Discrete Geometry, SS 2011 — exercise sheet 8+9

due date: Tuesday, June 14th, 2011, 14:00
due date for exercises 42–45: Tuesday, June 21st, 2011, 14:00

35. Tightness of the shatter function lemma (10 credits).

    Show that the bound

    $$\Phi_d(m) = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{d}$$

    on the shatter function is tight, by constructing a range space of VC-dimension d with a ground set of m elements and $\Phi_d(m)$ ranges.

    (Hint: An easy solution can be guessed by staring at the formula. Can you find different solutions, for example for $d = 1$? For $d = 2$?)

36. Preserving the VC-dimension (10 credits)

    Show that the operation of taking the symmetric difference $\oplus$ with a fixed set $A \subseteq X$ does not change the VC-dimension of a range space $(X, \mathcal{F})$:

    $$\mathcal{F}' := \{ R \oplus A \mid R \in \mathcal{F} \}$$

    Can it change the shatter function? Can it change the required size of ε-nets? Can it change the required size of ε-approximations?

37. (0 credits) Show that in a range space with VC-dimension $d = 1$, there is always an ε-net with at most $\max\{2, \lceil \frac{1}{\varepsilon} \rceil - 1\}$ elements, and this bound is tight.

38. (0 credits) Prove that $\Phi_d(m) \leq (em/d)^d$ for all $d \geq 1$ and $m \geq 1$.

39. Iterated ε-approximations (10 credits)

    (a) Prove: If N_1 is an ε_1-approximation of a finite range space $(X, \mathcal{F})$ and if N_2 is an ε_2-approximation of the induced range space $(N_1, \mathcal{F}|_{N_1})$, then N_2 is an $(\varepsilon_1 + \varepsilon_2)$-approximation of the original range space $(X, \mathcal{F})$.

    (b) If N_1 is only an ε_1-net (and not an ε_1-approximation), can one still conclude that N_2 is at least an $(\varepsilon_1 + \varepsilon_2)$-net?

    (c) If, on the other hand N_1 is an ε_1-approximation but N_2 is only an ε_2-net, can one conclude that N_2 is an $(\varepsilon_1 + \varepsilon_2)$-net?

40. Deviation from the mean (0 credits)

    Let $X = X_1 + X_2 + \cdots + X_n$ be the sum of n independent Bernoulli random variables with success probability $P[X_i = 1] = p$ (and $P[X_i = 0] = 1 - p$). Show that $P[X \geq \frac{1}{2}np] \geq \frac{1}{2}$, provided that $np \geq 8$, by applying the Chebysheff inequality, which says that $P[|X - \mu| \geq t\sigma] \leq 1/t^2$, for any random variable X with mean μ and variance σ.

41. VC-dimension of half-spaces (0 credits)

    Show that the range space of all half-spaces $\{a_1x_1 + a_2x_2 + \cdots + a_dx_d \leq b\}$ of $\mathbb{R}^d$ has VC-dimension $d + 1$.

    Hint for the upper bound: Take any $d + 2$ points $p_1, \ldots, p_{d+2}$ in $\mathbb{R}^d$. By rank considerations, the system of equations $\sum \lambda_i p_i = 0$, $\sum \lambda_i = 0$, has a nonzero solution $(\lambda_1, \ldots, \lambda_{d+2})$. Then the subset $\{ p_i \mid \lambda_i > 0 \}$ cannot be cut out by a half-space.

42. Lifting (10 credits)

   (a) Give a finite upper bound on the VC-dimension of the ranges of $\mathbb{R}^2$ that can be defined by quadratic equations $ax^2 + bxy + cy^2 + dx + ey + f \leq 0$, by using the mapping $(x, y) \mapsto (x^2, xy, y^2, x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}^5$ (the so-called *Veronese map*) and applying the previous exercise.

   (b) Extend the result to $d > 2$ dimensions.

   (c) Show that, for balls in $\mathbb{R}^d$, the general bound can be improved by using the lifting map $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, x_1^1 + \cdots + x_d^2)$.

   (d) (0 credits) Show that, for balls in d dimensions, the shatter function has actually the stronger bound $\Phi_{d+1}(n)$, and this is tight for n points in general position.

43. (0 credits) Show that the unit square cannot be expressed as $\{ (x, y) \mid p(x, y) \geq 0 \}$ with a single polynomial $p(x.y)$.

44. Finite VC-dimension under Boolean operations (8 credits)

   Suppose that $(X, \mathcal{F})$ is a range space of VC-dimension 5, and $(X, \mathcal{G})$ is a range space of VC-dimension 9. Find a finite bound on the VC-dimension of the range space

$$(X, \{ R \cap S \mid R \in \mathcal{F},\ S \in \mathcal{G} \}).$$

   Hint: try to bound the shatter function.

45. ε-nets and the cutting lemma (12 credits)

   (a) Let H be a finite set of lines in the plane. For a triangle T, let H_T be the set of lines intersecting the interior of T. let $\mathcal{T} \subseteq 2^H$ be the system of sets H_T for all triangles T. Show that the VC-dimension of $\mathcal{T}$ is bounded by a constant.

   (b) With the ε-net theorem, prove a weak form of the cutting theorem: For every finite set H of lines and every r, $1 < r < |H|$, there is a $\frac{1}{r}$-cutting for H consisting of $O(r^2 \log^2 r)$ triangles (or trapezoids).

46. Vertex neighborhoods (0 credits)

   For a given undirected graph $G = (V, E)$, let $N(v) = \{ u \in V \mid uv \in E \}$ denote the *neighborhood* of the vertex $v \in V$ (not including v itself). Show that there is a constant c such the set system $\{ N(v) \mid v \in V \}$ of vertex neighborhoods of any planar graph has VC-dimension bounded by c.