if each of the parts contains $\lceil \frac{|A|}{n} \rceil$ or $\lfloor \frac{|A|}{n} \rfloor$ elements of A.

Theorem 4. *Let n, d be positive integers, and let X be a d-colored set of points in general position in $\mathbb{R}^d$. Then there exists an equipartition of X into n subsets which simultaneously equipartition each of the color classes and whose convex hulls are pairwise disjoint.*

To see that Theorem 4 implies Conjecture 2 for the case $m = d$, observe that in this case the condition on X of admitting a partition into n pairwise disjoint d-colorful sets of size k implies that each color class has at least n elements. In an equipartition of a color class X_i, each part contains at least $\lfloor \frac{|X_i|}{n} \rfloor \geq 1$ points. Thus, each part of X contains all d colors. With $|X| = kn$ and an equipartition of X, we get n sets of size k that each contain at least one point of each of the d colors.

2 Preliminaries

In order to discretize Theorem 3, we start by employing a classical idea (see e.g. [2, Proof of Lemma 2]): We replace the points in X with small enough balls and then define measures on these. The problem with applying the continuous result is that the boundaries of the regions may cut through some balls, see Figure 1 (left). We will assign every such "ambiguous" point to one of the regions intersected by the ball centered at the point.

The following lemma shows that, if the radius ε of the balls is small enough, we will always get a partition of X with disjoint convex hulls, no matter how we resolve the ambiguities. In Section 3 we will prove that we can resolve these ambiguities in such a way that we get an equipartion of the full point set X as well as of each of the color classes X_i.

By general position, no ℓ-flat (affine subspace of dimension ℓ) with $\ell < d$ contains more than $\ell + 1$ points of X. When we replace the points by balls, we make their radius $\varepsilon > 0$ small enough so that no ℓ-flat with $\ell < d$ intersects more than $\ell + 1$ of these *balls*.

Lemma 5. *Let $P \subseteq \mathbb{R}^d$ be a finite set of points in general position, and let $\varepsilon > 0$ be chosen such that no ℓ-flat with $\ell < d$ intersects more than $\ell + 1$ balls $B_\varepsilon(x)$ of radius ε centered at points from P. Suppose we are given an affine hyperplane $H \subseteq \mathbb{R}^d$, and a partition of $P = P^+ \sqcup P^-$ satisfying*

$$P^+ \subseteq \{x \in P : B_\varepsilon(x) \cap H^+ \neq \emptyset\} \quad \textit{and} \quad P^- \subseteq \{x \in P : B_\varepsilon(x) \cap H^- \neq \emptyset\},$$

where H^+ and H^- are the open half-spaces determined by H. Then

$$\operatorname{conv} P^+ \cap \operatorname{conv} P^- = \emptyset.$$

Proof. For any ball $B_\varepsilon(x)$ intersecting a half-space, we know that the ball intersects the hyperplane H or its center x lies in the open half-space. The hyperplane H intersects at most d balls centered at points of P, and every ℓ-flat with $\ell < d$ intersects no more than $\ell + 1$ balls. Therefore, as all of the balls are sufficiently small, by a small movement of H, we can ensure that the centers of these balls are each on the appropriate side of H without intersecting any further balls. Thus, we get a hyperplane strictly separating the sets P^+ and P^-. Consequently, $\operatorname{conv} P^+ \cap \operatorname{conv} P^- = \emptyset$. □

In order to assign boundary points to regions we will set up a flow network with a fractional flow; from this we obtain an integer flow, which in turn will determine the assignment.

In a directed graph $D = (V, A)$ with a set of vertices V and a set of arcs A, a *flow* is a function $f: A \to \mathbb{R}$ that assigns to each arc a real number. The *excess* of the flow $f: A \to \mathbb{R}$ at the vertex v of the graph V is

$$\text{excess}(f, v) := \sum_{(u,v)\in A} f(u, v) - \sum_{(v,w)\in A} f(v, w).$$

To obtain an integer flow from a fractional one, we will use the following known result; see for instance [11, Corollary 11.2i].

Theorem 6. *Let $D = (V, A)$ be a directed graph, and let $p_A, q_A: A \to \mathbb{Z}$, and $p_V, q_V: V \to \mathbb{Z}$ be integer-valued functions on the arcs and on the vertices, respectively. If there is some flow $f: A \to \mathbb{R}$ on D such that*

$$p_A(a) \le f(a) \le q_A(a) \quad \text{for } a \in A \qquad \text{and} \qquad p_V(v) \le \text{excess}(f, v) \le q_V(v) \quad \text{for } v \in V,$$

then there is also an integer flow $f': A \to \mathbb{Z}$ that satisfies the same bounds.

Classical flow networks involve only a single vertex with negative excess (source) and a single vertex with positive excess (sink), conserving the flow at all other vertices. The network we consider has several sources and sinks. Additionally, these excesses as well as the "capacity" bounds on the arcs are not fixed but allowed to vary withing bounds. Such networks can easily be reduced to the classical situation by modifying the network; see for example [11, Chapter 11] for such transformations.

3 Proof of the main result

Proof of Theorem 4. Let n and d be positive integers, and let X be a d-colored set of points in general position in $\mathbb{R}^d$. Using the tools presented in Section 2, we now prove our claim that we can partition X into n sets of size $\lfloor \frac{|X|}{n} \rfloor$ or $\lceil \frac{|X|}{n} \rceil$ with pairwise disjoint convex hulls and which simultaneously equipartition the color classes. The proof is done in several steps.

(1) *From points to measures.*
We replace each point $x \in X$ by a ball $B_\varepsilon(x)$ centered at x, with $\varepsilon > 0$ a real number small enough such that no ℓ-flat with $\ell < d$ intersects more than $\ell + 1$ balls. With each ball centered at a point in X, we associate a uniformly distributed measure of 1. For each $i \in \{1, \dots, d\}$ and a measurable subset $A \subseteq \mathbb{R}^d$, let $\mu_i(A)$ be the total measure of balls centered at points in X_i that is captured by A. Clearly, $\mu_1, \dots, \mu_d$ are absolutely continuous finite measures on $\mathbb{R}^d$ with $\mu_i(\mathbb{R}^d) = |X_i|$. According to Theorem 3, there exists a partition of $\mathbb{R}^d$ into n convex regions $C_1, \dots, C_n$ which equipartitions the measures, that is,

$$\mu_i(C_j) = \frac{|X_i|}{n}$$

for all $i \in \{1, \dots, d\}$ and all $j \in \{1, \dots, n\}$.

(2) *A directed graph of incidences.*
In order to apply Lemma 5, we show the existence of an assignment of the points in X to the n regions $C_1, \dots, C_n$ such that for each point x assigned to a region C_j, $B_\varepsilon(x)$ intersects C_j,

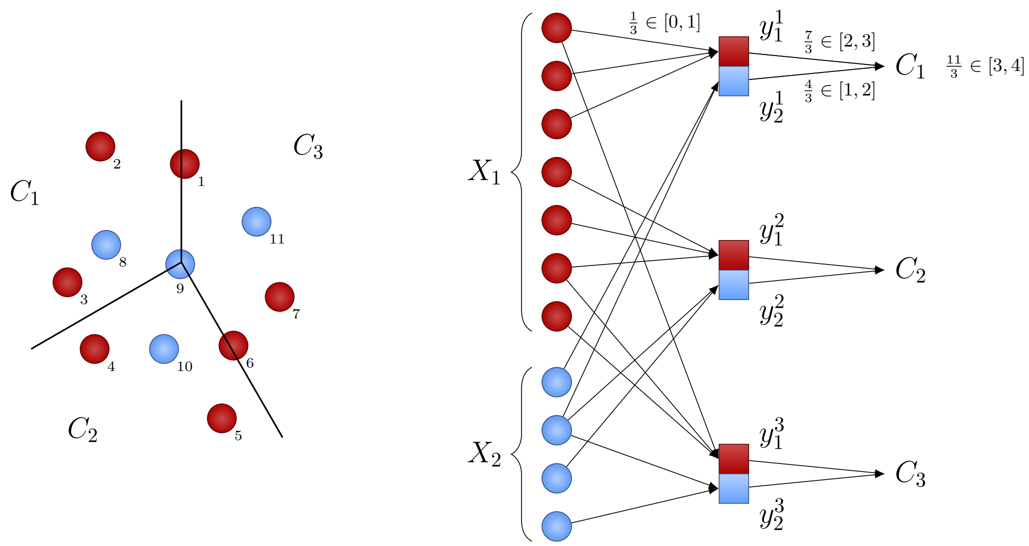

Figure 1: A configuration of 11 points/small balls with $d = 2$ colors in $d = 2$ dimensions, partitioned into $n = 3$ regions, and the corresponding directed graph D with some upper and lower bounds on the flow and its excess indicated as intervals.

while in total $\lfloor \frac{|X|}{n} \rfloor$ or $\lceil \frac{|X|}{n} \rceil$ points are assigned to each region. Such an assignment may be modeled as an integer flow from the points in X to the regions in the partition, where each $x \in X$ has an outflow of 1 and each region has an inflow of $\lfloor \frac{|X|}{n} \rfloor$ or $\lceil \frac{|X|}{n} \rceil$, the number of points assigned to it. To guarantee an equipartition of the color classes, we add a middle layer of vertices, one for each color and region, and set the constraints on these vertices and arcs accordingly.

We define the directed graph $D = (V, A)$ with $V = X \sqcup Y \sqcup Z$, where

$$Y = \left\{ y_i^j : 1 \leq i \leq d,\ 1 \leq j \leq n \right\}$$

contains a vertex y_i^j for each color i and each region C_j, and the set $Z = \{C_1, \ldots, C_n\}$ contains a vertex for each region. We have arcs from a point $x \in X$ to those vertices in Y corresponding to the color of x and the regions incident to the ball $B_\varepsilon(x)$ centered at x, as well as arcs from the vertices in Y to their respective region in Z. More precisely, the set of arcs is

$$\begin{aligned} A := &\left\{ (x, y_i^j) : 1 \leq i \leq d,\ 1 \leq j \leq n,\ x \in X_i,\ B_\varepsilon(x) \cap C_j \neq 0 \right\} \\ &\cup \ \left\{ (y_i^j, C_j) : 1 \leq i \leq d,\ 1 \leq j \leq n \right\}. \end{aligned}$$

For the vertices of D, we define lower bounds $p_V : V \to \mathbb{Z}$ and upper bounds $q_V : V \to \mathbb{Z}$ on the excess as follows:

$$\begin{aligned} p_V(x) &:= -1 & &= & q_V(x) &:= -1 \\ p_V(y) &:= 0 & &= & q_V(y) &:= 0 \\ p_V(C_j) &:= \left\lfloor \frac{|X|}{n} \right\rfloor & &\leq & q_V(C_j) &:= \left\lceil \frac{|X|}{n} \right\rceil \end{aligned}$$

For the arcs of D, we define lower bounds $p_A\colon A \to \mathbb{Z}$ and upper bounds $q_A\colon A \to \mathbb{Z}$ as follows:

$$p_A(x,y) := 0 \qquad < \qquad q_A(x,y) := 1$$
$$p_A(y_i^j, C_j) := \left\lfloor \frac{|X_i|}{n} \right\rfloor \quad \leq \quad q_A(y_i^j, C_j) := \left\lceil \frac{|X_i|}{n} \right\rceil$$

(3) *A fractional flow.*
We now construct a fractional flow $f\colon A \to \mathbb{R}$ by setting

$$f(x, y_i^j) := \mu_i(B_\varepsilon(x) \cap C_j) \qquad \text{and} \qquad f(y_i^j, C_j) := \frac{|X_i|}{n}.$$

The lower and upper constraints on the arcs are trivially satisfied,

$$p_A(a) \leq f(a) \leq q_A(a) \qquad \text{for all } \ a \in A.$$

With $\mu_i(B_\varepsilon(x)) = 1$ for all $x \in X_i$, we get

$$p_V(x) := -1 = \operatorname{excess}(f, x) = -\sum_{j=1}^{n} \mu_i(B_\varepsilon(x) \cap C_j) = -1 =: q_V(x).$$

With $\mu_i(C_j) = \frac{|X_i|}{n} = f(y_i^j, C_j)$ for a vertex $y_i^j \in Y$, the values yield

$$p_V(y_i^j) := 0 = \operatorname{excess}(f, y_i^j) = \sum_{x \in X} \mu_i(B_\varepsilon(x) \cap C_j) - f(y_i^j, C_j) = 0 =: q_V(y_i^j).$$

Lastly, for a $C_j \in Z$ we get

$$p_V(C_j) := \left\lfloor \frac{|X|}{n} \right\rfloor \leq \operatorname{excess}(f, C_j) = \sum_{i=1}^{d} f(y_i^j, C_j) = \frac{|X|}{n} \leq \left\lceil \frac{|X|}{n} \right\rceil =: q_V(C_j),$$

and consequently $p_V(v) \leq \operatorname{excess}(f, v) \leq q_V(v)$ for all $v \in V$.

(4) *Back to geometry.*
From the existence of this fractional flow, using Theorem 6, we obtain the existence of an integer flow on D, satisfying the constraints given by functions p_A, q_A and p_V, q_V. This in turn gives an assignment of points into sets of size $\lfloor \frac{|X|}{n} \rfloor$ and $\lceil \frac{|X|}{n} \rceil$, equipartitioning X. The middle layer of D ensures that each of the sets contains $\lfloor \frac{|X_i|}{n} \rfloor$ or $\lceil \frac{|X_i|}{n} \rceil$ points from the color class X_i, resulting in a simultaneous equipartition of X and all d color classes.

We now want that, for any two regions C_j and C_k, the sets of points P^+ assigned to C_j and P^- assigned to C_k have disjoint convex hulls. For each point x assigned to a region, $B_\varepsilon(x)$ intersects that region, by the definition of the arc set A. We may therefore apply Lemma 5 to the set $P = P^+ \sqcup P^-$ and conclude that the convex hulls of P^+ and P^- are disjoint. □

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