Discrete Geometry, SS 2011 — exercise sheet 2

due date: Tuesday, April 26th, 2011, 14:00

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9. (8 credits) Prove that the maximum number of halving edges in a set of $2n$ points in the plane increases strictly with n.

10. (9 credits) The recursive construction of a point set with $\Omega(n \log n)$ halving edges given in the lecture places three squeezed copies of a construction with $n/3$ points close to three rays emanating from a common center.

    Interpret this construction in the dual setting (see Exercise 5 of sheet 1), and show how to construct arrangements of n lines whose middle level has $\Omega(n \log n)$ vertices.

11. (13 credits)

    This exercise gives an alternative inductive construction for point sets with $\Omega(n \log n)$ halving edges, due to Erdős, Lovász, Simmons, and Straus.

    We start with a set P of n points that have f_n halving edges, and an injective mapping that assigns to every point $p \in P$ a halving edge $h(p)$ incident to p.

    (a) Find such a set with $n = 6$ points. (Why can there be no such set with $n = 2$ or $n = 4$ points? Can you find such a set with $n = 8$ points?)

    (b) The inductive construction makes a new set P' with $2n$ points, by replacing each point $p \in P$ by two points on the line through $h(p)$. Their positions on $h(p)$ are very close to p, on opposite sides of p.

    (c) Draw a figure.

    (d) Show that each halving edge $h(p)$ gives rise to three halving edges in P'. Each halving edge that is not one of the n edges $h(p)$ gives rise to two halving edges in P'. (Can P' have additional halving edges that are not included in these cases?)

    (e) Complete the argument for the inductive construction, and set up the recursion for f_n.

    (f) Conclude that there are point sets with $n = 3 \cdot 2^k$ points and (at least) $3(k+1) \cdot 2^{k-1} = \Omega(n \log n)$ halving edges, for $k \geq 1$.

12. (0 credits) Which of the above two constructions (Exercise 10 or Exercise 11) gives a better constant factor for the asymptotic $\Omega(n \log n)$ lower bound?

13. Perfect cross-matchings (0 credits)

    A *perfect cross-matching* of an even set of points is a partition into edges such that any pair of such edges crosses.

    Prove that a point set with $2n$ points has a perfect cross-matching if and only if it has exactly n halving edges.