10. Problem Set for

Advanced Algorithms II

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Wolfgang Mulzer, Katharina Klost

Due 26. June 2017, in class.

Problem 1 Arora's PTAS: The Number of Interfaces *10 points*

(a) Let S be a sequence of $2a$ portals, arranged on the boundary of a cell C. A *plane perfect matching* on S is a set of a pairwise disjoint curves in C with endpoints in S (there are no crossings and no common endpoints). Let C_a be the number of plane matchings on $2a$ portals. Argue that

$$C_0 = 1, \text{ and } C_a = \sum_{j=0}^{a-1} C_j C_{a-1-j}, \text{ for } a \geq 1.$$

(b) Let $f(x) = \sum_{a=0}^{\infty} C_a x^a$ be the *generating function* for $(C_a)_{a\in\mathbb{N}}$. Show that f obeys the functional equation $f(x) = 1 + xf(x)^2$.

(c) Show that

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Hint: We need $\lim_{x\to 0^+} f(x) = 1$.

(d) Define

$$\binom{\frac{1}{2}}{n} = \frac{\prod_{i=0}^{n-1}(1/2 - i)}{n!}.$$

Show that

$$\binom{\frac{1}{2}}{n} = \frac{(-1)^{n+1}}{4^n(2n-1)}\binom{2n}{n}.$$

(e) It is known that for $|y|$ small enough, $\sqrt{1+y} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}y^n$. Show that

$$f(x) = \sum_{a=0}^{\infty}\binom{2a}{a}\frac{x^a}{a+1}.$$

(f) Conclude that the number of possible interfaces during the dynamic programming phase of Arora's PTAS is $2^{O(m)}$.

Problem 2 Arora's PTAS: Portal-Respecting Tours *10 points*

For every $n \in \mathbb{N}$, give a nice instance of Euclidean TSP where the process of making an optimal tour π portal-respecting increases the length of π by at least a fixed constant factor, independent of ε and n.

Problem 3 Arora's PTAS: Bounding OPT *10 points*

Let $Q = \{q_1, \dots, q_n\}$ be a nice instance for Euclidean TSP, and let π be an optimal tour for Q. For $i \in \{0, \dots, L\}$, let v_i be the vertical line $v_i : x = i$ and h_i the horizontal line $h_i y = i$. Furthermore, let $t(v_i)$ and $t(h_i)$ be the number of intersections between π and v_i and h_i. Show that

$$\sum_{i=0}^{L} t(v_i) + \sum_{i=0}^{L} t(h_i) \leq 4\text{OPT}.$$

Hint: Let $e = q_1 q_2$ be an edge of π, and let $q_1 = (x_1, y_1)$ and $q_2 = (x_2, y_2)$. This means that the length of e is $c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Argue that e intersects at most $|x_1 - x_2| + |y_1 - y_2| + 2$ lines h_i, v_i, $i \in \{0, \dots, L\}$. Bound this by $4c$.

You may find the following inequality useful (prove it). For all $a, b \geq 0$, we have

$$a + b \leq \sqrt{2(a^2 + b^2)}.$$