Discrete Geometry, SS 2011 — exercise sheet 6

due date: Tuesday, May 24th, 2011, 14:00

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25. An application of the Semerédi-Trotter Theorem. (10 credits)

    Given a set of n points in the plane and an integer $k > 2$, prove that the number of lines that contain at least k point is $O(n^2/k^3 + n/k)$. Show that this bound is asymptotically tight.

26. The Sylvester-Gallai Theorem (0 credits)

    Consider a set L of $n \geq 2$ lines such that no two are parallel and they don't all go through the same point. An *ordinary crossing* is a point that belongs two exactly two lines of L.

    - Show that there is always at least one ordinary crossing.
    - Show that L determines at least n different intersection points.

    [There are actually many ordinary crossings: The strongest bound is from Csima and Sawyer (1993), who proved a lower bound of $\lceil 6n/13 \rceil$ for $n > 7$. Can you construct a family of arrangements with growing n and few ordinary points?]

27. Lower envelopes of functions (4 credits)

    Consider n distinct quadratic functions $f_i(x) = a_i x^2 + b_i x + c_i$ and their lower envelope (point-wise minimum) $g(x) = \min\{f_1(x), f_2(x), \ldots, f_n(x)\}$.

    We look at the sequence of the numbers of the functions that form the lower envelope, from left to right (from $x = -\infty$ to $+\infty$). Show that this sequence has the following property:

    (ii) there is no alternation of length four between two numbers. In other words, the sequence does not contain the pattern

    $$\ldots a \ldots b \ldots a \ldots b \ldots$$

    with $a \neq b$.

28. Lower envelopes of functions (6 credits)

    A Davenport-Schinzel sequence of order s is a sequence with the following properties:

    (i) Consecutive elements are distinct.

    (ii) There is no alternation of length $s + 2$ between two symbols.

    Let w be a Davenport-Schinzel sequence of order s over the elements $1, \ldots, n$. Construct n continuous functions $f_i \colon \mathbb{R} \to \mathbb{R}$ such that $|\{\, x \mid f_i(x) = f_j(x) \,\}| \leq s$ for all $i \neq j$ and the sequence of functions along the lower envelope is w.

29. Davenport-Schinzel sequences (10 credits)

    (a) Show that a Davenport-Schinzel sequence of order 2 with n symbols has length at most $2n - 1$, and this bound is tight.

    (b) Show that the length of a Davenport-Schinzel sequence of order s with n symbols is bounded in terms of n and s.