Discrete Geometry, SS 2011 — exercise sheet 3

due date: Tuesday, May 3rd, 2011, 14:00

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14. Duality (10 credits)

    (a) (4 credits) What is the set of lines dual to the points of a line segment? (See Exercise 5 of sheet 1.)

    (b) (6 credits) Consider an x-monotone concave polygonal chain C. (Concave means that the slope (=derivative) decreases from left to right.) The duals of the lines through the edges of C form a point set in the dual plane. Characterize the point sets that are obtained in this way.

    (c) (0 credits) The same question for a convex polygon C. Assume that C has no vertical edges.

15. Many point-line incidences (10 credits)

    (a) (4 credits)
    For a natural number k, take the grid $P = \{0, 1, \ldots, k-1\} \times \{0, 1, \ldots, 4k^2 - 1\}$ and the set L of lines with equations $y = ax + b$, for $a = 0, 1, \ldots, 2k-1$ and $b = 0, 1, \ldots, 2k^2 - 1$. Show that $|L| = |P| =: n$, and show that there are $\Omega(n^{4/3})$ pairs $(p, \ell) \in P \times L$ with $p \in \ell$.

    (b) (0 credits) How do these points and lines look in the dual plane? (See Exercise 5 of sheet 1.)

    (c) (6 credits) Modify the above example to obtain for all m, n with $m \le n^2$ and $n \le m^2$ a set P of n points and a set L of m lines that has $\Omega(m^{2/3}n^{2/3})$ pairs $(p, \ell) \in P \times L$ with $p \in \ell$.

16. Many regular polygons (10 credits)

    (a) (1 credit) Find point sets with n points that contain $\Omega(n^2)$ squares (i. e., quadruples of points forming the vertices of a square).

    (b) (3 credits) Find point sets with n points that contain $\Omega(n^2)$ regular triangles.

    (c) (3 credits) Let $\omega = e^{2\pi i/5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, and consider the set

    $$P_m := \left\{\, a + b\omega + c\omega^2 + d\omega^3 + f\omega^4 \mid a, b, c, d, f \in \{0, 1, \ldots, m\} \,\right\}$$

    Draw the set P_1 in the complex plane. (If you want, you can also draw $P_2, P_3, \ldots$, possibly with the help of a computer.) Prove that P_m contains only $n = O(m^4)$ points.

    (d) (0 credits) Prove that P_m contains $n = \Omega(m^4)$ points.

    (e) (3 credits) Prove that P_m contains $\Omega(m^8) = \Omega(n^2)$ regular pentagons.

    (f) (1 credit) Find point sets with n points that contain $\Omega(n^2)$ regular hexagons.

    (g) (0 credits) Are there point sets with n points that contain $\Omega(n^2)$ regular 7-gons?

    (h) (0 credits) For any point pattern H, prove that a set of n points in the plane contains at most n^2 subsets that are similar to H (obtained by translation, rotation, and scaling, and possibly reflection).

17. Many equal distances (0 credits)

    Prove that a d-dimensional point set where all pairwise distances are equal has at most $d + 1$ elements.