Discrete Geometry, SS 2011 — exercise sheet 10

due date: Tuesday, June 28th, 2011, 14:00

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47. Find a finite bound on the VC-dimension of the following range space: The ground set X is the set of lines in the plane. Every subset of lines that intersect a given segment xy in the plane forms a range.

48. Applying the ε-net theorem

    Use the ε-net theorem with the range space of the previous excercise to prove the following statement:

    Given a set P of $n \geq 2$ points in general position in the plane and a set L of m lines that avoid these points, one can always find a segment xy with $x, y \in P$ that is intersected by at most $O(m \log n/\sqrt{n})$ lines of L.

    (A stronger form of this lemma, without the $\log n$ term, has been shown in the lecture via $\frac{1}{r}$-cuttings.)

49. Dual range spaces (10 credits)

    The *incidence matrix* A of the finite range space $(X, \mathcal{F})$ is a 0-1-matrix whose columns are indexed by the elements $x \in X$ and whose rows are indexed by the ranges $R \in \mathcal{F}$. An entry of A is 1 if and only if $x \in R$.

    (a) How can one recognize a shattered subset in A?
    (b) Observe that the incidence matrix of the dual range space is the transposed matrix A^T.
    (c) Show that, if $(X, \mathcal{F})$ has VC-dimension d, then the VC-dimension of the dual range space is at least $\lfloor \log_2 d \rfloor$.
    (d) Give an upper bound on the VC-dimension of the dual range space in terms of d.

50. Weak ε-nets and the range space of convex sets. (10 credits)

    Let X be a set of n points in the plane in general position. For $0 < \varepsilon < 1$, let $\mathcal{R}^\varepsilon$ be the family of all convex sets that contain at least εn elements of X. a set T of points (not necessarily from X) is called a *weak ε-net* (with respect to convex sets) if every member of $\mathcal{R}^\varepsilon$ contains at least one element of T.

    Determine the maximum possible VC-dimension of the range space $(X, \{ R \cap X \mid R \in \mathcal{R}^\varepsilon \}$, when X varies over all sets of size n, in terms of n and ε.

51. Stabbing number (10 credits)

    (a) Show that a spanning tree T of a point set S with stabbing number k can be converted into a Hamiltonian cycle with stabbing number $\leq 2k$.
    Hint: Duplicate all edges of T. Take an Eulerian tour E of the resulting graph. Then start at an arbitrary place on E and take the points of S in the order in which they are first visited by E.
    (b) Show that a Hamiltonian cycle C of a point set S in general position with stabbing number k can be converted into a non-crossing Hamiltonian cycle without increasing the stabbing number.

52. Show that n points in the plane in general position determine at least $\Omega(\sqrt{n})$ segments that are "independent" in the sense that no line through one of them crosses any other segment.