7. Problem Set for

Advanced Algorithms II Spring 2017

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Due 6. June 2017, in the exercise session.

Problem 1 Cuckoo Hashing *10 Points*

(a) Let $k \in \{3, \dots, 2n\}$, and suppose we have a cuckoo hashing data structure that stores a set $S \subseteq \mathcal{U}$ and that was built using hash functions $h_0, h_1 : \mathcal{U} \to \{0, \dots, m-1\}$.

Let $x \in \mathcal{U} \setminus S$. Show that if $T(x) = k$, then $G_{h_0,h_1}(S \cup \{x\})$ contains an edge-simple path that starts with x and has at least $\lfloor k/3 \rfloor$ edges.

(b) Let $G = (V, E)$ be a connected graph with at least two cycles such that each connected component of every proper subgraph of G contains at most one cycle. Show that either (i) there are $u \neq v \in V$ such that G consists of three distinct paths with endpoints u and v that share no other vertices; or (ii) there are $u, v \in V$ such that G consists of a cycle C_1 that contains u, a cycle C_2 that contains v, and a (possibly empty) path π from u to v. The path π contains no vertices from C_1 or C_2 except for u and v. The cycles C_1 and C_2 share no vertices except possibly $u = v$, in case that π is empty.

Problem 2 Encoding Arguments I *10 Points*

Let $n \in \mathbb{N}$ and $x \in \{0,1\}^n$ a binary string of length n. A *run of length* k in x ist a contiguous substring of k ones in x, for $k \in \mathbb{N}$.

Using the uniform encoding lemma, derive a good upper bound for the probability that a random binary string of length n contains a run of length $\lceil \log n \rceil + s + 1$, for $s \in \mathbb{N}$.

Problem 3 Encoding Arguments II *10 Points*

Suppose we have a hash table T with n slots in which we would like to store a set $S \subset U$ with n elements. Conflicts are resolved using chaining. Furthermore, we assume that the hash function is chosen uniformly at random from the set of all functions $h : S \to \{0, \dots, n-1\}$.

For a slot i of T, we let Q_i denote the *number* of elements from S that are mapped to slot i by the hash function.

(a) What is the number H of possible hash functions $h : S \to \{0, \dots, n-1\}$? Argue that there is a canonical code that encodes all such hash functions with $\log H$ bits.

(b) Let h be a hash function that maps k elements from S into the same slot i. Show that h can be encoded using $\log H - k\log(k/e) + \log n$ bits.

Hint: $\binom{n}{k} \le (ne/k)^k$.

(c) Show that for any $s \ge 1$, the probability that there is a slot with at least $3\log n/\log\log n + s$ elements is at most $2^{-s/2}$.

(d) Show that

$$\mathbf{E}\left[\max_{i=0}^{n-1} Q_i\right] = O(\log n/\log\log n).$$