3. Problem Set for

# Advanced Algorithms II — Spring 2017

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**Due** 08. May 2017, in class

**Problem 1** Bit Complexity — *10 Points*

(a) Let $x$ be a integer that can be represented with $a$ bits and $y$ an integer that can be represented with $b$ bits. How many bits are needed at most to represent $x + y$? Explain.

(b) Let $x$ be a integer that can be represented with $a$ bits and $y$ an integer that can be represented with $b$ bits. How many bits are needed at most to represent $x \cdot y$? Explain.

(c) Let $A = (a_{ij})$ be an $n \times n$ matrix, such that all entries of $A$ can be represented with $a$ bits. The *determinant* of $A$, $\det(A)$, is given by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $S_n$ denotes the set of all permutations of $\{1, \dots, n\}$. Give a good upper bound on the number of bits needed to represent $\det(A)$. Explain your answer.

**Problem 2** Ellipsoid Algorithm — *10 Points*

Let $L : \max \mathbf{c}^T\mathbf{x}, A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, be a linear program with $n$ variables and $m$ constraints. A *efficient separation oracle* for $L$ is a poynomial-time algorithm that does the following: given a candidate solution $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq 0$, either (i) assert (correctly) that $\mathbf{x}$ is feasible for $L$, or (ii) produce a constraint of $L$ that certifies that $\mathbf{x}$ is not feasible, i.e., a row $j$ of $A$ such that $\mathbf{a}_j \cdot \mathbf{x} > \mathbf{b}$.

(a) Explain the following statement: "If an efficient separation oracle is available, then the ellipsoid algorithm runs in polynomial time."

(b) The *minimum cost aroborescence* problem problem is as follows: given a directed graph $G = (V, E)$, a *root* $r \in V$, and a positive *cost* $c_e$ for each edge $e \in E$, find a subset $F \subseteq E$ of directed edges such that (i) $(V, F)$ contains a directed path from $r$ to every other vertex, and (ii) the total cost of $F$ is minimized. Edmonds showed that this problem is captured by the following LP:

$$\min \sum_{e \in E} c_e x_e, \text{subject to}$$

$$\sum_{\substack{(u,v) \in E,\\ u \in S, v \in V \setminus S}} x_e \geq 1, \text{for all } S \subseteq V \text{ with } r \in S$$

$$x_e \in [0, 1], \text{ for all } e \in E.$$

What is the number of constraints of this LP? Show that the minimum cost arborescence problem can be solved in polynomial time.

**Problem 3** Variants of Farkas' Lemma *10 Points*

In class, we have seen the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then *either* there is a $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \leq \mathbf{b}$, *or* there is a $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T \mathbf{b} < 0$.

Prove that the following two statements are equivalent to the lemma from class:

(a) Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. *Either* there is an $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$, *or* there is a $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T \mathbf{b} < 0$.

(b) Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. *Either* there is an $\mathbf{x} \in \mathbb{R}^m$ with $A\mathbf{x} \leq \mathbf{b}$, *or* there is an $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}^T A = \mathbf{0}^T$ and $\mathbf{y}^T \mathbf{b} < 0$.