9. Problem Set for

Advanced Algorithms II — Spring 2017

Wolfgang Mulzer, Katharina Klost

Due 19. June 2017, in class.

Problem 1 MAX-3SAT — *10 points*

MAX-3SAT is the following problem: suppose we have a propositional formula

$$\Psi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$$

in CNF with n variables and m clauses, such that each clause contains exactly three literals with pairwise distinct variables. Our task is to find an assignment $\alpha : \{x_1, \dots, x_n\} \to \{0, 1\}$ that satisfies the maximum number of clauses.

(a) Show that MAX-3SAT is NP-hard.

(b) Show that there is always an assignment α that satisfies at least $(7/8)m$ clauses. *Hint*: Consider a random α.

(c) For any $\varepsilon > 0$, give a Las-Vegas approximation algorithm that provides a $(7/8-\varepsilon)$-approximation for MAX-3SAT and runs in expected polynomial time.

(d) (*extra credit, 5 points*) Give a Las-Vegas approximation algorithm that provides a 7/8-approximation for MAX-3SAT and runs in expected polynomial time.

Problem 2 Metric *st*-TSP: Hoogeveen's algorithm — *10 points*

Let $V = \{1, \dots, n\}$ be a set of vertices, and let $d : \{1, \dots, n\} \times \{1, \dots, n\} \to \mathbb{R}_{\geq 0}$ be a metric on V. Let $s, t \in V$ be two distinct vertices. A *TSP-path* from s to t is a path that starts in s, ends in t, and visits every vertex from V exactly once. In the *metric st-TSP-problem*, we are looking for a TSP-path from s to t of minimum total length.

(a) Adapt the Christofides heuristic for the metric *st*-TSP-problem.

Hint: An *Eulerian st-path* is a path that starts in s, ends in t, and visits every edge exactly once. A connected graph contains an Eulerian *st*-path if and only if s and t are the only vertices with odd degree. Compute an MST T for K_n with edge lengths $d(i, j)$ and set W to the set of all vertices with "wrong" degree in T. Let M be a minimum cost perfect matching for W. Proceed as in the Christofides heuristic.

(b) Let Π_{st} be an optimal TSP-path from s to t. Show that $w(T) \leq w(\Pi_{st})$.

(c) Let F be the edge *multiset* $T \cup \Pi_{st}$. Argue that the set of vertices with odd degree in (V, F) is exactly W.

(d) Let $w_1, w_2, \ldots, w_{2m}$ be the vertices of W in the order as they appear in Π_{st}. For $i = 1, \ldots m$, let F_i be the sequence of edges on Π_{st} between w_{2i-1} and w_{2i}. Argue that $w(M) \leq \sum_{i=1}^{m} w(F_i)$.

(e) Let $F' = F \setminus \bigcup_{i=1}^{m} F_i$. Argue that (V, F') is Eulerean. Conclude that there are two perfect matchings M_1 and M_2 on W with $w(M_1) + w(M_2) \leq w(F')$.

(f) Conclude that $3w(M) \leq w(F) \leq 2w(\Pi_{st})$, and that the algorithm gives a 5/3-approximation for the minimum st-TSP-problem.

Problem 3 Nice Inputs for Euclidean TSP *10 points*

Prove the lemma from class that in order to obtain a PTAS for Euclidean TSP, it suffices to consider nice inputs.

Suppose that for each fixed $\varepsilon > 0$, there exists an algorithm $\mathcal{A}$ with the following property: given $n \in \mathbb{N}$, such that n is a power of two, and a set Q of at most n vertices of the $4n^2 \times 4n^2$ grid, the algorithm $\mathcal{A}$ runs for $n^{O(1/\varepsilon)}$ steps and returns a TSP tour for Q with length at most $(1 + \varepsilon)\mathrm{OPT}_Q$.

Then, for each $\varepsilon > 0$, there is an algorithm $\mathcal{A}'$ with the following property: given a set $P \subset \mathbb{R}^2$ of n points in the plane, the algorithm $\mathcal{A}'$ runs for $n^{O(1/\varepsilon)}$ steps and returns a TSP tour for P with length at most $(1 + \varepsilon)\mathrm{OPT}_P$.