11. Problem Set for

Advanced Algorithms II — Spring 2017

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Due 3. July 2017, in class.

Problem 1 k-Center *10 points*

Let $S \subset \mathbb{R}^2$ be a set of n *sites* in the plane, representing n houses, and let $k \in \mathbb{N}$. We would like to find a set $C \subset \mathbb{R}^2$, $|C| = k$, of k *centers*, where we can station our ambulances, such that the maximum distance between a site and the closest center is minimized. In other words, we would like to minimize

$$r(C) = \max_{s \in S} \min_{c \in C} \|s - c\|.$$

Consider the following greedy algorithm.

```
  input: S ⊂ ℝ², k ∈ ℕ, r ∈ ℝ⁺
1 C ← ∅
2 while S ≠ ∅ do
3 |   Choose an arbitrary s ∈ S and add it to C.
4 |   Remove all sites in S with distance at most 2r from s.
5 if |C| ≤ k then
6 |   return C
7 else
8 |   return There is no C' ⊂ ℝ², |C'| = k, with r(C') ≤ r.
```

(a) Show that if the algorithm adds more than k elements to C, then there is no $C' \subset \mathbb{R}^2$, $|C'| \leq k$, with with $r(C') \leq r$ (line 8). Conclude that the algorithm computes a 2-approximation for an optimum solution, provided that it is called with $r = \mathrm{OPT}$.

Hint: Let $C^* \subset \mathbb{R}^2$, $|C^*| = k$, with $r(C^*) = r$. For $c \in C^*$, let $N(c)$ be the set of all sites in S at distance at most r from c. Show that if the algorithm chooses a site $s \in N(c)$, then it will remove all of $N(c)$ from S. Conclude that the algorithm terminates after k iterations.

(b) Suppose we modify the algorithm as follows:

```
input: S ⊂ ℝ², k ∈ ℕ
C ← ∅
Choose an arbitrary s ∈ S and add it to C.
while |C| ≤ k do
|   Choose a site s ∈ S with maximum distance from C.
|   Add s to C.
return C
```

Show that this algorithm gives a 2-approximation of the optimum.

Hint: Argue that the modified algorithms behaves like the original algorithm on input (S, k, OPT).

Problem 2 MAX-3SAT: Deterministic Approximation *10 points*

MAX-3SAT is the following problem: suppose we have a propositional formula

$$\Psi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

in CNF with n variables and m clauses, such that each clause contains exactly three literals with pairwise distinct variables.

In Problem 9.1, you showed that there is a randomized 7/8-approximation algorithm for MAX-3SAT. Now, we will develop a deterministic algorithm for this problem.

(a) Let $\alpha : \{x_1, \dots, x_n\} \to \{0, 1\}$ be a random assignment for Ψ, and let Y the random variable that counts the number of clauses that α satisfies in Ψ. Show that

$$\mathbf{E}_\alpha[Y] = \frac{1}{2}\mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 0\right] + \frac{1}{2}\mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 1\right].$$

Conclude that

$$\mathbf{E}_\alpha[Y] \leq \max\left\{\mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 0\right], \mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 1\right]\right\}.$$

(b) Give a deterministic polynomial-time algorithm to find a 7/8-approximate solution to MAX-3SAT.

Hint: Use (a) in a greedy fashion. Explain how to compute $\mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 0\right]$ and $\mathbf{E}_\alpha\left[Y \mid \alpha(x_1) = 1\right]$ in polynomial time. Then, iterate over $x_2, \dots, x_n$.

Problem 3 Minimum Hitting Set *10 points*

Let V be a set with n elements, and let $\mathcal{S} = \{S_1, \dots, S_m \mid S_i \subseteq V, S_i \neq \emptyset\}$ be a set system on V. A *hitting set* for $\mathcal{S}$ is a set $H \subseteq V$ with $H \cap S_i \neq \emptyset$, for $i = 1, \dots, m$. In the MINIMUMHITTINGSET problem, we need to find a hitting set of minimum size.

Give an algorithm that solves MINIMUMHITTINGSET in time $O(f(c, k)\,\text{poly}(n, m))$, where $c = \max_{i=1}^{m} |S_i|$ is the maximum size of a set in $\mathcal{S}$ and k is the size of a minimum hitting set