Discrete Geometry, SS 2011 — exercise sheet 5

due date: Tuesday, May 17th, 2011, 14:00

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21. The unbalanced case of the Szemerédi-Trotter bound (0 credits)

    Prove that $I(n, n^{0.8}) = O(n^{1.2})$ by using an $\frac{1}{r}$-cutting with an appropriate choice of r, and applying the weak incidence bounds $I(m, n) \le \min\{n\sqrt{m} + m, m\sqrt{n} + n\}$.

22. Cells in arrangements (10 credits)

    (a) Show that n lines partition the plane into at most $1 + \binom{n+1}{2}$ cells, with equality for general position.

    (b) What is the maximum number of bounded edges and unbounded rays in such a partition? What is the number of vertices?

    (c) What is the number of cells that have no bottom-most point? (Under the assumption of general position). What is the number of cells that have a bottom-most point?

    (d) What are the analogous bounds on the number of vertices, edges, faces, and cells for planes in 3-space?

23. Incidences (10 credits)

    (a) Show that m points and n unit circles in the plane have at most $O(m\sqrt{n} + n)$ incidences.
    Hint: Count pairs of points on each circle versus all pairs of points.

    (b) Show that m points and n arbitrary circles in the plane have at most $O(\min\{n\sqrt{m}+ m, mn^{2/3} + n\})$ incidences.

24. Powers of sums of independent Bernoulli random variables (10 credits)

    Let $X = X_1 + X_2 + \cdots + X_n$ be the sum of n independent random variables, each taking the value 1 with probability p and the value 0 with probability $1 - p$.

    (a) Calculate the expected value $E[X^2]$.

    (b) Show that, for an integer $d \ge 0$,

    $$E\left[\binom{X}{d}\right] = p^d \binom{n}{d}.$$

    (You can try the proof by a combinatorial interpretation.)

    (c) Conclude that $E[X^d] \le C_d (np)^d$ whenever $np \ge d$, for a suitable constant C_d.