5. Problem Set for

Advanced Algorithms II

Spring 2017

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Due 22. May 2017, in class

Problem 1 Minimax Theorem *10 Points*

Finish the proof of the minimax theorem by showing that the two linear programs are in fact dual to each other.

Problem 2 Borůvka's Algorithm *10 Points*

Let $G = (V, E)$ be a connected undirected weighted graph with n nodes and m edges. Without loss of generality, suppose that all edges weights are pairwise distinct.

Borůvka's algorithm for finding a minimum spanning tree (MST) consists of a sequence of *Borůvka phases*. In each phase, each node picks its incident edge with the minimum weight. The resulting edge set F constitutes a spanning forest of G. The components of this forest are contracted, resulting loops are deleted, and if there are multiple edges, we keep only the edge of smallest weight. The algorithm stops as soon as there is only one node left. The algorithm returns the union of all edge sets that were contracted throughout its execution.

(a) Show that Borůvka's algorithm computes an MST.

(b) Show that a Borůvka phase can be implemented in time $O(m')$, where m' is the number of edges in the current graph.

(c) Conclude that the algorithm runs in $O(\min\{m \log n, n^2\})$ steps.

(d) Recall that Prim's algorithm can find an MST in G in time $O(n \log n + m)$. Combine the algorithms of Prim and Borůvka in order to obtain an algorithm that finds MST(G) in $O(m \log \log n)$ steps.

Hint: First, perform a suitable number of Borůvka phases and then switch to Prim.

Problem 3 MST-Verification *10 Punkte*

Let $G = (V, E)$ be a connected undirected weighted graph with n nodes and m edges, such that all edge weights are pairwise distinct.

Let $A \subseteq E$ be acyclic. An edge $e \in E$ is *A-light*, if either (i) $A \cup \{e\}$ is acyclic; or (ii) the unique cycle C in $A \cup \{e\}$ contains an edge that has a larger weight than e.

(a) Show that A is an MST for G if and only if $E \setminus A$ contains no A-light edges.

(b) Show that $e \in E$ is A-light if and only if $e \in \text{MSF}(V, A \cup \{e\})$, where MSF denotes the minimum spanning *forest*.