STABLE FRAMES IN MODEL CATEGORIES

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Abstract. We develop a stable analogue to the theory of cosimplicial frames in model cagegories; this is used to enrich all homotopy categories of stable model categories over the usual stable homotopy category and to give a different description of the smash product of spectra which is compared with the known descriptions; in particular, the original smash product of Boardman is identified with the newer smash products coming from a symmetric monoidal model of the stable homotopy category.

1. Introduction

Model categories are a convenient framework for "doing homotopy theory". Despite their not very complicated definition, they are rather powerful - many of the constructions known in topology can actually be carried out in model categories or their associated homotopy categories, for example suspensions and cofiber sequences, which gives lots of extra structure one can exploit.
The theory of cosimplicial frames, developed in [DK80] and [Hov99], shows that model categories are, in fact, closely related to the homotopy theory of topological spaces respectively simplicial sets:

Theorem 1.1. *The homotopy category of any model category is naturally enriched over the usual homotopy category of CW-complexes.*

Here, enriched is to be understood in the sense of the modules of [Hov99, 4.1]. A model category is called *stable* if the suspension functor on its homotopy category is an equivalence. In [Hov99, 8.11], Hovey raises the question whether there is a stable analogue of the above theorem:

Question 1.2. *Can the homotopy category of any stable model category be naturally enriched over the usual stable homotopy category?*

We will answer this question in the affirmative. In fact, we obtain an even stronger result: The stable homotopy category is the homotopy category of "the stable model category on one generator". This slogan already appears in [SS02], and in fact the methods used there are sufficient to define the enrichment functor on objects. We refine the methods so we can handle morphisms. Unfortunately, the precise statement of our main theorem is quite technical, so we defer the precise statement to 5.9.
Under technical constraints on the model category, the above question is also answered in [Dug06]. Our approach has the advantage that it does not need any technical assumptions on the model categories involved - not even functorial factorizations - and that the construction is more natural than the one used in [Dug06].

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1.1. **Fixing definitions.** For convenience, we will use a few slightly nonstandard definitions. For a model category $\mathcal{C}$, we do not consider the usual homotopy category which has the same objects as $\mathcal{C}$, but instead the equivalent full subcategory spanned by the cofibrant objects only. We just write $\mathrm{Ho}(\mathcal{C})$ for this category again. This definition has the advantage that in the definition of the derived functor of a left Quillen functor $\mathcal{C} \to \mathcal{D}$ no cofibrant replacement is needed, hence the derived functor is actually *equal* to the original functor on objects. Furthermore, this means that the derived functor of the composition of two left Quillen functors is equal to the composition of the two derived functors, which streamlines our proof. Using [Hov99, 1.3.7], we could get the proof working with the usual definitions, but this does not seem to be worth the trouble.

2. Sequential spectra

In this section, we will describe the category of spectra we want to work with. Working with this concrete model for $\mathcal{SHC}$ is crucial; the constructions we want to do depend not only on the homotopy category, but on the model category itself, and will fail to work for many models; in particular, we cannot replace simplicial sets by topological spaces in the following definition.

2.1. **The category of sequential spectra.**

Definition 2.1. *Let S^1 be the simplicial circle $\Delta[1]/\partial\Delta[1]$. A* sequential spectrum *or just* spectrum *of simplicial sets is a sequence $\{X_n\}_{n\geq 0}$ of pointed simplicial sets together with pointed maps $\sigma_n : X_n \wedge S^1 \to X_{n+1}$. A map of spectra $f : X \to Y$ is a sequence of pointed maps $f_n : X_n \to Y_n$ such that the obvious diagrams commute. The resulting category of sequential spectra of simplicial sets will be denoted as* Sp.

Denote by $Ev_n : \mathrm{Sp} \to \mathrm{SSet}$ the functor of evaluation in degree n and by $F_n : \mathrm{SSet} \to \mathrm{Sp}$ its left adjoint; the spectrum $F_n(X)$ is the free spectrum on X in degree n. The spectrum $F_0 S^0$ which has the n-sphere S^n in level n with structure maps the identifications $S^n \wedge S^1 \cong S^{n+1}$, is called the *sphere spectrum* and will be denoted by $\mathbb{S}$. This is the spectrum which will later on play the role of the unit.

2.2. **The homotopy theory of spectra.** We have a level model structure on Sp which is induced by the model structure on SSet_*: A map $f : X \to Y$ of spectra is a *level weak equivalence* respectively *level fibration* if all f_n are weak equivalences respectively fibrations of simplicial sets, and f is a cofibration if f_0 is a cofibration and the induced map $X_{n+1} \cup_{X_n \wedge S^1} Y_n \wedge S^1 \to Y_{n+1}$ is a cofibration for all n.
For stable homotopy theory, the homotopy category of this model category is too large; speaking loosely, it should not matter what happens in low dimensions, but it certainly does for level weak equivalences. To repair this, define the homotopy groups of a spectrum A as

$$\pi_k(A) = \operatorname*{colim}_n(\pi_{k+n}|A_n|)$$

for any *integer* k where the colimit is taken over the maps

$$\pi_{k+n}|A_n| \stackrel{-\wedge S^1}{\to} \pi_{k+n+1}[A_n \wedge S^1| \stackrel{\sigma_n}{\to} \pi_{k+n+1}|A_{n+1}|$$

A map $f : A \to B$ of spectra induces maps on the homotopy groups $\pi_k(f) : \pi_k(A) \to \pi_k(B)$ since f induces compatible maps $\pi_{k+n}|A_n| \to \pi_{k+n}|B_n|$. Call a map of spectra a π_*-*isomorphism* if it induces isomorphisms on all homotopy groups.

Theorem 2.2. *There is a model structure on* Sp *with weak equivalences the* π_*-*isomorphisms and with the same cofibrations as in the level model structure.*

Proof. See [BF78] or [GJ99, X]. □

From now on, we write $\mathcal{SHC}$ for Ho(Sp) with its stable model structure.
The fibrant objects in this model structure are exactly the levelwise Kan Ω-spectra, i.e., those spectra of levelwise Kan fibrant simplicial sets where all adjoint structure maps $A_n \to \Omega A_{n+1}$ are weak equivalences of simplicial sets, and the acyclic fibrations are the level acyclic fibrations since we did not change the cofibrations when we stabilized.

3. Cosimplicial frames

The proof of our main theorem relies on the technique of frames in a model category first developed in [DK80]; we will give a short overview, mostly based on [Hov99, Section 5].
There is one easy definition of a cosimplicial frame: A cosimplicial object in a (pointed) model category $\mathcal{C}$ is a frame if and only if the associated adjunction $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$ is a Quillen pair. This is the correct definition; however, there is an equivalent description of frames which is easier to handle since it makes no reference to the associated adjunction; only intrinsic properties of the cosimplicial object will be used.

3.1. Cosimplicial objects.

The basic fact underlying the theory of cosimplicial frames is the following:

Proposition 3.1. *For any cocomplete, pointed category* $\mathcal{C}$*, the category of adjunctions* $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$ *is equivalent to the category* $\mathcal{C}^\Delta$ *of cosimplicial objects in* $\mathcal{C}$*.*

Proof. This is standard; see for example [Hov99, 3.1.6] for details. □

For less awkward notation, we make the following definition:

Definition 3.2. *For a cosimplicial object* X *in* $\mathcal{C}$*, we write* $(X \wedge -, \mathrm{Map}(X, -))$ *for the associated adjunction* $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$*.*

The category of cosimplicial objects in a cocomplete, pointed category $\mathcal{C}$ is also a simplicial category in a natural way:

Definition 3.3. *For a cosimplicial object* X *in* $\mathcal{C}$ *and a pointed simplicial set* K*, define a cosimplicial object* $X \wedge_S K$ *by*

$$(X \wedge_S K)_n = X \wedge (K \wedge \Delta[n]_+)$$

with cosimplicial structure maps induced by the cosimplicial structure map of the cosimplicial object $K \wedge \Delta[-]_+$ *under the functor* $X \wedge -$*.*

Using the equivalent language of adjunctions $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$, this construction takes the following form: A cosimplicial object X represents an adjunction $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$, and we can precompose this adjunction with the adjunction $(K\wedge -, (-)^K) : \mathrm{SSet}_* \rightleftharpoons \mathrm{SSet}_*$ to obtain another adjunction $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$, and this adjunction is represented by the cosimplicial object $X \wedge_S K$.

Proposition 3.4. *This smash product is part of a simplicial structure on $\mathcal{C}^\Delta$.*

Proof. The simplicial mapping spaces are defined as

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}_{\mathcal{C}^\Delta}(X \wedge_S \Delta[n]_+, Y)$$

with simplicial structure maps induced from the cosimplicial structure maps of the cosimplicial object of cosimplicial objects $X \wedge_S \Delta[-]_+$. See [Qui67, II.1] for a description of the right adjoints $(-)^K : \mathcal{C}^\Delta \to \mathcal{C}^\Delta$. □

If $\mathcal{C}$ is a model category, the category $\mathcal{C}^\Delta$ also carries a model structure. Note the different meanings of $X \wedge_S K$ and $X \wedge K$: The first is a cosimplicial object, the latter an object of $\mathcal{C}$. Also recall that $X \wedge \Delta[n]_+ \cong X_n$.

Definition 3.5. *Let $f : X \to Y$ be a morphism of cosimplicial objects in a model category $\mathcal{C}$. The map f is*

- *a weak equivalence if for all n, the map $f_n : X_n \to Y_n$ is a weak equivalence*
- *a (acyclic) Reedy cofibration if the induced maps*

$$X \wedge \Delta[n]_+ \coprod_{X\wedge\partial\,\Delta[n]_+} Y \wedge \partial\,\Delta[n]_+ \to Y_n$$

  are (acyclic) cofibrations in $\mathcal{C}$ for all n
- *a (acyclic) Reedy fibration if it has the corresponding right lifting property with respect to (acyclic) Reedy cofibrations.*

Remark 3.6. It is easy to check with this definition that an object Y is Reedy fibrant if and only if the induced map $Y \to c(Y_0)$ is a Reedy fibration and Y_0 is fibrant, where $c(Y_0)$ denotes the constant cosimplicial object on Y_0.We will often make use of this.

Of course, this defines a model structure such that the two possibly different notions of acyclic cofibrations (resp. the two possibly different notions of acyclic fibrations) agree:

Theorem 3.7. *With these classes of weak equivalences, cofibrations and fibrations, the category $\mathcal{C}^\Delta$ is a model category.*

Proof. This is [Hov99, 5.2.5], or see [Ree]. □

Note that a cosimplicial object X is cofibrant if and only if $X \wedge -$ preserves cofibrations - by definition, it preserves the generating cofibrations $\partial\,\Delta[n]_+ \to \Delta[n]_+$, and hence all cofibrations.
Now C^Δ is a model category and a simplicial category; however, the SM7-axiom for a simplicial model category fails. We only have the following:

Proposition 3.8. *Let $f : X \to Y$ be a Reedy cofibration of cosimplicial objects in a model category $\mathcal{C}$ and let $i : K \to L$ be a cofibration of simplicial sets. Then the pushout-product map*

$$f\Box i : X \wedge_S L \coprod_{X\wedge_S K} Y \wedge_S K \to Y \wedge_S L$$

is a cofibration which is trivial if f is.

Proof. See [RSS01, 7.4] or [Hov99, 5.4.1] (or rather its pointed analogue [Hov99, 5.7.1]); or use the methods in the proof of Proposition 5.4. □

This is close to the SM7-axiom, but the pushout-product need not be a weak equivalence when i is a trivial cofibration.
Now we can characterize frames:

Proposition 3.9. *Let X be a cosimplicial object in a model category $\mathcal{C}$. Then $X \wedge - : \mathrm{SSet} \to \mathcal{C}$ is left Quillen if and only if X is Reedy cofibrant and for all standard maps $\Delta[n]_+ \to \Delta[m]_+$ of standard simplices, $X \wedge \Delta[n]_+ \to X \wedge \Delta[m]_+$ is a weak equivalence in $\mathcal{C}$.*

Proof. See [Hov99, 3.6.8]. □

Note that $X \wedge \Delta[n]_+ \to X \wedge \Delta[m]_+$ is a weak equivalence if and only if all cosimplicial structure maps of X are weak equivalences if and only if the induced map $X \to c(X_0)$ is a level weak equivalence. This leads to the following definition:

Definition 3.10. *A cosimplicial object in $\mathcal{C}$ is* homotopically constant *if all cosimplicial structure maps are weak equivalences. It is a* cosimplicial frame *or just* frame *if it is Reedy cofibrant and homotopically constant.*

By the above proposition, frames correspond to Quillen pairs $\mathrm{SSet}_* \rightleftharpoons \mathcal{C}$. We also have the following:

Proposition 3.11. *For any simplicial set K and any frame X, the cosimplicial object $X \wedge_S K$ is again a frame.*

Proof. A cosimplicial object X is a frame if and only if the corresponding adjunction is Quillen. Since the adjunction $(K \wedge -, (-)^K) : \mathrm{SSet}_* \rightleftharpoons \mathrm{SSet}_*$ is Quillen, the composite of this adjunction with $(X \wedge -, \mathrm{Map}(X, -))$ is Quillen if X is a frame; hence $X \wedge_S K$ is a frame. □

Since we will mainly use smashing with S^1, we introduce simpler notation:

Definition 3.12. *For a cosimplicial object X, we write ΣX for the cosimplicial object $X \wedge_S S^1$ and $\Omega(-)$ for the right adjoint of Σ.*

For frames, the simplicial mapping spaces in $\mathcal{C}^\Delta$ also carry homotopical information (in general, they do not because SM7 fails):

Proposition 3.13. *For cosimplicial objects A and B in $\mathcal{C}$ with A a cosimplicial frame and B Reedy fibrant, we have a natural isomorphism*

$$\pi_n \mathrm{Map}(A, B) \cong [A \wedge_S S^n, B]$$

where $[-, -]$ denotes the morphism sets in $\mathrm{Ho}(\mathcal{C}^\Delta)$.

Proof. The point is that $A \wedge_S \Delta[-]$ is a cosimplicial frame on A (it is a bicosimplicial object in $\mathcal{C}$) if A is a frame. Using Proposition 3.8, this is easy to see. The claim now follows from [Hov99, 6.1.2] which states that the mapping spaces obtained from frames have the correct homotopy type. □

3.2. **Frames.** Now we have defined frames and seen some basic properties, we want to put together some results concerning existence and uniqueness of frames. The main aim of this section is to prove Theorem 3.18. For this end, we need to develop some more theory. Theorem 3.18 subsumes most of the properties of frames in a very compact form; it is not formulated in [Hov99], but all the ingredients for the proof can be found there.

Proposition 3.14. *Let $\mathcal{C}$ be a pointed model category and X a cofibrant object of $\mathcal{C}$. Then there is a left Quillen functor $L : \mathrm{SSet}_* \to \mathcal{C}$ with an isomorphism from $L(\Delta[0])$ to X; or equivalently, there is a frame A on X, i.e., such that $A_0 \cong X$. If X is fibrant, we may choose A to be Reedy fibrant.*

Proof. This is basically a consequence of the factorizations in $\mathcal{C}^\Delta$; see [Hov99, 5.2.8]. □

Proposition 3.15. *Let X, Y be cofibrant-fibrant objects of $\mathcal{C}$, $f : X \to Y$ a morphism in $\mathcal{C}$. Then there are frames A on X, B on Y together with a morphism $F : X \to Y$ covering f.*

Proof. This is essentially [Hov99, 5.5.1]. □

Proposition 3.16. *Let A, B be frames and $f, g : A \to B$ be two maps such that $f_0, g_0 : A_0 \to B_0$ represent the same morphism in $\mathrm{Ho}(\mathcal{C})$. Then $f = g$ in $\mathrm{Ho}(\mathcal{C}^\Delta)$.*

Proof. This is similar to [Hov99, 5.5.2], though not quite the same. Let $ev_0 : \mathcal{C}^\Delta \rightleftharpoons \mathcal{C} : c$ denote the adjunction given by evaluation in degree 0 and constant cosimplicial object. This is a Quillen pair by the discussion after [Hov99, 5.2.7]. The assumptions of the proposition are such that for the derived functor ev_0^L, we have $ev_0^L(f) = ev_0^L(g)$. Since B is homotopically constant, the map $B \to c(ev_0(B))$ is a weak equivalence; hence the counit of the derived adjunction is an isomorphism for frames. Since $c^R(ev_0^L(f)) = c^R(ev_0^L(g))$, it follows $f = g$ in $\mathrm{Ho}(\mathcal{C}^\Delta)$ as desired. □

We will also need the following:

Proposition 3.17. *Let A, B be frames, $f, g : A \to B$ maps which are equal in $\mathrm{Ho}(\mathcal{C}^\Delta)$. Then the derived natural transformations $f^L, g^L : A \wedge^L - \to B \wedge^L -$ are equal*

Proof. See [Hov99, 5.5.2]. □

This means that we can regard $\mathrm{Ho}(Fr(\mathcal{C}))$ as the category of left Quillen functors $\mathrm{SSet}_* \to \mathcal{C}$, localized at the natural weak equivalences, and thus as a category of derived left Quillen functors $\mathrm{Ho}(\mathrm{SSet}_*) \to \mathrm{Ho}(\mathcal{C})$: We send an object X of $Fr(\mathcal{C})$ to the functor $X \wedge^L - : \mathrm{Ho}(\mathrm{SSet}_*) \to \mathrm{Ho}(\mathcal{C})$ and a morphism $f : X \to Y$ to the derived natural transformation; the proposition above implies that this is well-defined.

Now we can prove the main theorem announced at the beginning of this chapter:

Theorem 3.18. *Let $\mathrm{Ho}(Fr(\mathcal{C}))$ be the full subcategory of $\mathrm{Ho}(\mathcal{C}^\Delta)$ determined by the cosimplicial frames. Then evaluation in degree 0 $ev_0 : \mathcal{C}^\Delta \to \mathcal{C}$ induces an equivalence of categories $\mathrm{Ho}(Fr(\mathcal{C})) \to \mathrm{Ho}(\mathcal{C})$. Furthermore, the suspension functor $\Sigma : \mathcal{C}^\Delta \to \mathcal{C}^\Delta$ restricts to a functor $\Sigma : \mathrm{Ho}(Fr(\mathcal{C})) \to \mathrm{Ho}(Fr(\mathcal{C}))$ which is an equivalence if $\mathcal{C}$ is stable.*

Proof. First note that $ev_0 : \mathcal{C}^\Delta \to \mathcal{C}$ is left Quillen, so we indeed get a functor $ev_0 : \mathrm{Ho}(\mathcal{C}^\Delta) \to \mathrm{Ho}(\mathcal{C})$.
Let X be an object of $\mathrm{Ho}(\mathcal{C})$, i.e., a cofibrant object of $\mathcal{C}$. By 3.14, we find a frame A on X; this means $ev_0^L(A) \cong X$, which proves essential surjectivity.
Let $g : X \to Y$ be a morphism in $\mathrm{Ho}(\mathcal{C})$. We may up to isomorphism assume that X, Y are cofibrant-fibrant; then g is represented by an actual morphism $f : X \to Y$ in $\mathcal{C}$; by 3.15, we find frames A, B on X and Y with a map $F : A \to B$ covering f; now, $ev_0^L(F) = g$. Hence ev_0^L is full.
Now let $f, g : A \to B$ be two maps in $\mathrm{Ho}(\mathcal{C}^\Delta)$ such that $ev_0^L(F) = ev_0^L(G)$; we may again up to isomorphism assume A, B to be cofibrant-fibrant and that f, g are represented by actual morphisms $F, G : A \to B$ in $\mathcal{C}^\Delta$. Then 3.16 implies that $f = g$. Hence ev_0^L is faithful; this proves the first claim.
For the second statement, Lemma 3.19 below shows that Σ restricts to a functor $\Sigma : \mathrm{Ho}(Fr(\mathcal{C})) \to \mathrm{Ho}(Fr(\mathcal{C}))$ as claimed. Clearly, on the homotopy category level we have $ev_0 \circ \Sigma \cong \Sigma \circ ev_0$ where the right-hand Σ is the suspension in $\mathrm{Ho}(\mathcal{C})$. Since $\mathcal{C}$ is stable, all involved fucntors except the left-hand Σ are equivalences; hence Σ is an equivalence as well. □

3.3. **Frames and the suspension functor.** Let $\mathcal{C}$ be a model category. We write (Σ, Ω) for the adjoint pair $(- \wedge_S S^1, (-)^{S^1})$ on $\mathcal{C}^\Delta$.

Lemma 3.19. *For any model category $\mathcal{C}$, the functor $\Sigma : \mathcal{C}^\Delta \to \mathcal{C}^\Delta$ is a Quillen functor in the Reedy model structure and preserves cosimplicial frames.*

Proof. By setting $K = L = S^1$ in Proposition 3.8, we see that Σ preserves cofibrations and acyclic cofibrations. Furthermore, ΣX is the cosimplicial object associated to the functor $X \wedge (S^1 \wedge -) : \mathrm{SSet}_* \to \mathcal{C}$ which is left Quillen as composition of two left Quillen functors, thus ΣX is a cosimplicial frame. □

Unfortunately, even if the underlying model category is stable, Σ is usually not a Quillen equivalence. To remedy this failure, we define another class of "weak equivalences" (which will in general NOT be part of a model structure):

Definition 3.20. *A map $f : X \to Y$ of cosimplicial objects in a model category is a* realization weak equivalence *if for all cosimplicial frames A, the induced map $[A, X] \to [A, Y]$ is an isomorphism in the homotopy category of $\mathcal{C}^\Delta$.*

The definition is made to fit into a potential model structure where the frames are the cofibrant objects; such a model structure exists under the usual conditions which allow localization, see [RSS01] and [Dug01]. For us, it is mainly an auxiliary construction which is helpful to prove Proposition 3.24.

Lemma 3.21. *All weak equivalences are realization weak equivalences.*

Proof. This is clear since weak equivalences $X \to Y$ induce isomorphisms $[A, X] \to [A, Y]$ for all A. □

Lemma 3.22. *Let X, Y be frames and $f : X \to Y$ a realization weak equivalence. Then f is a weak equivalence.*

Proof. We can assume without loss of generality that X and Y are Reedy fibrant. By definition, f induces an isomorphism $[Y, X] \to [Y, Y]$. The preimage of the identity of Y is easily checked to be a homotopy inverse for f. □

Realization weak equivalences also have the expected behaviour with respect to the suspension and loop functor:

Proposition 3.23. *Let X be a cosimplicial frame over a stable model category $\mathcal{C}$ and Y a Reedy fibrant cosimplicial object. Then a map $f : \Sigma X \to Y$ is a realization weak equivalence if and only its adjoint $\tilde{f} : X \to \Omega Y$ is a realization weak equivalence.*

Proof. Let A be a cosimplicial frame. There is a commutative diagram

$$\begin{tikzcd} {[A, X]} \arrow[r, "{[A,\tilde{f}]}"] \arrow[d, "\Sigma"'] & {[A, \Omega Y]} \arrow[d, "\cong"'] \\ {[\Sigma A, \Sigma X]} \arrow[r, "{[\Sigma A, f]}"] & {[\Sigma A, Y]} \end{tikzcd}$$

The map on the left is an isomorphism since $\mathcal{C}$ is stable, A and X are frames and the homotopy category of frames is equivalent to the homotopy category of $\mathcal{C}$.
If f is a realization weak equivalence, the bottom map is an isomorphism since ΣA is a frame, so the top map is an isomorphism as well; hence $\tilde{f}$ is a realization weak equivalence. Conversely, if $\tilde{f}$ is a realization weak equivalence, the top map is an isomorphism, hence the lower map also is for any frame A. By Theorem 3.18, any frame B is up to homotopy of the form ΣA; this implies the claim. □

Proposition 3.24. *Let $f : X \to Y$ be a map of Reedy fibrant cosimplicial objects which is both a realization weak equivalence and a Reedy fibration. Then f has the right lifting property with respect to cosimplicial frames.*

Proof. By definition of a realization weak equivalence and since we have sufficient cofibrancy and fibrancy conditions, each map $A \to Y$ with A a frame admits a lift up to homotopy $A \to X$, i.e. an actual map $A \to X$ making the diagram commutative up to homotopy. It is a standard fact about model categories that in such a triangle, with the right-hand map a fibration, one can change a lift up to homotopy within its homotopy class to an actual lift; see for example in the proof of [Hov99, 6.3.7]. □

4. Spectra and adjunctions

In this section, we describe why we want to work with the category Sp: It is easy to describe left adjoints starting in SSet or SSet_*, and this is inherited by Sp. This is completely category-theoretical and has nothing to do with homotopy theory or model structures. We already know how to describe adjunctions out of SSet_* and natural transformations between them. For spectra, the point is that one can write a spectrum as a coequalizer of free spectra in a canonical way:

Proposition 4.1. *For a spectrum A, there is a coequalizer diagram*

$$\bigvee_n F_n A_{n-1} \wedge S^1 \underset{H}{\overset{T}{\rightrightarrows}} \bigvee_n F_n A_n \longrightarrow A$$

where H is induced by the structure maps of A and T is induced by the maps $F_n A_{n-1} \wedge S^1 \to F_{n-1} A_{n-1}$ adjoint to the identity of $A_{n-1} \wedge S^1$.

Proof. This is straightforward. We have maps $F_nA_n \to A$ adjoint to the identity of A_n, the wedge of these maps is the map $\bigvee_n F_nA_n \to A$. To check the universal property, note that a map $\bigvee_n F_nA_n \to B$ is adjoint to a sequence of maps $A_n \to B_n$; this is a map of spectra if and only if the map $\bigvee_n F_nA_n \to B$ is compatible with the two coequalizer maps. $\square$

To formulate our theorem, we need another definition:

Definition 4.2. *A* Σ-cospectrum *in* $\mathcal{C}^\Delta$ *is a sequence of cosimplicial objects* X_n *together with structure maps* $\Sigma X_n \to X_{n-1}$*; a morphism of cospectra* $X \to Y$ *is a sequence of morphisms* $X_n \to Y_n$ *compatible with the structure maps, like in a spectrum. Denote the resulting category as* $C^\Delta(\Sigma)$*.*

Theorem 4.3. *For a cocomplete category* $\mathcal{C}$*, the category* $\mathcal{C}^\Delta(\Sigma)$ *of* Σ*-cospectra is equivalent to the category* $Ad(\mathrm{Sp}, \mathcal{C})$ *of adjunctions* $\mathrm{Sp} \rightleftharpoons \mathcal{C}$ *with natural transformations as morphisms.*

Proof. This is straightforward. Given an adjunction $L : \mathrm{Sp} \rightleftharpoons \mathcal{C} : R$, form the cospectrum with n-th object the cosimplicial object associated to the left adjoint $L \circ F_n$. For essential surjectivity, use the above coequalizer diagram to define the left adjoint out of a Σ-cospectrum X. The right adjoint R associated to X is given by $R(A)_n = \mathrm{Map}(X_n, A)$ with structure maps induced by the structure maps of X. Compare [SS02, 6.5]. $\square$

Again, to avoid awkward notation, we make the following definition:

Definition 4.4. *For a* Σ*-cospectrum* X*, we write* $(X \wedge -, \mathrm{Map}(X, -))$ *for the associated adjunction. We denote the* m*-th cosimplicial level of the cosimplicial object* X_n *by* $X_{n,m}$*.*

Note that none of this depended on actual properties of SSet_* or $- \wedge S^1$ besides the fact that $- \wedge S^1$ is a left adjoint; one may define L-cospectra in an arbitrary cocomplete category $\mathcal{C}$ with an adjunction $L : \mathcal{C} \rightleftharpoons \mathcal{C} : R$ as sequences of objects X_n of $\mathcal{C}$ with structure maps $LX_n \to X_{n+1}$, and all of the above remains true for this category of spectra. In all cases of interest of us - in particular for Σ-cospectra - the underlying category $\mathcal{C}$ is actually a model category and L is left Quillen, and then cospectra form a model category again. Since this will be important to us, we give an explicit definition:

Definition 4.5. *Let* $\mathcal{C}$ *be a model category and let* $L : \mathcal{C} \rightleftharpoons \mathcal{C} : R$ *be a Quillen pair. A* cospectrum *with respect to this data is a sequence* $X_0, X_1, \ldots$ *of objects of* $\mathcal{C}$ *together with structure maps* $\sigma_n : LX_n \to X_{n-1}$*. A morphism* $f : X \to Y$ *of cospectra is a sequence of maps* $f_n : X_n \to Y_n$ *such that for all n, the obvious diagram commutes. We denote this category by* $\mathcal{C}(L)$ *and will call the objects L-cospectra.*
A L-cospectrum up to degree k *is a sequence of objects* X_0*,* X_1*,...,*X_k *in* $\mathcal{C}$ *together with structure maps* $LX_m \to X_{m-1}$ *for* $m = 1 \ldots k$*. A morphism* $X \to Y$ *is a sequence of maps* $X_m \to Y_m$ *compatible with the structure maps as above. We denote the resulting category as* $\mathcal{C}(L, k)$*.*

Both these constructions again yield model categories in a natural way:

Theorem 4.6. *There is a level model structure on* $\mathcal{C}(L)$ *and on* $\mathcal{C}(L, k)$ *for any* k *where a map* $f : X \to Y$ *is a*

- *weak equivalence resp. cofibration if and only if all f_n are weak equivalences resp. cofibrations*
- *a (trivial) fibration if f_0 is and for all $n \geq 0$, the induced map $X_n \to Y_n \times_{RY_{n-1}} RX_{n-1}$ is a (trivial) fibration.*

Proof. This is certainly not new; however, there seems to be no actual proof of this in printing. Note that we do not assume that $\mathcal{C}$ is cofibrantly generated; the theorem holds for any model category. However, the proof of the model axioms, just using the model axioms in $\mathcal{C}$, is straightforward (though not precisely short); note that one needs to check that "trivial fibration" as stated is indeed the same as a fibration and a weak equivalence. $\square$

5. Stable frames in model categories

5.1. **Stable frames.** The following is the stable analogue of 3.9:

Theorem 5.1. *Let X be a Σ-cospectrum in the model category $\mathcal{C}$. Then the adjoint pair $X \wedge - : Sp \rightleftharpoons \mathcal{C} : \mathrm{Map}(X, -)$ is a Quillen pair if and only if all X_n are cosimplicial frames and the structure maps $\Sigma X_n \to X_{n-1}$ are weak equivalences.*

Proof. See [SS02, 6.5]. $\square$

Definition 5.2. *A Σ-cospectrum X which is levelwise a frame and has weak equivalences $\Sigma X_n \to X_{n-1}$ is a* stable frame *on the object $X_{0,0}$.*

Theorem 5.3. *Let $\mathcal{C}$ be a stable model category and A a cofibrant-fibrant object of $\mathcal{C}$. Then there is a stable frame X on A, i.e., such that $X_{0,0} \cong A$; or equivalently, there is a left Quillen functor $L : Sp \to \mathcal{C}$ with $L(\mathbb{S}) \cong A$. This frame can be chosen to be fibrant in the model category $\mathcal{C}^\Delta(\Sigma)$.*

Proof. Choose a Reedy fibrant frame X_0 on A. By Theorem 3.18, we can find a Reedy fibrant frame X_1 together with a weak equivalence $\Sigma X_1 \to X_0$. Iterating this construction provides a stable frame, which may be replaced fibrantly in $\mathcal{C}^\Delta(\Sigma)$ without changing $A = X_{0,0}$ by construction of the factorizations in $\mathcal{C}^\Delta(\Sigma)$. $\square$

For studying derived natural transformations, we will need that two homotopic maps of stable frames induce the same derived natural transformations. For this end, we need some compatibility between the model structures on $\mathcal{C}^\Delta(\Sigma)$, $\mathcal{C}$ and Sp. The next proposition is our stable equivalent of [Hov99, 5.4.1] and 3.8, and the proof is virtually the same as the one given there.

Proposition 5.4. *Let $\mathcal{C}$ be a model category. Assume $f : X \to Y$ is a cofibration in $\mathcal{C}^\Delta(\Sigma)$ and $g : A \to B$ is a cofibration of spectra. Then the induced pushout-product map in $\mathcal{C}$ $f \square g : X \wedge B \coprod_{X \wedge A} Y \wedge A \to Y \wedge B$ is a cofibration which is trivial if f is.*

Proof. We may assume that g is one of the generating cofibrations $F_m \partial \Delta[n] \to F_m \Delta[n]$ using [Hov99, 4.2.4].
In this case, the induced map is the map

$$X_{m,n} \coprod_{X_m \wedge \partial \Delta[n]_+} Y_m \wedge \partial \Delta[n]_+ \to Y_{m,n}$$

which is a cofibration by definition of the Reedy cofibrations between cosimplicial objects if f is a cofibration. If f is acyclic, it is also acyclic; see [Hov99, 5.2.5]. $\square$

Corollary 5.5. *Let B be a cofibrant spectrum. Then the functor $-\wedge B : \mathcal{C}^\Delta(\Sigma) \to \mathcal{C}$ preserves cofibrations and acyclic cofibrations and hence has a left derived functor.*

Proof. Set $A = *$ in 5.4. □

Corollary 5.6. *Let X, Y be stable frames and $F, G : X \to Y$ two homotopic maps. Then F and G induce the same derived natural transformations between the derived functors of $X \wedge -$ and $Y \wedge -$.*

Proof. Let A be a cofibrant spectrum. We have two maps $F(A), G(A) : X \wedge A \to Y \wedge A$. We claim that these two maps represent the same map in $\mathrm{Ho}(\mathcal{C})$: Since $-\wedge A$ has a derived functor by the preceding corollary, we get a diagram of functors

$$\begin{tikzcd} \mathcal{C}^\Delta(\Sigma) \arrow[r, "-\wedge A"] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{Ho}(\mathcal{C}^\Delta(\Sigma)) \arrow[r, "-\wedge^L A"] & \mathrm{Ho}(\mathcal{C}) \end{tikzcd}$$

which commutes up to a natural isomorphism. We want to see that F and G go to the same map via the clockwise composition; but since the left vertical map sending them to their homotopy classes already sends them to the same map, the claim follows. □

Unsurprisingly, weak equivalences between stable frames induce natural weak equivalences:

Proposition 5.7. *Let $f : X \to Y$ be a weak equivalence of stable frames. Then the derived natural transformation $f\wedge - : X\wedge - \to Y \wedge -$ is a natural weak equivalence, i.e., a weak equivalence for all cofibrant spectra A.*

Proof. By [Hov99, 1.3.18], we may as well check the corresponding statement for the right adjoints, i.e., that for a fibrant object Z of $\mathcal{C}$, the map

$$\mathrm{Map}(f, Z) : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a π_*-isomorphism. The functor

$$\mathrm{Map}(-, Z) : (\mathcal{C}^\Delta)^{op} \to \mathrm{SSet}_*$$

preserves weak equivalences between frames if Z is fibrant. Since $\mathrm{Map}(Y, Z)$ and $\mathrm{Map}(X, Z)$ are levelwise of the form $\mathrm{Map}(Y_n, Z)$ and $\mathrm{Map}(X_n, Z)$ for frames X_n and Y_n and f is a levelwise weak equivalence, the map

$$\mathrm{Map}(f, Z) : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a level weak equivalence. This is what we wanted to prove. Note that $\mathrm{Map}(Y, Z)$ and $\mathrm{Map}(X, Z)$ are Ω-spectra, hence the notions of level weak equivalence and π_*-isomorphism agree. □

Definition 5.8. *For a stable model category $\mathcal{C}$, let $SF(\mathcal{C})$ denote the full subcategory of $\mathcal{C}^\Delta(\Sigma)$ given by all stable frames and $\mathrm{Ho}(SF(\mathcal{C}))$ the full subcategory of the homotopy category of $\mathcal{C}^\Delta(\Sigma)$ given by stable frames.*

Now we have carried together enough information to prove the stable analogue of Theorem 3.18.

Theorem 5.9. *Let X be a cofibrant object of $\mathcal{C}$, Y a cofibrant-fibrant object; let ωX be a stable frame on X and ωY a fibrant stable frame on Y.*
a) Any map $f : X \to Y$ extends (nonuniquely) to a map $F : \omega X \to \omega Y$ and hence to a natural transformation $\omega X \wedge - \to \omega Y \wedge -$ covering f on the sphere spectrum.
b) Let $f' : X \to Y$ be homotopic to f. Then any F and F' constructed from f and f' as in a) are homotopic and hence induce the same derived natural transformation between the derived functors of $\omega X \wedge -$ and $\omega Y \wedge -$.
c) If f is a weak equivalence, any map $\omega X \to \omega Y$ as in a) is a natural weak equivalence.
d) Evaluation in degree $(0, 0)$ induces an equivalence of categories

$$ev_{(0,0)} : \mathrm{Ho}(SF(\mathcal{C})) \overset{\cong}{\to} \mathrm{Ho}(\mathcal{C})$$

from the homotopy category $\mathrm{Ho}(SF(\mathcal{C}))$ of stable frames in $\mathcal{C}$ to $\mathrm{Ho}(\mathcal{C})$.

Proof. For a), we first extend f to a map $F_0 : \omega^0 X \to \omega^0 Y$: Since $\omega^0 Y$ is Reedy fibrant and homotopically constant, the map $\omega^0 Y \to cY$ is an acyclic fibration. We also have a map $\omega^0 X \to cY$ adjoint to f, and this map lifts to a map $F_0 : \omega^0 X \to \omega^0 Y$ since $\omega^0 X$ is cofibrant and $\omega^0 Y \to cY$ is an acyclic fibration.
Now we want to produce a map $F_1 : \omega^1 X \to \omega^1 Y$ extending F_0 to a map of cospectra up to degree 1 which is nothing else but a lift in the diagram

$$\begin{tikzcd} & & \omega^1 Y \arrow[d] \\ \omega^1 X \arrow[r] & \Omega\omega^0 X \arrow[r, "F_0"] & \Omega\omega^0 Y \end{tikzcd}$$

where the maps $\omega^1 X \to \Omega\omega^0 X$ and $\omega^1 Y \to \Omega\omega^0 Y$ are the structure maps. Since ωY is fibrant, the map on the right is a realization weak equivalence by Proposition 3.23 and a Reedy fibration; hence we find a lift in this diagram by Proposition 3.24. Proceeding like this, we find maps $F_n : \omega^n X \to \omega^n Y$ which form a morphism of cospectra and cover f. This proves a).
For b), it is by the preceding corollary enough to see that the homotopy type of a map $F : \omega X \to \omega Y$ is determined by the homotopy type of the restriction of F to $f : X \to Y$. By [Hov99, 5.5.2] or Theorem 3.18, it suffices to see that the homotopy type of F is determined by the homotopy type of $F_0 : \omega^0 X \to \omega^0 Y$ since the homotopy type of F_0 is determined by f.
Let $ev_0 : \mathcal{C}^\Delta(\Sigma) \to \mathcal{C}^\Delta$ denote the left Quillen functor given by evaluation in degree 0. We get an induced map of the simplicial mapping spaces $\mathrm{Map}(\omega X, \omega Y) \to \mathrm{Map}(\omega^0 X, \omega^0 Y)$. On π_0, this map induces $[\omega X, \omega Y] \to [\omega^0 X, \omega^0 Y]$. The maps F and F' go to the same element in $[\omega^0 X, \omega^0 Y]$ by assumption; hence we are finished if we can see that the map $\mathrm{Map}(\omega X, \omega Y) \to \mathrm{Map}(\omega^0 X, \omega^0 Y)$ is a homotopy equivalence (and thus induces an isomorphism on π_0).
Let $\omega^{\leq n} X$ resp. $\omega^{\leq n} Y$ denote the partial cosimplicial cospectrum obtained by only taking the first $n+1$ objects of ωX resp. ωY. We get a pullback square as follows, where the unnamed maps are induced from the structure maps of ωX and ωY and

the mapping spaces are those of $\mathcal{C}^{\Delta}$:

$$\begin{tikzcd}
\mathrm{Map}(\omega^{\leq 1}X, \omega^{\leq 1}Y) \arrow[rr, "ev_1"] \arrow[d, "ev_0"'] & & \mathrm{Map}(\omega^1 X, \omega^1 Y) \arrow[d] \\
\mathrm{Map}(\omega^0 X, \omega^0 Y) \arrow[r, "\Omega"'] & \mathrm{Map}(\Omega\omega^0 X, \Omega\omega^0 Y) \arrow[r] & \mathrm{Map}(\omega^1 X, \Omega\omega^0 Y)
\end{tikzcd}$$

The map on the right is a fibration since the map $\omega^1 Y \to \Omega\omega^0 Y$ is a fibration and $\mathrm{Map}(\omega^1 X, -)$ is right Quillen; note that this does not follow from 3.8, but requires an argument that $\omega^1 X \wedge_S -$ preserves acyclic cofibrations; compare [Hov99, 5.4.3]. By [Hov99, 6.1.2], we have that the induced map $\pi_n \mathrm{Map}(\omega^1 X, \omega^1 Y) \to \pi_n \mathrm{Map}(\omega^1 X, \Omega\omega^0 Y)$ is just the map $[\Sigma^n \omega^1 X, \omega^1 Y] \to [\Sigma^n \omega^1 X, \Omega\omega^0 Y]$ induced by the structure map of Y, which is an isomorphism since $\Sigma^n \omega^1 X$ is a frame and $\omega^1 Y \to \Omega\omega^0 Y$ is a realization weak equivalence. So the map on the right induces an isomorphism on all homotopy groups with basepoint the zero map. To see that it is in fact a π_*-isomorphism, we have to extend this to all basepoints.
Since we can find a frame Z with $\Sigma Z \simeq \omega^1 X$, we see that $\mathrm{Map}(\omega^1 X, \omega^1 Y) \simeq \mathrm{Map}(\Sigma Z, \omega^1 Y) \simeq \Omega \mathrm{Map}(Z, \omega^1 Y)$ is a loopspace up to weak equivalence; and similarly $\mathrm{Map}(\omega^1 X, \Omega\omega^0 Y) \simeq \Omega \mathrm{Map}(Z, \Omega\omega^0 Y)$ is a loopspace, and the induced map between the two spaces is up to homotopy Ω of the map $\mathrm{Map}(Z, \omega^1 Y) \to \mathrm{Map}(Z, \Omega\omega^0 Y)$. But in a loopspace, all components are weakly equivalent in a way respected by loop maps, hence we can conclude that $\mathrm{Map}(\omega^1 X, \omega^1 Y) \to \mathrm{Map}(\omega^1 X, \Omega\omega^0 Y)$ is a π_*-isomorphism and hence an acyclic fibration. Then the pullback map $\mathrm{Map}(\omega^{\leq 1}X, \omega^{\leq 1}Y) \to \mathrm{Map}(\omega^0 X, \omega^0 Y)$ is an acyclic fibration as well.
Now consider for any n the square

$$\begin{tikzcd}
\mathrm{Map}(\omega^{\leq n}X, \omega^{\leq n}Y) \arrow[rr, "ev_n"] \arrow[d] & & \mathrm{Map}(\omega^n X, \omega^n Y) \arrow[d] \\
\mathrm{Map}(\omega^{\leq n-1}X, \omega^{\leq n-1}Y) \arrow[r, "ev_{n-1}"'] & \mathrm{Map}(\omega^{n-1} X, \omega^{n-1} Y) \arrow[r] & \mathrm{Map}(\omega^n X, \Omega\omega^{n-1} Y)
\end{tikzcd}$$

Again, this is a pullback square and the map on the right is an acyclic fibration, so the map on the left also is. Hence, for any n, the map $\mathrm{Map}(\omega^{\leq n}X, \omega^{\leq n}Y) \to \mathrm{Map}(\omega^{\leq n-1}X, \omega^{\leq n-1}Y)$ forgetting the degree n-part is an acyclic fibration. Furthermore, we have $\lim_n \mathrm{Map}(\omega^{\leq n}X, \omega^{\leq n}Y) = \mathrm{Map}(\omega X, \omega Y)$. Thus the map $\mathrm{Map}(\omega X, \omega Y) \to \mathrm{Map}(\omega^0 X, \omega^0 Y)$ is also an acyclic fibration as a limit of acyclic fibrations, proving our claim.
For c), first note that F_0 is a weak equivalence since it is a map between homotopically constant cosimplicial objects covering the weak equivalence f in degree 0. Now we look at the commutative diagram

$$\begin{tikzcd}
\Sigma\omega X_1 \arrow[r, "\Sigma F_1"] \arrow[d] & \Sigma\omega Y_1 \arrow[d] \\
\omega X_0 \arrow[r, "F_0"'] & \omega Y_0
\end{tikzcd}$$

The two vertical maps and F_0 are weak equivalences, so ΣF_1 also is. By Theorem 3.18 and since ωX_1 and ωY_1 are frames, this means F_1 is a weak equivalence. By iterating this argument, we find that F is a weak equivalence, which induces a

natural weak equivalence between the functors $\omega X \wedge -$ and $\omega Y \wedge -$.
For d), first note that $ev_{(0,0)}$ indeed induces a functor $ev : \mathrm{Ho}(SF(\mathcal{C})) \to \mathrm{Ho}(\mathcal{C})$ since $ev_{(0,0)} : \mathcal{C}^{\Delta}(\Sigma) \to \mathcal{C}$ is just the functor $- \wedge \mathbb{S}$ which has a derived functor by Corollary 5.5. Since one can build a stable frame on any cofibrant-fibrant object of $\mathcal{C}$ and every object of $\mathrm{Ho}(\mathcal{C})$ is isomorphic to such an object, we get that ev is surjective on isomorphism classes of objects.
That ev is full is just a reformulation of part a). Given a morphism $g : A \to B$ in $\mathrm{Ho}(\mathcal{C})$, we may assume that A and B are cofibrant-fibrant and we obtain an actual morphism $f : A \to B$ in $\mathcal{C}$. Let X be a stable frame on A, Y a fibrant stable frame on X. By a), f extends to a map $F : X \to Y$ and $ev_{(0,0)}(F) = f$, hence $ev(F) = g$.
Finally, that ev is faithful follows directly from part c) since two maps of stable frames which are homotopic in degree $(0, 0)$ are homotopic. $\square$

6. Enrichments

We adopt the definitions of modules over a monoidal category from [Hov99, 4.1]: this is how the homotopy category of a stable model category will be enriched over $\mathcal{SHC}$.

Now we construct our enrichment functor (or, rather, module functor). Let $\mathcal{C}$ be a stable model category. We have a functor

$$- \wedge - : SF(\mathcal{C}) \times \mathrm{Sp} \to \mathcal{C}$$

Lemma 6.1. *This functor has a derived functor* $\Phi : \mathrm{Ho}(SF(\mathcal{C})) \times \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$

Note that there is no claim that $- \wedge -$ is a Quillen bifunctor; we just want an induced functor on the homotopy categories.

Proof. Straightforward. $\square$

Now choose an inverse ω to the equivalence $ev_{\mathbb{S}} : \mathrm{Ho}(SF(\mathcal{C})) \to \mathrm{Ho}(\mathcal{C})$. In the terminology of [Hov99], ω is what one might call a *stable framing* for $\mathcal{C}$; the choice of ω boils down to choosing, for each cofibrant object A of $\mathcal{C}$, a fibrant stable frame ωA with a weak equivalence $A \to \omega A \wedge \mathbb{S} \cong (\omega A)_{0,0}$. The following definition depends on the choice of ω, but in no essential way.
Now we can define the enrichment functor:

Definition 6.2. *The enrichment functor*

$$\otimes : \mathrm{Ho}(\mathcal{C}) \times \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$$

is given as the composition

$$\mathrm{Ho}(\mathcal{C}) \times \mathcal{SHC} \xrightarrow{\omega \times Id} \mathrm{Ho}(SF(\mathcal{C})) \times \mathcal{SHC} \xrightarrow{\Phi} \mathrm{Ho}(\mathcal{C})$$

Theorem 6.3. *For* $\mathcal{C} = \mathrm{Sp}$*, the functor* $\otimes : \mathcal{SHC} \times \mathcal{SHC} \to \mathcal{SHC}$ *makes* $\mathcal{SHC}$ *into a monoidal category with unit* $\mathbb{S}$*. For arbitrary* $\mathcal{C}$*,* $\otimes : \mathrm{Ho}(\mathcal{C}) \times \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$ *makes* $\mathrm{Ho}(\mathcal{C})$ *into a closed* $\mathcal{SHC}$*-module with respect to the monoidal structure on* $\mathcal{SHC}$ *given by* $\otimes$*. A left Qullen functor* $\mathcal{C} \to \mathcal{D}$ *between stable model categories induces an* $\mathcal{SHC}$*-module functor* $\mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$*.*

Proof. Consider the categories $\mathrm{Ho}(SF(\mathcal{C}))$ and $\mathrm{Ho}(SF(\mathrm{Sp}))$, regarded as functor categories. We obtain a functor $\mathrm{Ho}(SF(\mathcal{C})) \times \mathrm{Ho}(SF(\mathrm{Sp})) \to \mathrm{Ho}(SF(\mathcal{C}))$ by composition of derived Quillen functors $\mathcal{SHC} \to \mathcal{SHC}$ with derived Quillen functors $\mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$ and horizontal composition of natural transformations. Because

of our conventions regarding left Quillen functors and the homotopy category, the composition of their derived functors is strictly associative, and the identity of $\mathcal{SHC}$, which is a left derived functor, acts as strict identity on $\mathrm{Ho}(SF(\mathcal{C}))$. Hence $\mathrm{Ho}(SF(\mathrm{Sp}))$ is monoidal and $\mathrm{Ho}(SF(\mathcal{C}))$ is a $\mathrm{Ho}(SF(\mathrm{Sp}))$-module. Clearly, composition with derived left Quillen functors induces (strict) $\mathrm{Ho}(SF(\mathrm{Sp}))$-module functors $\mathrm{Ho}(SF(\mathcal{C})) \to \mathrm{Ho}(SF(\mathcal{D}))$.
Since $\mathcal{SHC}$ is equivalent to $\mathrm{Ho}(SF(\mathrm{Sp}))$, we can, after choosing inverse equivalences, pull the monoidal structure from the latter category over to $\mathcal{SHC}$; this destroys strict associativity and strict unitality, but it is still a coherent monoidal product, and this is actually the definition we have given above. That the result is again a monoidal category is certainly no surprise; the proof is just a long, tedious and uninspired diagram chase. The argument that left Quillen functors induce $\mathcal{SHC}$-module functors is similar. $\square$

In particular, we have constructed a smash product on $\mathcal{SHC}$; the obvious question is whether we have actually constructed something new. The following Theorem says that this is not so.

Theorem 6.4. *Let $\mathcal{C}$ be a monoidal stable model category with pairing $\Box$ and unit U. Choose a left Quillen functor $F : \mathrm{Sp} \to \mathcal{C}$ sending $\mathbb{S}$ to a cofibrant replacement of U. Then the following holds:*

- *The composition*
$$\mathrm{Ho}(\mathcal{C}) \times \mathcal{SHC} \xrightarrow{Id \times F} \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{C}) \xrightarrow{-\Box-} \mathrm{Ho}(\mathcal{C})$$
is a possible model for the enrichment functor for $\mathcal{C}$.
- *The derived functor $F : \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$ is strong monoidal. In particular, if $\mathcal{C}$ is any symmetric monoidal model for stable homotopy theory, then F induces a strong monoidal equivalence.*

At first glance, this seems to be an extremely strong statement, but it actually is not, and it is already mainly known: In [Shi01], it is proven that symmetric spectra are in a certain sense initial among all stable monoidal model categories, and an analogue of our theorem holds for $\mathrm{Ho}(\mathrm{Sp}^\Sigma)$ instead of $\mathrm{Ho}(\mathrm{Sp})$. Hence the only new statement we make is that our smash product on $\mathcal{SHC}$ is compatible with the one from Sp^Σ.

Proof. For i), we construct a particular inverse $\omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(SF(\mathcal{C}))$ to evaluation at the sphere spectrum as follows: For any cofibrant object X of $\mathcal{C}$, the functor $X\Box-$ is left Quillen, and we set $\omega(X) = X\Box F(-) : \mathrm{Sp} \to \mathcal{C}$. By definition, $\omega(X)(\mathbb{S}) \cong X$ in the homotopy category. A morphism $f : X \to Y$ in $\mathcal{C}$ induces a natural transformation $\omega X \to \omega Y$ covering, up to homotopy, f on the sphere spectrum; on the homotopy category, these constructions hence yield a functor $\omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(SF(\mathcal{C}))$ inverse to evaluation at the sphere spectrum, and we may use this particular inverse to construct the enrichment. Now, for an object X of $\mathrm{Ho}(\mathcal{C})$ and a spectrum A, we have $X \otimes A = \omega(X)(A) = X\Box F(A)$ by definition, and the claim follows.
For ii), we have to produce a natural isomorphism $F(A \otimes B) \cong F(A)\Box F(B)$. By naturality of the enrichment with respect to left Quillen functors, we have an isomorphism $F(A \otimes B) \cong F(A) \otimes B$, and by part i) we may arrange things such that

$F(A) \otimes B = F(A) \Box F(B)$. This isomorphism is natural in both variables; it remains to check the commutativity of various coherence diagrams, cf. [ML98, XI.2]. This is left to the reader; the only point is that one can always assume that one of the involved objects is the sphere spectrum since all involved functors and natural transformations are determined by their behaviour on the sphere spectrum. $\Box$

Corollary 6.5. *Let $\mathcal{C}$ be any symmetric monoidal model for the stable homotopy category. The smash product on* Ho(Sp) *we have constructed is, under the Quillen equivalence* $\mathrm{Sp} \to \mathcal{C}$ *sending* $\mathbb{S}$ *to the unit of* $\mathcal{C}$*, equivalent to the one in* $\mathrm{Ho}(\mathcal{C})$*.*

Remark 6.6. The proof of the Theorem actually proves something slightly stronger: There is no need for a monoidal structure on $\mathcal{C}$, one only needs a monoidal structure on $\mathrm{Ho}(\mathcal{C})$ which is induced by Quillen functors and natural transformations, but which may be not associative or unital on the nose - as for example our construction of the smash product in $\mathcal{SHC} = \mathrm{Ho}(\mathrm{Sp})$.

6.1. **Compatibility with the triangulated structure.** Both $\mathcal{SHC}$ and $\mathrm{Ho}(\mathcal{C})$ are triangulated categories; so the enrichment functor $\mathcal{SHC} \times \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ ought to be compatible with this structure. The following theorem tells us this is indeed so:

Theorem 6.7. *The functor* $\otimes : \mathrm{Ho}(\mathcal{C}) \times \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$ *is biexact: For any object* A *of* $\mathcal{SHC}$ *and any* X *in* $Ho(\mathcal{C})$*, the functors*

$$- \otimes A : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$$

and

$$X \otimes - : \mathcal{SHC} \to \mathrm{Ho}(\mathcal{C})$$

preserve triangles and are additive.

Proof. This is proven similarly to the corresponding unstable result in [Hov99]. $\Box$

7. An explicit description of the smash product on $\mathcal{SHC}$

In this chapter, we want to give an explicit description of the smash product on $\mathcal{SHC} = \mathrm{Ho}(\mathrm{Sp})$ as we have constructed it above. We obtain a comparison of our smash product with the original smash product on $\mathcal{SHC}$ by Boardman.

7.1. **Quillen endofunctors of spectra.** Given two spectra A and B, the following is the most naive candidate for $A \wedge B$: Choose a function $q : \mathbb{N} \to \mathbb{N}$ which is monotone, $q(n) \leq n$ and such that $q(n+1) - q(n)$ is at most 1. Then $p = Id - q : \mathbb{N} \to \mathbb{N}$ has the same properties and $p + q = Id$. Furthermore, we demand that both p and q are unbounded. Then for spectra A and B, we define the *naive smash product with respect to q* $A \wedge_q B$ levelwise as

$$(A \wedge_q B)_n = A_{q(n)} \wedge B_{p(n)}$$

with the following structure maps: If $q(n+1) = q(n)$, we use the structure map of B to obtain a map $A_{q(n)} \wedge B_{p(n)} \wedge S^1 \to A_{q(n+1)} \wedge B_{p(n+1)}$; else we use the structure map of A after commuting the S^1 past the $B_{p(n)}$. Clearly, this is a functorial construction - both $A \wedge_q -$ and $- \wedge_q B$ are functors $\mathrm{Sp} \to \mathrm{Sp}$.

Remark 7.1. One might want to make up for the "commuting the S^1 past the B factor" in some way, and this in in fact appers in the original definition in topological spaces; however, there is no natural way to do this in our simplicial context; and all our arguments go through without a problem in this regard.

Of course our aim is to see that $A \wedge_q B$ is a model for $A \otimes B$. To see this, it is enough to see that $A \wedge_q -$ is left Quillen for cofibrant A sending $\mathbb{S}$ to A. That $A \wedge_q -$ is left Quillen is easy to check by noting that $A \wedge_q -$ commutes with colimits and is hence a left adjoint and then writing down the associated Σ-cospectrum which happens to be a stable frame if q is unbounded. To see that $A \wedge_q \mathbb{S} \cong A$, note that $- \wedge_q \mathbb{S}$ is also left Quillen, thanks to the fact that also p is unbounded, and clearly $\mathbb{S} \wedge_q \mathbb{S} \cong \mathbb{S}$;: hence $- \wedge_q \mathbb{S}$ is weakly equivalent to the identity and our claim follows. We obtain the following:

Theorem 7.2. *The functor* $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$*,* $(A, B) \to A \wedge_q B$*, represents the smash product functor* $\mathcal{SHC} \times \mathcal{SHC} \to \mathcal{SHC}$*.*

Proof. This is clear by the preceding discussion and the construction of the smash product via Quillen functors. Note that a map $f : A \to A'$ induces a natural transformation $A \wedge_q - \to A' \wedge_q -$. □

This can be used to give a direct proof that the smash product is symmetric: After all, $A \wedge_q B \cong B \wedge_p A$, and both are models for the smash product $A \wedge B$. The following, similar statement is also interesting in its own right:

Proposition 7.3. *Given two left Quillen functors* $F, G : \mathrm{Sp} \to \mathrm{Sp}$*, the derived functors satisfy* $FG \cong GF$*.*

Proof. The derived functors of F and G are determined up to isomorphism by $A = F(\mathbb{S})$ and $B = G(\mathbb{S})$; hence we may assume $F = A \wedge_q -$ and $G = B \wedge_p -$. To see that the derived functors FG and GF are isomorphic, it suffices to see that $F(G(\mathbb{S})) \cong G(F(\mathbb{S}))$; however, $F(G(\mathbb{S})) \cong F(B) = A \wedge_q B \cong B \wedge_p A \cong G(A) \cong G(F(\mathbb{S}))$ as desired. □

7.2. **The original definition of the smash product.** We will follow [Ada95] in our description of the smash product.

The basic idea is very similar to the one outlined above, with one subtle difference regarding the structure maps. Choose functions $p, q : \mathbb{N} \to \mathbb{N}$ as above. Given two topological spectra A and B, we again define a spectrum $A \wedge_q B$ with n-th space $A_{p(n)} \wedge B_{q(n)}$, but with slightly different structure maps. In the topological setting, there is a "multiplication by -1"-map τ on S^1; regarding S^1 as the one-point compactification of $\mathbb{R}$, this is just the map sending x to $-x$. Now, If $q(n+1) = q(n) + 1$, we just use the structure map of B to obtain a map $A_{p(n)} \wedge B_{q(n)} \wedge S^1 \to A_{p(n+1)} \wedge B_{q(n+1)}$; however, if $q(n+1) = q(n)$, we first permute the S^1 past the $B_{q(n)}$, then use τ to obtain a self-map $A_{p(n)} \wedge S^1 \wedge B_{q(n)} \to A_{p(n)} \wedge S^1 \wedge B_{q(n)}$, and then use the structure map of A to obtain a map to $A_{p(n+1)} \wedge B_{q(n+1)}$. The *classical smash product* $- \wedge - : \mathrm{Ho}(\mathrm{Sp}^{Top}) \times \mathrm{Ho}(\mathrm{Sp}^{Top}) \to \mathrm{Ho}(\mathrm{Sp}^{Top})$ constructed in [Ada95] has the following basic property:

Theorem 7.4. *For arbitrary* p*,* q*, there is a natural isomorphism* $A \wedge_q B \to A \wedge B$*.*

The following is easy to check:

Proposition 7.5. *For cofibrant spectra* A*,* B*, the functors* $A \wedge_q - : \mathrm{Sp}^{Top} \to \mathrm{Sp}^{Top}$ *and* $- \wedge_q B : \mathrm{Sp}^{Top} \to \mathrm{Sp}^{Top}$ *are left Quillen and send the topological sphere spectrum* $\mathbb{S}^{Top}$ *to* A *resp.* B *up to weak equivalence.*

The associativity and commutativity isomorphisms for the smash product are then obtained by making intelligent choices for p and q. In particular, things are arranged such that the associativity, unit and commutativity isomorphisms stem from natural transformations of the overlying Quillen functors. Thus we may apply 6.4 (or, rather, the remark following the proof) to obtain the following:

Theorem 7.6. *Let $F : \mathrm{Sp} \to \mathrm{Sp}^{Top}$ denote the geometric realization functor. Then F induces a monoidal equivalence $\mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Sp}^{Top})$ where the first category is equipped with the smash product we have constructed and $\mathrm{Ho}(\mathrm{Sp}^{Top})$ is equipped with the classical smash product just described.*

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