Emo HM

Rotating a convex Polygon

in

a convex Polygon

Diplomarbeit vorgelegt von Roman Waupotitsch

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Graz, im Mai 1988

Ich versichere, diese Arbeit selbststaendig verfasst, andere als die angegebenen Hilfsmittel nicht benutzt und mich auch sonst keiner unerlaubten Hilfsmittel bedient zu haben.

Table of contents

1. Introduction

Let P and Q^o be two convex polygons in the plane. Suppose now that polygon P is fixed, and that Q^o may be varied by rotation, translation and expansion in size.
The images of Q^o with respect to the three operations are called rotation-translation-expansion-images of Q (RTE - images for short). An RTE-image Q is said to be in allowed position if no point of Q lies outside of P.
The primary goal of this paper is to find that RTE-image Q in allowed position that realizes the largest expansion of Q^o. We will call such an RTE-image RT - optimal in P.
To this end we consider the following idea. First consider a T - optimal image Q_o^* of Q^o, i.e. a displacement of Q^o in allowed position with maximal expansion factor. We will see that Q_o^* is unique, provided that no two edges of P are parallel, see figure 1.

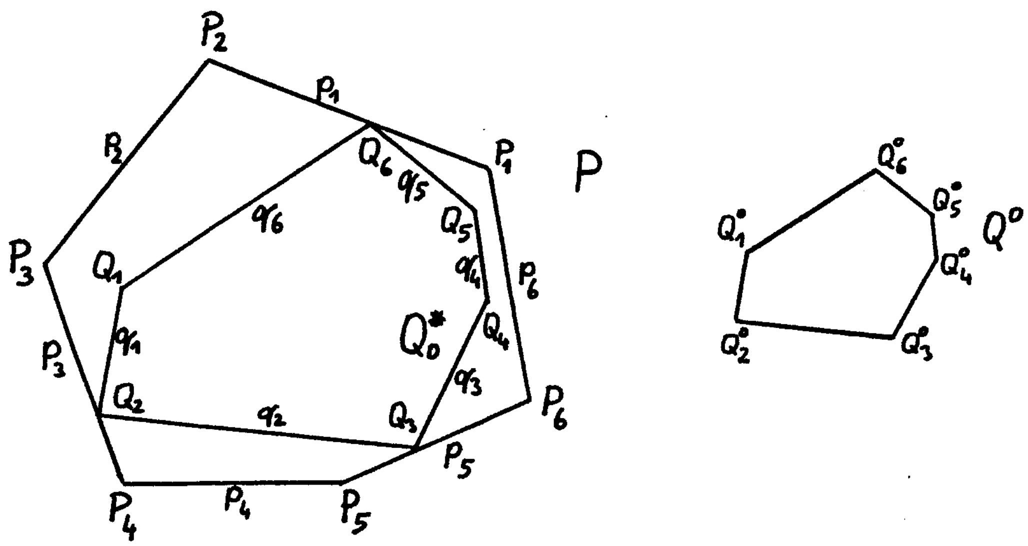


figure 1

Then we start to rotate Q_o^*, while (i) keeping it in allowed position, and (ii) expanding it as much as possible. In other words, for every image Q_φ obtained from Q^o by rotation around the origin by any angle Θ in the interval $[0,2\pi)$, we look for the T-optimal image Q^*_Θ of Q_Θ (we use $[a,b)$ for real numbers a and b to denote the set of real numbers c with $a \leq c < b$). We will see that the range $0 \leq \Theta < 2\pi$ can be partitioned into <u>rotation intervals</u> $[\Theta',\Theta'')$ such that all images Q^*_Θ, $\Theta' \leq \Theta \leq \Theta''$, can be obtained from $Q^*_{\Theta'}$ by rotation and expansion around a fixed center $M(\Theta',\Theta'')$ (without translation!). This center $M(\Theta',\Theta'')$ will be called the <u>rotation center</u> of Q^o for the interval $[\Theta',\Theta'')$.

In an algorithmic treatment we will see that we can proceed from rotation interval to rotation interval and that the number of rotation intervals is bounded by $O(kn^2)$ (where k is the number of vertices in Q^o and n is the number of vertices in P). The algorithm for "rotating" Q inside P has a space requirement proportional to (k + n) and a time complexity $O(k^2n^2 \log n)$ which means that we can solve the following problems within these bounds:

(1) Find a (or all) RT-optimal images of Q^o in P.

(2) Construct a datastructure for Q^o and P, such that for every angle of rotation Θ of Q^o the T-optimal image Q^*_Θ can be computed in $O(\log kn)$ time.

(3) Find the largest expansion of Q^o that can be completely rotated inside P (using only translation without changing its size).

The first problem has been studied before by Sharir [SH] who gives a $O(kn^2 * \log^2 kn)$ solution which outperforms our result unless k is very small, i.e. unless $k \in O(\log n)$. This is true for example, for constant k or n.

It is not clear whether or not Sharir's algorithm is linear in space (i.e. $O(k + n)$); because in the preversion of his paper [SH] he assumes that he will use a divide and conquer algorithm, and for reasons which will be more reasonable later in this paper, we expect a $O(kn^2)$ solution.

One section of this paper is dedicated to a paper of Bernard Chazelle ([CH]) in which a optimal solution for the translation-rotation restricted problem (i.e no expansion of polygon Q is allowed) is shown. The reason why we present this paper is, that Chazelle uses two very powerful means of the computational geometry, namely, the "duality", i.e. the transformation of E^2 into a dual space, in which every line of E^2 is represented as point; and the "Divide and Conquer" method.

2. Preliminary definitions and remarks

First we will introduce some terminology related to the geometric objects we are talking about.

definition 1: We define a *point* as a pair $V = (x,y)$ of coordinates in the plane E^2. A line segment v is the convex hull of two points $V_1 = (x_1,y_1)$ and $V_2 = (x_2,y_2)$, i.e., the set $v = \{\lambda(x_1,y_1) + (1+\lambda)(x_2,y_2) \mid 0 \leq \lambda \leq 1\}$. V_1 and V_2 are called the endpoints of the line segment v.

All along this paper we will use capital letters for points and lower case letters for line segments.

definition 2: A *polygon* R is a sequence $(V_1,\ldots,V_m)$, $m \geq 3$, of pairwise different points. V_i, $1 \leq i \leq m$, are called the *vertices* of R and the line segments v_i with endpoints V_i and V_{i+1} (for $1 \leq i \leq m-1$) or V_m and V_1 (for $i = m$) are called the *edges* of polygon R. R is a *convex polygon* if for all edges v_i, $1 \leq i \leq m$, all vertices of R except for the endpoints of v_i, lie on one side of the line through v_i.

definition 3: All along this paper Q^o will be a convex polygon with k vertices $Q_1^o,\ldots,Q_k^o$ and edges $q_1^o,\ldots,q_k^o$. P will be a convex polygon with n vertices $P_1,\ldots,P_n$ and edges $p_1,\ldots,p_n$. Both, the vertices of Q and the vertices of P are sorted in counterclockwise order on the boundary of polygon Q, and P respectively.

For the sake of simplicity we assume that no two edges of P are parallel.

definition 4: Let h_i be the closed halfplane which is bounded by line l_i - this is the line through edge p_i - and contains all vertices of polygon P. We will say that a point G lies inside P ($G=(x_G,y_G) \in P$) or a set N of points lies inside P ($N=\{N_1,N_2\dots\} \subseteq P$) if point G, or every point of set N respectively lies in the intersection H_p of the halfplanes h_1 to h_n i.e. $H_p = h_1 \cap h_2 \cap \dots \cap h_n$.

definition 5: We can apply one or several of the following transformations concurrently on polygon Q^o (the mathematical operations on each vertex of Q^o will be described for vertex $Q_i^o = (x_i^o, y_i^o)$):

(a) *T - transformation:* Q^* is a *translation image* (*T - image* for short) of Q^o, if $Q^* = Q^o + v^o$. ($Q_i^* = Q_i^o + (\Delta x^o, \Delta y^o) = (x_i^o+\Delta x^o, y_i^o+\Delta y^o)$).

(b) *R - transformation:* Q_θ is the *rotation image* (*R - image*) of Q^o, if $Q_\theta = R^o(0,\theta)Q^o$ and $R^o(0,\theta)$ is the rotation-matrix for a rotation around the origin by angle θ, $0 \leq \theta < 2\pi$

$$(Q_{i\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} * Q_i^o =$$

$$= (x_i^o * \cos\theta - y_i^o * \sin\theta,\ x_i^o * \sin\theta + y_i^o * \cos\theta)).$$

(c) *E - transformation:* Q' is the *expansion image* (*E - image*) of Q°, if $Q' = \mu^{\circ}(0)Q^{\circ}$ and $\mu^{\circ}(0)$ is the expansion factor for the expansion of Q° around the origin. ($Q_i' = \mu(0)Q_i^{\circ} = (\mu^{\circ}(0)x_i^{\circ}, \mu^{\circ}(0)y_i^{\circ})$).

Every polygon $Q^{*}_{\Theta}{}'$ of the form $Q^{*}_{\Theta}{}' = \mu^{\circ}(M)R^{\circ}(M,\Theta)Q^{\circ} + v^{\circ}$ will be called a *RTE - image* of Q°. $\mu^{\circ}(M)R^{\circ}(M,\Theta)Q^{\circ}$ is the *RE-image* of polygon Q° when Q° is RE-transformed *around* M, with $\mu^{\circ}(M)R^{\circ}(M,\Theta)Q^{\circ} = (\mu^{\circ}(0)R^{\circ}(0,\Theta))(Q^{\circ}-M) + M$.
And therefore the complete formula for the RTE-image of vertex Q_i° is $Q^{*}_{\Theta}{}' = (x^{*}_{\Theta i}{}', y^{*}_{\Theta i}{}')$ with

$$x^{*}_{\Theta i}{}' = \mu^{\circ}(0) * ((x_i^{\circ} - x_M) * \cos\Theta - (y_i^{\circ} - y_M) * \sin\Theta) + x_M + \Delta x^{\circ}$$ and

$$y^{*}_{\Theta i}{}' = \mu^{\circ}(0) * ((x_i^{\circ} - x_M) * \sin\Theta + (y_i^{\circ} - y_M) * \cos\Theta) + y_M + \Delta y^{\circ}.$$

We will speak of the *orientation* of polygon Q when we mean the angle on which polygon Q° has been rotated to obtain $Q^{*}_{\Theta}{}'$.

We will write Q_{Θ} or Q instead of $Q^{*}_{\Theta}{}'$ if the meaning is clear.

We are now able to describe how polygon $Q^{*}_{\Theta}{}'$ can be calculated from Q° by simultaneously applying v°, $R^{\circ}(M,\Delta\Theta)$ and $\mu^{\circ}(M)$ on Q°. But as we will see later in this paper we will have to calculate various RTE-images of Q° for various orientations Θ. And this RTE-images will not be calculated by applying v°, $R^{\circ}(M,\Delta\Theta)$ and $\mu^{\circ}(M)$ on Q° (except for the first one), but by applying the corresponding function v, $R(M,\Delta\Theta)$ and $\mu(M)$ to an other RTE-image.

definition 6: Assume that we are given a polygon $Q^{*}_{\Theta'}$ at orientation Θ'.
We define

(i) $R(M_{\Theta',\Theta''},\Delta\Theta)$ as the rotation of polygon $Q^{*}_{\Theta'}$ to polygon $Q^{*}_{\Theta'+\Delta\Theta}$ around the not yet defined rotation center $M_{\Theta',\Theta''}$.

(ii) $\mu(M_{\Theta',\Theta''})$ as the quotient of the expansion of Q at orientation $\Theta'+\Delta\Theta$ divided by the expansion of Q at orientation Θ'.

(iii) v determines the vector for which polygon $Q^{*}_{\Theta'+\Delta\Theta}$ is T- transformed after rotation and expansion.

For a detailed mathematical description replace μ°, R°, v°, and Q° in definition 5 by μ, R, v, and $Q^{*}_{\Theta'}$ respectively.

definition 7: If vertex Q_i of Q is in contact with edge p_j of P, i.e. $Q_i \in p_j = \{\lambda(x_j,y_j) + (1-\lambda)(x_j,y_j) \mid 0 \leq \lambda \leq 1\}$, it is said to make an *obstacle contact $C_{i,j}$* with that edge.
An obstacle contact of vertex Q_i of Q with vertex P_j of P (i.e. $(x_i,y_i) = (x_j,y_j)$) is called a *double obstacle contact $DC_{i,j}$* and will be counted as two obstacle contacts, one for edge p_{j-1} and the second for p_j.

definition 8: Polygon Q is said to be in *optimal position or T - optimal*, see figure 1, for any orientation Θ of Q and under the assumption that no two edges of P are parallel if

(i) the expansion μ is maximal for orientation Θ and an arbitrary translation v under the condition that

(ii) Q is in *allowed position*, i.e. that each vertex of polygon Q is inside polygon P ($Q \subseteq P$).

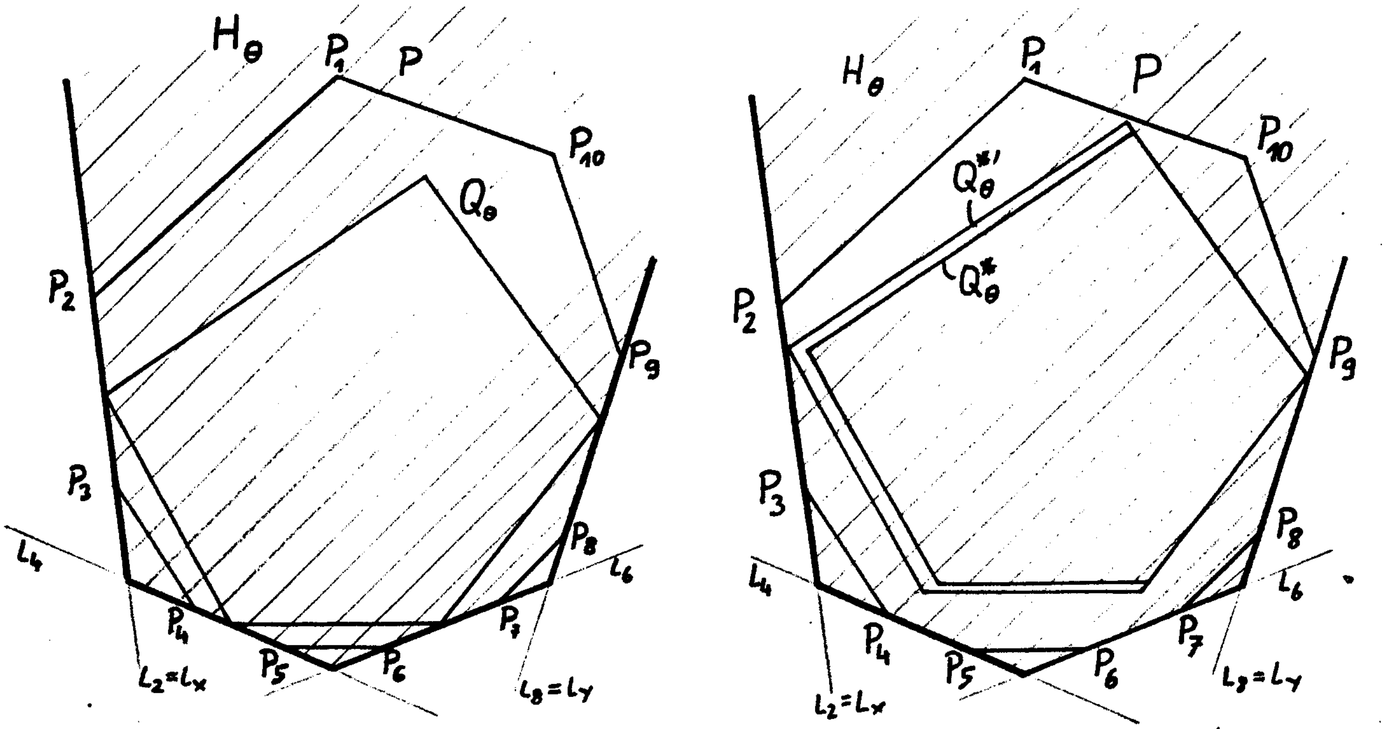
From definition 8 we can conclude that a T-optimal polygon Q cannot be translated (T- transformed) such that it is in allowed position after translation. But this immediately implies that the T-optimal position of polygon Q, for every orientation Θ is unique.

lemma 1: Assume that polygon Q is in T-optimal position in polygon P and Θ is the orientation of Q. We define H_Θ as the intersection of those closed halfplanes h_i (h_i defined as above) for which the corresponding edge p_i makes an obstacle contact with a vertex of Q at orientation Θ.
We claim that this area H_Θ must be closed for Q being T-optimal.

proof: We assume that Q is T-optimal and H_Θ is not closed.
Let L_Θ be the set of lines l_i through those edges p_i which make the obstacle contacts with vertices of Q. We take those two lines l_x and l_y of L_Θ which are not intersected by lines of L_Θ at both sides of that point at which a vertex of polygon Q makes its

obstacle contact with them (see figure 2).
It is easy to verify that polygon Q can be T-transformed parallel to either l_x or l_y for any sufficiently small Δv (choose Δv such that Q is in allowed position after the movement) and afterwards E-transformed. But this is a contradiction to our assumption (see figure 2).

□



$L_\theta = \{l_2, l_4, l_6, l_8\}$, $H_\theta = h_2 \cap h_4 \cap h_6 \cap h_8$

figure 2

It is obvious that the smallest set L_θ which implies a closed area H_θ contains three lines (i.e. the ordinal of L_θ is 3, $|L_\theta| = 3$), and results in a triangle shaped H_θ.

<u>*corollary 2:*</u> If polygon Q is in T-optimal position in polygon P it makes at least three obstacle contacts.

If polygon Q makes three obstacle contacts ($|L_\Theta| = 3$), non of which is a double obstacle contact it is said to be in *general position*.

> problem-definition: Rotate (RE-transform) a convex polygon Q^0 inside a convex polygon P by an angle of 2π such that the RTE-image Q_Θ' is in T-optimal position for every orientation Θ, with $0 \leq \Theta \leq 2\pi$.

In the first moment it seems that this problem cannot be solved efficiently since an infinite number of various orientations of polygon Q exist. But we have found a method with which the range $[0,2\pi]$ can be partitioned into a finite number of rotation intervals such that we can rotate polygon Q around a fixed rotation center M for each interval. We will see that for any orientation inside of such a rotation interval the obstacle contacts do not change (i.e. no vertex of Q which makes an obstacle contact escapes from its edge during rotation, and Q makes no additional obstacle contact with P). With help of this rotation center we do not need to T-transform polygon Q, except in the initial part. We will see in the following section that a unique rotation center can only be found for situations where polygon Q makes 3 obstacle contacts (i.e. $|L_\Theta| = 3$), except in special cases.

3. Calculating the rotation center

definition 9: Let $Q_{\Theta'}$ be in T-optimal position in polygon P at orientation Θ'. As we have seen in corollary 2 polygon $Q_{\Theta'}$ makes three or more obstacle contacts with P. We define $M_{\Theta'\Theta''} \in E^2$ (not necessarily a point inside P) as the rotation center for the interval $[\Theta',\Theta'']$ if $Q_{\Theta'}$ can be *RE-transformed* around $M_{\Theta'\Theta''}$ in the interval of orientation $[\Theta',\Theta'']$, with $\Theta' < \Theta'' \leq 2\pi$, such that Q_Θ is T-optimal for every orientation Θ with $\Theta' \leq \Theta \leq \Theta''$.

note: We do not apply any T-transformation on Q!

In the remainder of this paper we will use the short form M instead of $M_{\Theta'\Theta''}$ if the boundaries of the interval $[\Theta',\Theta'']$ are unique.

observation 3: During an RE-transformation of polygon Q around M, as described above, each vertex of Q which makes an obstacle contact with an edge p_j of P will slide along the boundary of P (especially along edge p_j) either in clockwise or counterclockwise direction. The expansion $\mu(M)$ from $Q_{\Theta'}$ to $Q_{\Theta''}$ is thereby determined by the prolongation (or shortening respectively) of the euclidian distance $d(M,Q_{\Theta' i}) = ((x_M-x_i)^2+(y_M-y_i)^2)^{1/2}$ with $M = (x_M,y_M)$ and $Q_{\Theta' i} = (x_i,y_i)$. Since this prolongation must be equal for every vertex of Q, not only for those vertices which make an obstacle contact, it is easy to verify that also the vertices of Q which do not make an obstacle contact with P move on an imaginary straight halfline, say i.

Our task is it now to show the construction of a rotation center.
We will therefore proceed in constructing the set of rotation centers for two obstacle contacts C_{ai} and C_{bj}. C_{ai} and C_{bj} are two of the obstacle contacts which polygon Q makes with polygon P. We need not watch the remaining edges and vertices of both polygons for the following studies.

> *note:* We will use the term "rotation center" also in the case when polygon Q makes just two obstacle contacts with two edges of P, although in this situation actually an infinite number of rotation centers exist.

One rotation center for these obstacle contacts can be found if we intersect the perpendicular to p_i in Q_a with the perpendicular to p_j in Q_b.

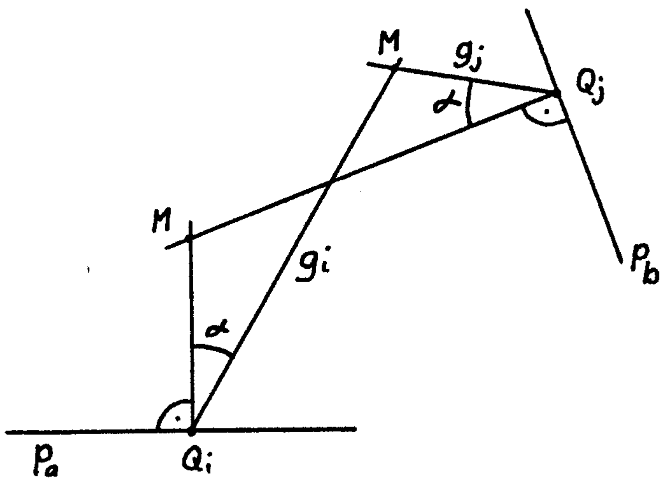


figure 3

But also the point of intersection M of two lines g_i and g_j which have the same angle α measured from the

perpendicular to p_i (p_j) through g_i (g_j), see figure 3 is a rotation center for C_{ai} and C_{bj}.
And therefore we can imply that an infinite number of rotation centers for two obstacle contacts exist. Additionally it can be checked out that the angle of intersection of g_i and g_j is equal for every angle α, and so, with help of the following lemma 4 we can prove that all rotation centers for these two vertices lie on a circle.

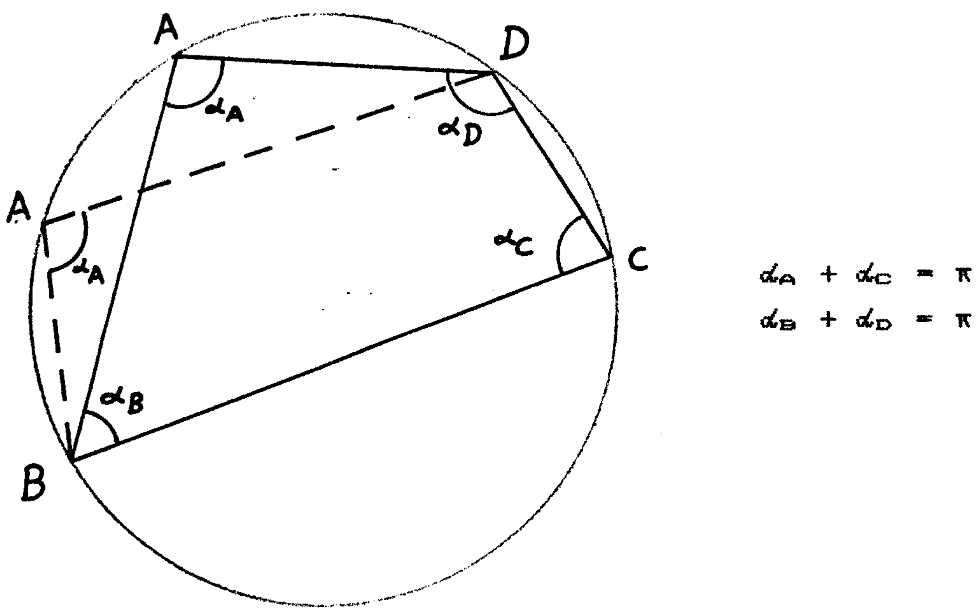


figure 4

The following lemma can be found in every formulary.

lemma 4: A circle which touches the four points of a quadrangle can be drawn, if and only if the sum of the angles of the quadrangle lying diagonally opposite is π (see figure 4).

corollary 5: Let Q be a quadrangle with vertices A, B, C and D and let α_A, α_B, α_C and α_D be the corresponding inner angles of the quadrangle. Furthermore let K be its circumference. Then K remains the circumference if vertex A is moved along the arc of the circle between B and D, and angle α_A does not change during this movement, since angle α_C does not change, see figure 4.
But vice versa assume that we are given three points A, B and D $\in E^2$. Then all other points A' $\in E^2$ which meet the same condition as A (i.e. the angle between the two vectors (AB) and (AD) is equal to the angle between the two vectors (A'B) and (A'D)) lie on a circle .

note: The special case when the angle between (AB) and (AC) is $\pi/2$ is the well known circle of Thales!

The construction of the center of this circle is the same as the construction of the circumference of triangle Δ(ABD).
By exchanging the part of B and D by the two vertices Q_a and Q_b and exchanging the part of A by the rotation center M we can conclude lemma 6:

lemma 6: If we are given two edges of P, p_i and p_j, and two vertices Q_a and Q_b making two obstacle contacts C_{ai}, and C_{bj} respectively we can calculate an infinite number of rotation centers for these two obstacle contacts. These rotation centers lie on a circle which may be constructed in the following way:

a) Intersect the perpendiculars to p_i and p_j in Q_a and Q_b respectively and

b) Calculate the circumference for the three points Q_a, Q_b and the point of the intersection of the perpendiculars.

□

note : In the remainder of this paper we will write $CIRC_{aibj}$ for the circle on which the rotation centers for the two obstacle contacts C_{ai} and C_{bj} lie.

Before we continue with the explanation of the construction of M for three obstacle contacts let us evaluate $\mu(M)$ in dependence of the angle between the vectors (Q_a, M) and (Q_a, P_{j+1}) for the obstacle contacts defined in lemma 5.
Let us therefore give the following definition:

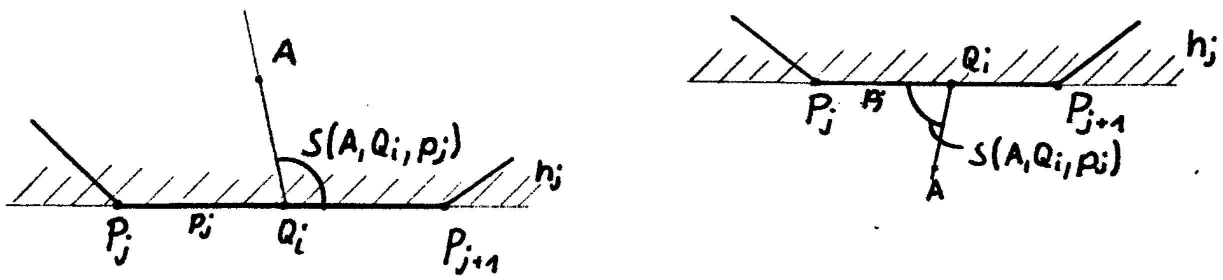


figure 5

definition 10: Let p_j be any edge of P which makes an obstacle contact with vertex Q_i of Q (we assume that Q is in allowed position). We define $S(A, Q_i, p_j)$, with $A \in E^2$ as the angle between the vectors

(i) (Q_i, A) and (Q_i, P_{j+1}) if A lies in h_j or
(ii) (Q_i, A) and (Q_i, P_j) if A lies in the complement area E^2/h_j.

lemma 7: We are given two edges of P, p_i and p_j, and two vertices Q_a and Q_b making two obstacle contacts C_{ai} and C_{bj} respectively. The expansion of Q, which is the prolongation of its edges after the rotation from orientation Θ' to Θ'' is

$$\mu(M) = \sin S(M,Q_i,p_j) \ / \ \sin(\Theta''-\Theta'+S(M,Q_i,p_j))$$

for $0 \leq \Theta''-\Theta' < \pi-S(M,Q_i,p_j)$.

observation 8: In the case when $S(M,Q_i,p_j)$ is less than $\pi/2$ and $(\Theta''-\Theta')$ is less then 2β polygon Q becomes smaller after rotation, and the minimum is received for $(\Theta''-\Theta') = S(M,Q_i,p_j)$.

lemma 9: Let Q be in T-optimal position in P. We claim that the angle $S(M,Q_i,i_i)$ between vector(Q_i,M) and the imaginary halfline i_i is equal to angle $S(M,Q_j,p_k)$ for any vertex Q_i which makes no obstacle contact with P and vertex Q_j which makes the obstacle contact C_{jk}.

proof: In observation 3 we have seen that the prolongation of the distance $d(M,Q_i)$ for every point of Q depends on $\mu(M)$. And this together with the fact that $\mu(M)$ depends on the angle $S(M,Q_i,p_k)$ proves our claim.

Thus we can construct the rotation center for a polygon Q in general and T-optimal position in polygon P in the following way:

algorithm 1: a) For every pair of edges of polygon Q making obstacle contacts with vertices of polygon P find the circles of the rotation center.

b) Find the common point of intersection M of these three circles. I.e. choose that point where all three circles intersect.

observation 10: Let Q be in general and T-optimal position in P and let it make the obstacle contacts C_{ae}, C_{bf} and C_{cg}. W.l.o.g. $1 \leq a < b < c \leq k$ and $1 \leq e < f < g \leq n$. Let M be the rotation center for this situation. And furthermore let h_{xy} be the closed halfplane which (i) is bounded by l_{xy}, this is the line through two vertices of Q making an obstacle contact (i.e. $x,y \in \{a,b,c\}$ and $x <> y$), and (ii) contains the three vertices Q_a, Q_b and Q_c. Let $C(h_{xy})$ be the open halfplane $R^2 \backslash h_{xy}$.

It is easy to see that circle $CIRC_{aebf}$ for the obstacle contacts C_{ae} and C_{bf} can intersect the circle $CIRC_{aecg}$ for C_{ae} and C_{cg} either in the intersection of h_{ab} and h_{ac} or in the intersection of $C(h_{ab})$ and $C(h_{ac})$. Therefore the rotation center M must lie inside of area B (see figure 6a) which has the following definition:

$$B = (h_{ab} \cap h_{ac} \cap h_{bc}) \cup (C(h_{ab}) \cap C(h_{bc})) \cup (C(h_{ab}) \cap C(h_{ac})) \cup (C(h_{bc}) \cap C(h_{ac})).$$

observation 11: Let Q be in general and T-optimal position in P with the obstacle contacts defined as in observation 10. And let X be the point of intersection

of the straight line through Q_a and Q_b and the circle $CIRC_{bfcg}$.
Then we can observe that Q_a lies inside this circle if the angle between the vectors (Q_a, Q_b) and (Q_a, Q_c) is smaller than the angle between the vectors (X, Q_b) and (X, Q_c). And since rotation center M cannot lie inside P if Q_a lies inside $CIRC_{bfcg}$ because $CIRC_{bfcg}$ does not intersect area $h_{ab} \cap h_{ac} \cap h_{bc}$, and since the equality of the angles $S(X, Q_b, p_f) = S(Q_a, Q_b, p_f)$ implies that $S(Q_a, Q_c, p_g) > S(Q_a, Q_b, p_f)$ we can result that M lies inside P if and only if

$S(Q_a, Q_b, p_f) \geq S(Q_a, Q_c, p_g)$ and
$S(Q_b, Q_c, p_g) \geq S(Q_b, Q_a, p_e)$ and
$S(Q_c, Q_a, p_e) \geq S(Q_c, Q_b, p_f)$.

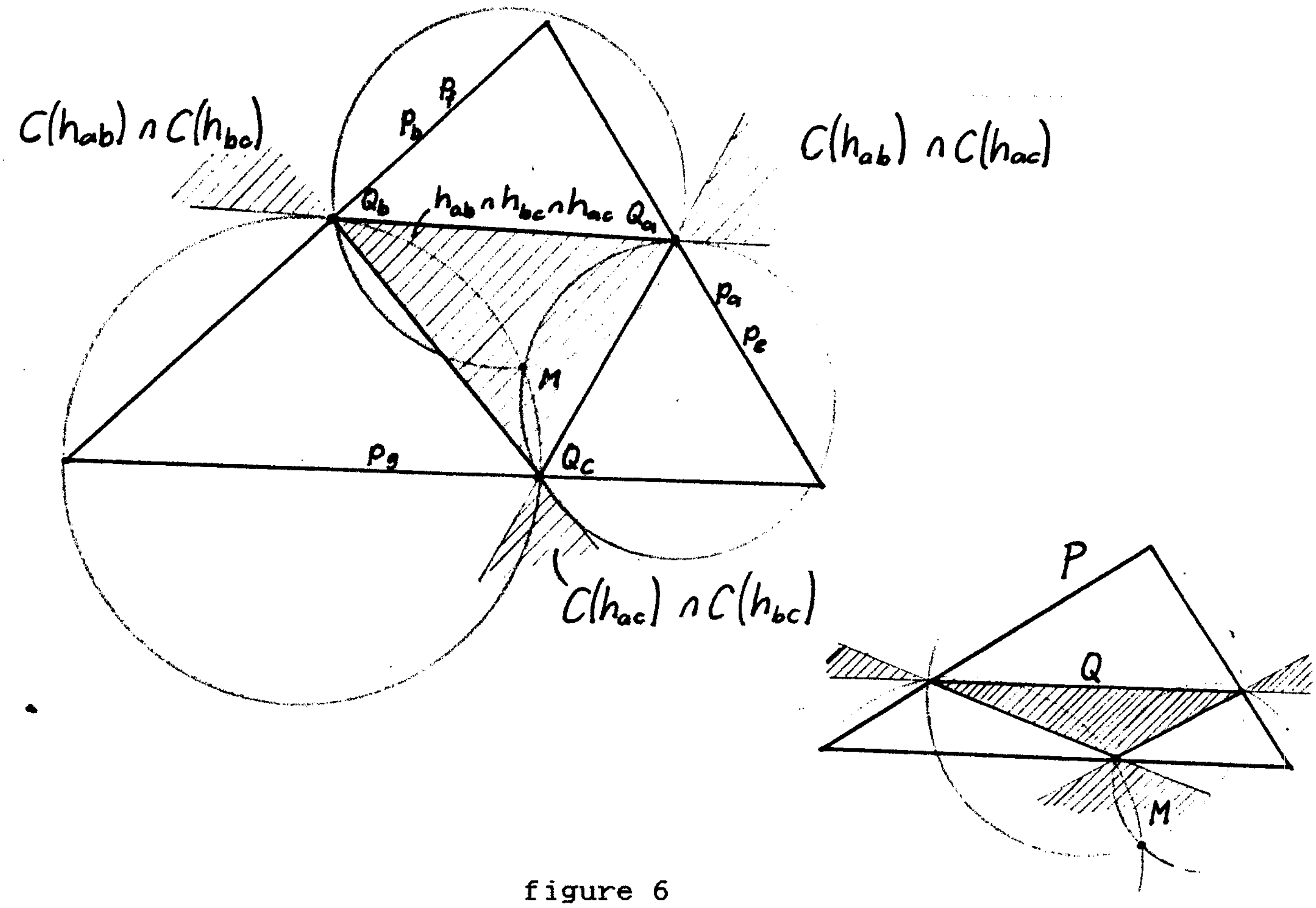


figure 6

In this case every vertex moves in counterclockwise order on the boundary of P. Otherwise M is in the exterior of P, see figure 6b, and that vertex for which the factor S does not meet the unequality moves in clockwise order.

lemma 12: Let Q be in general and T-optimal position in P and let it make the obstacle contacts C_{ae}, C_{bf} and C_{cg}. W.l.o.g. $1 \leq a < b < c \leq k$ and $1 \leq e < f < g \leq n$. Furthermore let M be the rotation center for this situation.
Then we claim that only one point of $\{Q_a, Q_b, Q_c\}$ can move in clockwise direction on the boundary of P when polygon Q is RE-transformed around M.

proof: W.l.o.g we assume that Q_a moves in clockwise direction. This implies that $S(Q_a, Q_b, p_f) < S(Q_a, Q_c, p_g)$. Let us now assume that Q_c is the second point that moves clockwise. Then $S(Q_c, Q_a, p_e)$ must be smaller than $S(Q_c, Q_b, p_f)$. On the other hand $S(Q_c, Q_a, p_e)$ must be bigger than $S(Q_a, Q_c, p_g)$ because otherwise the edges p_e, p_f, p_g would not form a triangle. But this implies that $S(Q_a, Q_b, p_f) < S(Q_a, Q_c, p_g) < S(Q_c, Q_a, p_e)$ and therefore $S(Q_a, Q_b, p_f) < S(Q_c, Q_b, p_f)$. But since $S(Q_a, Q_b, p_f)$ is obviously bigger than $S(Q_c, Q_b, p_f)$ this is a contradiction.
With the same argumentation we can prove that the lemma is also true if we assume that Q_b moves clockwise.

□

We have seen that for three obstacle contacts a unique rotation center M can be found. However, this is not possible for four obstacle contacts (except in special cases).

How can such a "four obstacle contact situation" occur?

Let us assume that Q is in general and T-optimal position in P, and let M be the rotation center, and Θ' be the orientation of Q. If we now RE-transform polygon Q around M one of the following events will happen:

a) One of the vertices of Q which was not in contact with a edge of P at orientation Θ', say Q_i, moves on his imaginary line i_i closer and closer to an edge of P, say p_j and, finally, makes an obstacle contact with p_j at orientation Θ''. The point where Q_i makes it obstacle contact with p_j is the intersection of halfline i_i with edge p_j.

b) One of the vertices of Q which is one of the three obstacle contact at orientation Θ' (and also during the rotation) reaches the end of the edge with which it is in contact with before situation a) occurs. In this case it makes a double obstacle contact.

In the moment when Q makes a fourth obstacle contact with P the rotation has to be interrupted because in both described situations polygon Q would not be in allowed position for any further RE-transformation around M.

In the following we will describe how the rotation center can be calculated in all possible "four obstacle contact situations".

observation 13: Assume that Q is in general and T-optimal position. Let C_{ae}, C_{bf} and C_{cg} be the three obstacle contacts that the vertices of polygon Q make with edges of polygon P, with $1 \leq a < b < c \leq k$ and $1 \leq e < f < g \leq n$.

1) A fourth vertex of Q, say Q_d makes the fourth obstacle contact with with p_j. For these four obstacle contacts we cannot find an unique rotation center. Therefore we must omit one of the obstacle contacts for the calculation of M.

   a) Q_d makes an obstacle contact with an edge already involved in another obstacle contact, w.l.o.g. we assume $p_j = p_f$. In this case it is obvious that Q_b and Q_d are vertices adjacent to the same edge of Q.
   The new rotation center must be calculated for the three obstacle contacts C_{ae}, C_{df}, and C_{cg}, see figure 7.

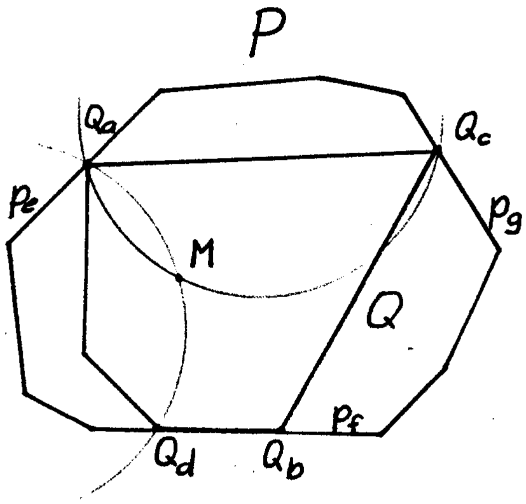


figure 7

   b) W.l.o.g $f < j < g$.
   The new rotation center cannot be M_{fgj} (this is the rotation center calculated for the obstacle contacts on edges p_f, p_g and p_j), since rotating Q on M_{fgj} would have vertex Q_a leaving the

interior of polygon P (i.e. $S(M_{fgi}, Q_b, p_f) > S(M_{fgi}, Q_a, p_e)$).
If we decide to use M_{egi} as new rotation center, Q_b will escape from p_f and polygon Q can be translated (T-transformed) in P and therefore rotating on M_{egi} does not meet the condition of optimality.
The resulting decision for the rotation center is M_{efi} and we can formulate the general condition for the calculation of M for this kind of a four obstacle contact situation:

> Select those three edges of L_e for which the intersection of the corresponding area H_e forms a triangle, and the edge making the new obstacle contact is one of them.

2) W.l.o.g. assume that vertex Q_a moves in counterclockwise direction (i.e. $S(Q_a, Q_b, p_f) > S(Q_a, Q_c, p_g)$) and makes a double obstacle contact with P_{e+1}.

   a) The point of intersection of the three halfplanes $h_{e+1} \cap h_g \cap h_f$ forms a triangle:
   In this case the vertex Q_a is not further considered to be in contact with p_e. And we calculate the common rotation center M_{e+1fg}.

   b) otherwise:
   We choose vertex Q_a as new rotation center M, see figure 8. It can easily be verified that RE-transforming Q on M keeps the optimality of Q (vertex Q_b escapes from p_f).

note: If $S(Q_a, Q_b, p_f) < S(Q_a, Q_c, p_g)$ in a) p_{e+1} must be replaced by p_e and in b) P_{e+1} must be replaced by P_e.

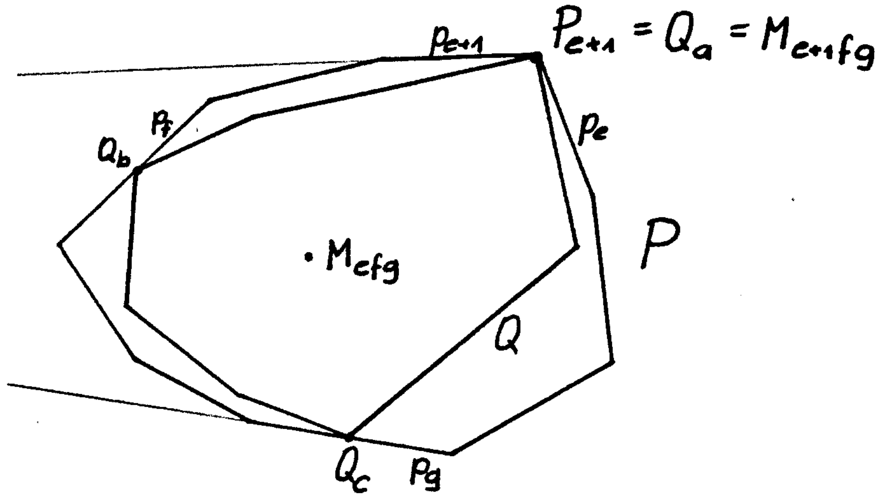


figure 8

3) One vertex of polygon Q which is also the rotation center, say Q_m, is in contact with vertex P_m of Polygon P. Q_b is in contact with p_f. During RE-transformation vertex Q_b reaches the end of p_f and makes also a double obstacle contact with P_{f+1}, see figure 9. Choose Q_m as new rotation center if $S(Q_m, Q_b, p_{f+1}) > S(Q_b, Q_m, p_{m+1})$. Choose Q_b otherwise.

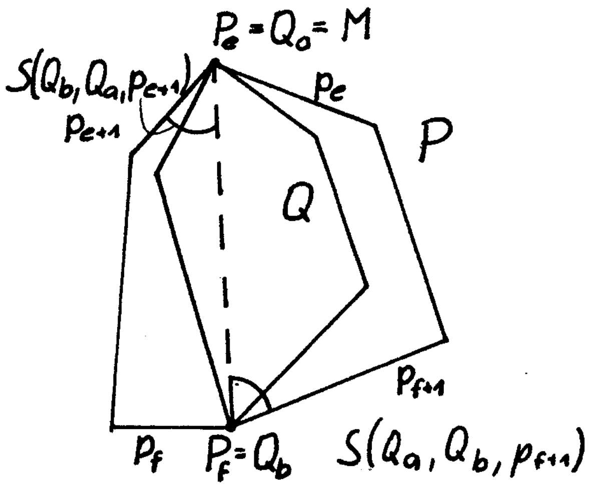


figure 9

observation 14: During the RE-transformation it is not possible that polygon Q is in T-optimal position in P and makes more than four obstacle contacts, because, as we have seen in observation 13 every four obstacle contact situation does just exist four one orientation of Q, not for an interval of orientation like three obstacle contacts.

definition 11: A *rotation interval RI* is the interval of orientation in which one triple of obstacle contacts does not change when we apply an RE-transformation on Q.

4. On the number of various rotation intervals

definition 12: Let l_i be the line through p_i and let Q_m and Q_{m+1} be two consecutive vertices of Q. Q_m and Q_{m+1} being in contact with p_i. W.l.o.g assume that Θ is the orientation of Q at which both Q_m and Q_{m+1} can be in contact with l_i. Furthermore let α be the angle between the vectors $(Q_m P_i)$ and $(Q_m Q_{m-1})$. I_{mi} is the interval of orientation $[\Theta, \Theta+\alpha]$ in which vertex Q_m can be in contact with l_i and polygon Q lies inside of h_i.

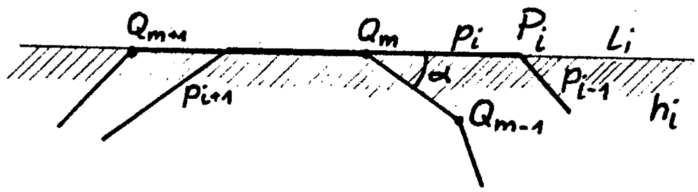


figure 10

definition 13: Let C_{mi} be an obstacle contact of vertex Q_m of polygon Q with edge p_i of polygon P, and let p_j be any other edge of P.
R_{mij} is the set of vertices which contains all those vertices Q_b of Q for that the intersection of the interval of orientation I_{bj} with the interval I_{mi} is not empty:

$$R_{mij} = \{Q_b \in Q \mid I_{bj} \cap I_{mi} <> 0\}.$$

definition 14: If a vertex Q_i of polygon Q makes an obstacle contact with an edge p_i and simultaneously vertex Q_j is in contact with p_m these contacts are said to form a *pair of obstacle contacts*.

lemma 15: Let C_{ai} be an obstacle contact and let p_j be any other edge of P. Additionally let R_{aij} be the set of vertices defined as in definition 13.

a) R_{aij} is a set of consecutive vertices of polygon Q.
b) The intersection of the sets R_{aij} and R_{bij}, with Q_b any other vertex of Q, is either empty or contains one vertex.

proof:
ad a) Trivially true.
ad b) If vertex Q_c is a vertex of R_{aij} and the interval I_{cj} is completely contained in I_{ai}, I_{cj} can obviously not intersect any other interval I_{bi}, for $Q_b \in Q \setminus \{Q_a, Q_c\}$ (figure 11a).

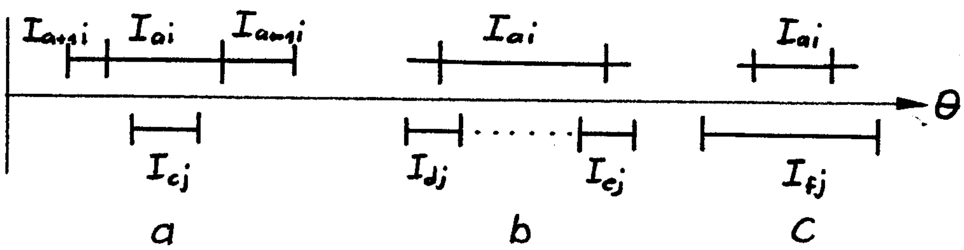


figure 11

On the other hand at most two intervals, say I_{dj} and I_{ej} can overlap I_{ai} (one at the upper bound of I_{ai}, and the second on the lower bound). And since I_{bi} can just intersect either I_{dj} or I_{ej} (see figure 11b) also in this case the lemma is true.
The third case where $I_{ai} \subseteq I_{fi}$ is a special case of the previous one (figure 11c).

□

lemma 16: For each pair of edges of P at most O(k) pairs of obstacle contacts can be formed. And therefore the maximum number of various double obstacle contact situations during rotation is $O(kn^2)$.

proof: Since each of the vertices of polygon Q can be in contact with an edge of P. And since each vertex implies an appropriate set of vertices R_{aij} for any edge p_i of P, an intuitive number of $O(k^2)$ pairs of obstacle contact could be assumed.
The sum of obstacle contacts for a pair of edges, say p_i and p_j is equal to the sum of vertices in the sets R_{xij}, with x = 1,...,k. The number of vertices for each R_{xij} can be split into the number of vertices for which the interval I_{yj} is completely contained in I_{xi} and those for which the interval I_{yj} additionally intersects the intervals I_{zi}, with $z \in \{1,\ldots,k\}\setminus\{x,y\}$ of one or more other sets R_{zij}. With help of lemma 15 we see immediately that the sum of vertices of the first kind is less than k. And since k sets R_{xij} for a fixed pair of edges p_i and p_j exist, each with at most two vertices of the second kind we can result that $\Sigma_{x=1..k}|R_{xij}| \leq 3k$, where $|R_{xij}|$ is the number of vertices of R_{xij}. This and the fact that n(n-1)/2 pairs of edges of P exist proves the lemma.

□

definition 15: Let Q be a polygon in general and T-optimal position at orientation Θ, and let C_{ae}, C_{bf} and C_{cg} be the obstacle contacts of vertices of polygon Q with edges of polygon P. We define Θmax_{aebfcg} as the maximal orientation of the intersection of the intervals I_{ae}, I_{bf} and I_{cg}, i.e. $\Theta max_{aebfcg} = \max\{\Theta \mid \Theta \in I_{ae} \cap I_{bf} \cap I_{cg}\}$.

lemma 17: Let Q be a polygon in general and T-optimal position at orientation Θ, and let C_{ae}, C_{bf} and C_{cg} be the obstacle contacts of vertices of polygon Q with edges of polygon P. Let μ^o be the expansion of Q. Furthermore let μ_1^* be the expansion of Q at orientation $\Theta+\Delta\Theta < \Theta max_{aebfcg}$ if Q does not make a fourth obstacle contact during the RE-transformation from Θ to $\Theta+\Delta\Theta$. And let μ_2^* be the expansion of Q at orientation $\Theta+\Delta\Theta$ if the obstacle contacts of Q change (i.e. the rotation center changes) during the RE-transformation from Θ to $\Theta+\Delta\Theta$.
We claim that μ_1^* is greater than μ_2^* .

proof: As we have seen in observation 13 the rotation center has to be changed when a fourth obstacle contact occurs since otherwise polygon Q would not be in allowed position for any further rotation. W.l.o.g. we assume that C_{dh} is the fourth obstacle contact, and furthermore we assume that M_{fgh} is the rotation center for this situation. If we now RE-transform polygon Q for an angle $\Delta\Theta$ (the expansion of Q at this orientation $\Theta+\Delta\Theta$ is μ_2^*), and then omit edge p_h from P but do not change the expansion of Q, Q can be T-transformed and therefore is not T-optimal. And this proves the lemma.

□

observation 18: Assume that we are given two lines l_1 and l_2 in E^2 and a convex polygon Q two vertices of which make an obstacle contact with these two lines, say Q_a with l_1 and Q_b with l_2. Let us now apply an E-transformation on Q around any point in the plane (either with $\mu > 1$ or $\mu < 1$) and then T-transform polygon Q such that Q makes the two initial obstacle contacts again. It is easy to verify that these two transformations (first E then T-transformation) can be substituted by one E-transformation around the point L_{12}, i.e. the point of intersection of l_1 and l_2. Additionally we can observe that during such an E-transformation each vertex Q_i ($1 \leq i \leq k$) moves on an imaginary line (L_{12}, Q_i). Therefore every vertex Q_i can make an obstacle contact with any other line only if this line intersects the halfline from L_{12} through Q_i at that side of Q_i where Q_i moves to.

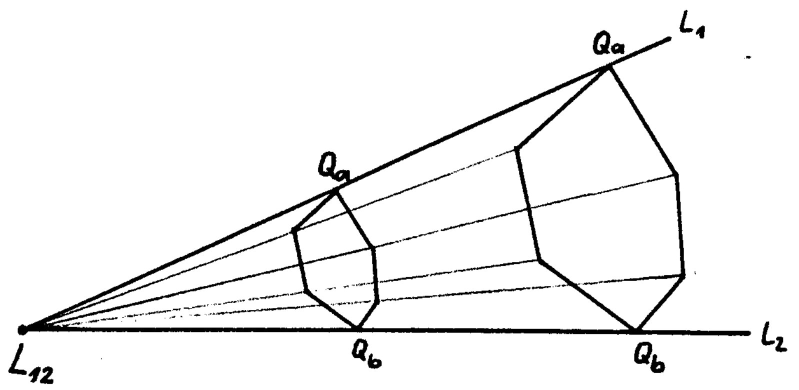


figure 12

lemma 19: Let Q be a convex polygon in T-optimal position in polygon P, and let it make the three obstacle contacts C_{ae}, C_{bf} and C_{cg} at orientation Θ' and let M be the corresponding rotation center. Furthermore let p_h be a fourth line of P with $f < h < g$ and let Q_d be a fourth vertex of Q with $b < d < c$ such that Θmax_{dhae}, which is the maximum orientation of the intersection of the intervals I_{dh} and I_{ae}. We assume that Θmax_{dhae} is greater than Θ'. Additionally we assume that the imaginary halfline i_d does not intersect line l_h, i.e. the line through p_h. We claim that polygon Q and polygon P cannot make the double obstacle contact DC_{aedh} for any orientation Θ with $\Theta' \leq \Theta \leq \Theta max_{dhae}$ such that Q is in allowed position. (For any orientation θ > θmax_{dhae} the lemma is obviously true).

proof: Let us first assume that polygon Q does not make a fourth obstacle contact with P during the RE-transformation around M for any orientation θ smaller than θmax_{dhae}. With this assumption polygon Q, trivially, cannot make the obstacle contacts C_{ae} and C_{dh} concurrently even if it is TE-transformed at any orientation θ with $\theta' \leq \theta \leq \theta max_{dhae}$. We have seen in observation 18 that a TE-transformation can be substituted by an E-transformation around the intersection of two lines. One of these lines is obviously l_e, the line through h_e. The second line could, for example be l_f, the line with which Q_b makes its obstacle contacts. In this case the lemma is true because the line segment (L_{ef}, Q_d) does not intersect line l_h. And the lemma is also true if we use any other line of E^2 through any vertex of polygon Q when the line intersects the line segment (L_{ef}, L_{eg}).

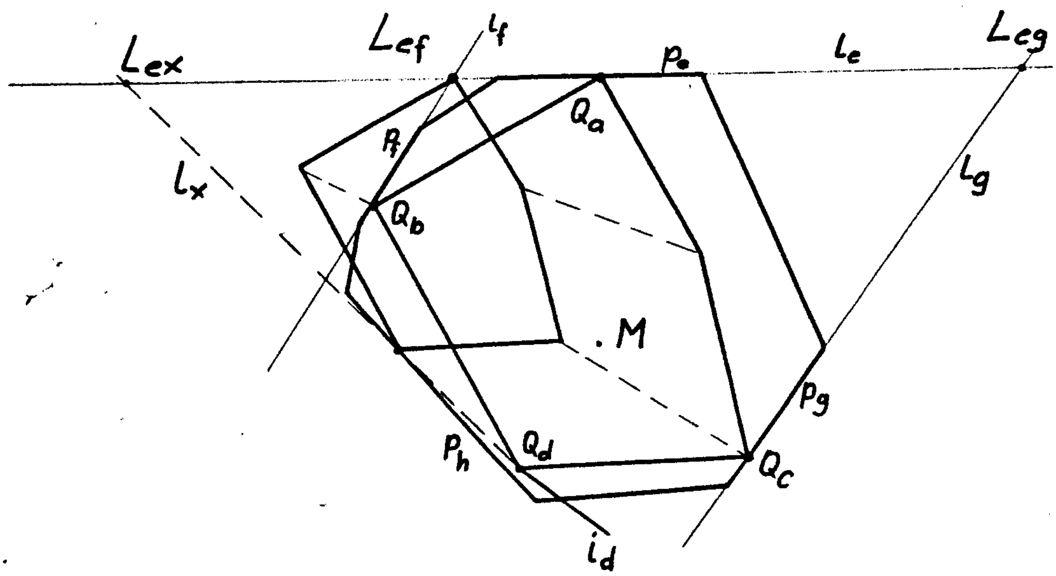


figure 13

But we cannot E-transform polygon Q around any point of l_a outside of (L_{af}, L_{ag}) because Q cannot be in allowed position after the transformation. Especially we cannot E-transform polygon Q around the intersection L_{ax}, when l_x is a line through Q_d which intersects edge p_h (see figure 13).
But also if we assume that the obstacle contacts change one or several times during the RE-transformation from Θ' to Θmax_{dhae} this lemma is true. Because at every orientation Θ, with $\Theta' \leq \Theta \leq \Theta max_{dhae}$ polygon Q (the one which changed its obstacle contacts) is a TE-transformation of that polygon which makes the original obstacle contacts at Θ (in lemma 17 we have seen the expansion of latter polygon is smaller than that of the original). And as we have seen above such a TE-transformation image cannot make the obstacle contacts C_{ae} and C_{dh} concurrently.

□

corollary 20: Let polygon Q be in any four obstacle contact situation, and let C_{ae}, C_{bf}, C_{cg}, C_{dh} be the four obstacle contacts, w.l.o.g. $a < b < c < d$. With RI we denote the rotation interval the upper bound of which is determined by this four obstacle contact situation. Assume that C_{cg} is the obstacle contact which vanishes when polygon Q is rotated any further. Then, as a implication of lemma 19, we can say that C_{cg} cannot form a pair of obstacle contacts in any other rotation interval with C_{ae}, such that C_{ae} or C_{cg} vanishes at the end of the rotation interval.

In the case when a double obstacle contact forms two obstacle contacts this double obstacle contact must be split into two single ones.

This implies that every time a rotation interval changes one of the $O(kn^2)$ pairs of obstacle contact which vanishes cannot reoccur and therefore we can claim:

theorem 1: The maximal number of rotation intervals is $O(kn^2)$.

We will now show that a number of rotation intervals proportional to kn^2 is also a lower bound. To this end we will first describe the movement of a vertex of Q which we call "swinging".

definition 16: Assume that we are given polygon Q at orientation Θ in T-optimal position making the obstacle contacts C_{ae}, C_{bf}, and C_{cg}. We first define the set RC_{Θ} as the set of all consecutive rotation intervals – the first is $[\Theta,\Theta+\Delta\Theta]$ – which have the property that either the rotation center for every rotation interval lies inside of P, or the rotation center for every interval lies outside of P.
We say that vertex Q_m *swings* on the edges p_i to p_{i+m}, if for some consecutive sets RC – starting with RC_{Θ} – the rotation center skips from the inside of P to the outside and then back again to the inside etc., such that vertex Q_m moves from p_i to p_{i+m} for the rotation intervals in one set RC by never escaping from the boundary of P. I.e. Q_m makes all the double obstacle contacts $DC_{m\ i+1}$ to $DC_{m\ i+m-1}$ if M is inside of P. In the consecutive set of rotation intervals for which M

is outside of P, Q moves back from p_{i+m} to p_i .

lemma 21: The lower bound of the number of rotation intervals is $\Omega(kn^2)$.

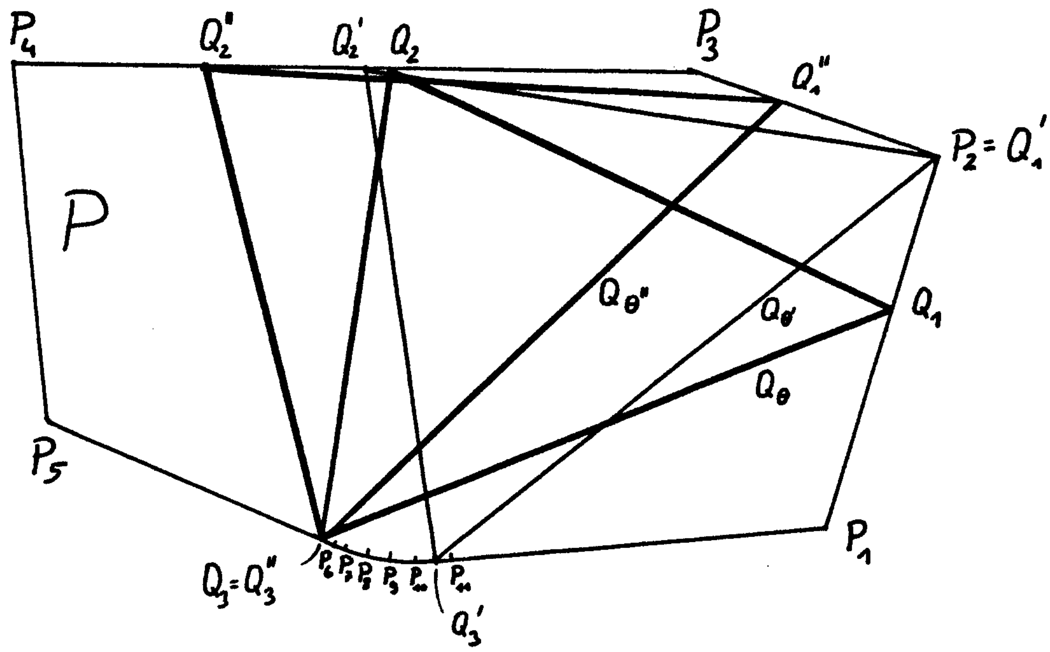


figure 14

proof: Let Q be a regular convex k-gon and P a polygon having the following properties:

(i) Q_a swings on the edges p_d to p_{d+x} while

(ii) the vertices Q_b and Q_c – w.l.o.g. b < c – alternately restrict the last rotation interval of the sets RC by making double obstacle contacts, and no other vertex, except Q_a makes an additional obstacle contact with polygon P in all rotation intervals.

(iii) Q_b and Q_c make their double obstacle contacts with edges p_f to p_{f+y}, and p_g and p_{g+z} respectively.

(iv) every constant x, y, and z has size cn for any constant $c < 1$.

It is easy to verify that each time Q_c makes its double obstacle contact the rotation center for the following rotation interval must lie inside of P. And the swinging of vertex Q_m is achieved by alternately making the difference $S(Q_m, Q_b, p_f) - S(Q_m, Q_b, p_f)$ positive and then negative (cf. figure 14 as an example).

And therefore the following consideration proves the lemma:
Vertex Q_m swings $y+z$ times over x edges which implies that we have $2cn$ sets RC, each of which has cn rotation intervals. And since polygon Q is a regular k-gon, every of the k vertices of Q implies these $2c^2n^2$ rotation intervals and gives us the resulting amount of $2c^2kn^2$ rotation intervals which is $\Omega(kn^2)$.

□

5. The rotation algorithm

We are now ready to give the complete algorithm to solve the problem of rotating polygon Q in polygon P.

algorithm 2:

a) Find the initial T-optimal position (for $\Theta = 0$) of polygon Q which can be achieved by TE-transformation. To solve this problem we can use a method proposed by Edelsbrunner and Welzl [EW] which reduces the problem to linear programing [ME] and solves it in O(k+n) time. Therefore we have to transform our initial problem to get the linear-program-definition:

Each vertex Q_j ($j = 1,...,k$) of polygon Q has coordinates μQ_j+v and these vertices must lie in the intersection H_P of the closed halfplanes h_i ($i = 1,...,n$) (defined as in definition 4) such that μ is maximal. Or more formally: Maximize μ such that

$$\mu Q + v \subseteq P$$

with an arbitrary translation v.

This is a system of unequations with kn unequations which can be reduced to n unequations in the following way:

Every restriction is a line through an edge of P, and as we have seen in section 4, for any fixed orientation of Q each of these lines can only be in contact with one vertex of Q. And therefore, for each line only that unequation which restricts the location of the point which can be in contact with it, must be considered, because all other unequations for this line (i.e. the unequations for the remaining $k - 1$ vertices of Q) are redundant.

The n unequations can be found in the following way (W.l.o.g. we assume that p_1 can be in contact with Q_1, i.e. that l_1 is a restriction for Q_1, and the adjacent edge to p_1 in clockwise order cannot be in contact with Q_1):

The first unequation is obviously $\mu Q_1 + v \leq p_1$ and the variables are initialized to $i = 1$ and $j = 2$.
Repeat the following instructions until the vertex of Q for which p_n is the restriction is found.

If Q_j can be in contact with p_i (i.e. $Q \subseteq h_i$ when Q_j is in contact with l_i).
(i) add the unequation $\mu Q_j + v \leq p_i$ to the set of unequations
(ii) proceed to the next edge of P (i.e. $i = i+1$)
otherwise proceed to the next vertex of Q (i.e. $j = j + 1$)

This sequence calculates the correct unequations because, if p_i ($i \in \{1,...,n-1\}$) can be in contact with Q_j ($j \in \{1,..,k\}$) no edge p_l, with $l \in \{i+1,...,n\}$, can be in contact with any vertex Q_m, with $m \in \{1,..,j-1\}$.

b) Repeat the following instructions until polygon Q has been rotated for 2π.

(i) Find the rotation center M for the current orientation of Q.

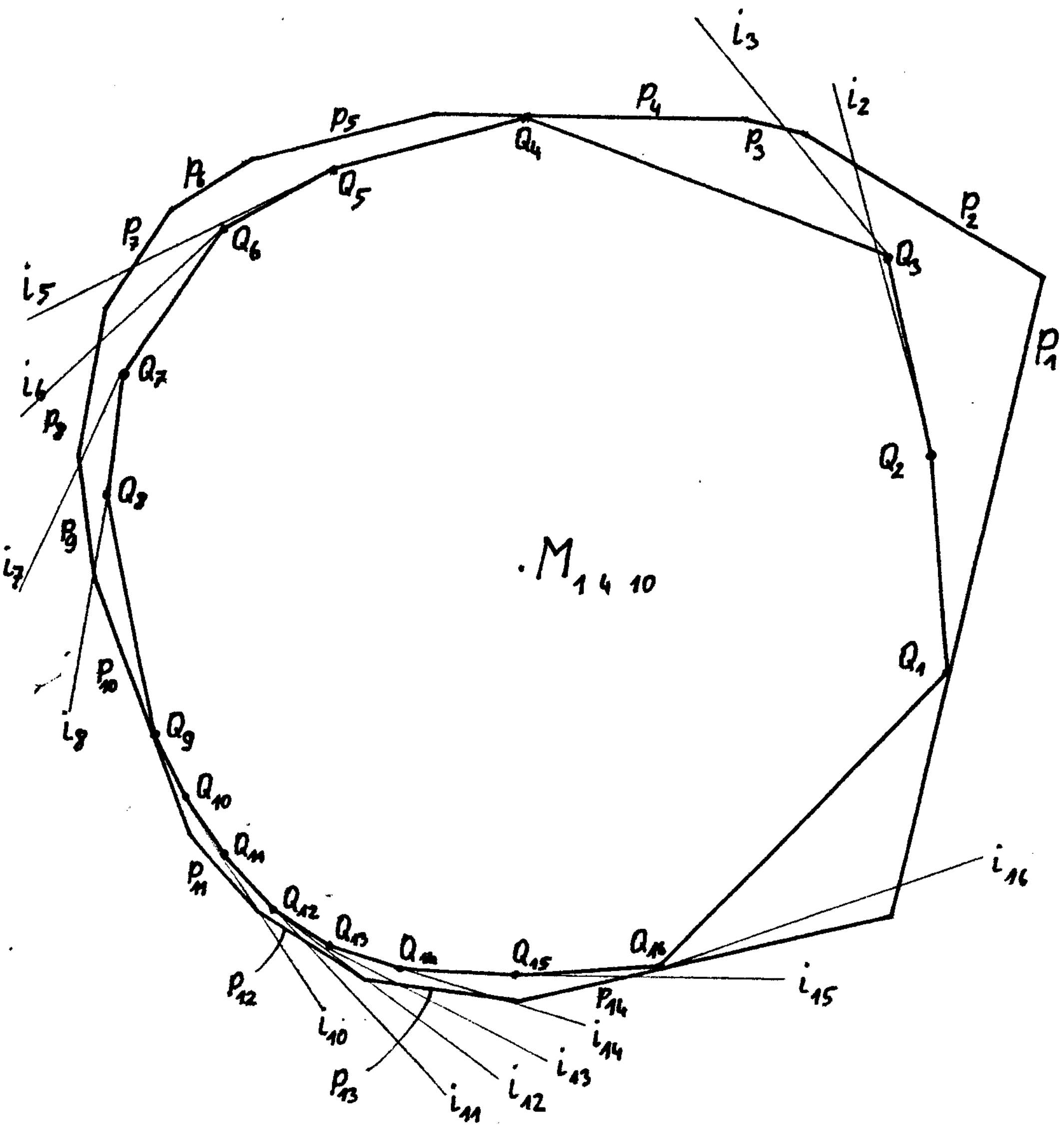


figure 15

(ii) Calculate the orientation of Q for which the next four obstacle contact situation occurs when Q is rotated around M.

I.e. for every vertex Q_i of Q find that edge of P (with binary search) with which the imaginary line i_i (defined in observation 3) intersects (see figure 15) when Q is RE-transformed around M (w.l.o.g. let p_j be this edge). Calculate the orientation at which this obstacle contact $C_{i,j}$ will occur. Finally, when the orientations of intersection for all lines i_i, with i = 1,...,n are calculated take the minimal orientation. (It is obvious that for the three vertices of Q, which slide on the boundary of P, the imaginary line i must be replaced by the corresponding edge of P).

theorem 2: With help of algorithm 2 a convex polygon Q can be rotated in the interior of another convex polygon P such that Q is T-optimal for every angle of rotation. And furthermore algorithm 2 needs a time proportional to $kn^2T(k,n)$. Where $T(k,n)$ is the time used to find the next four obstacle contact situation(see algorithm 2b(ii)), $T(k,n) = \min\{n \log k, k \log n\}$.

6. Chazelle's approach

Before we proceed with the discussion of the applications of our algorithm we will now give a short description of Chazelle's paper which solves the polygon containment problem if only translation and rotation and no expansion of polygon Q is allowed (for the sake of simplicity of reading this paper we will adherence to the notation used in our paper). Our main attention is thereby directed to two algorithmic elements, namely, "duality" and "divide and conquer".

In that part of the computer algorithms which is called the computational geometry it is often very difficult to observe how lines in the plane behave when we apply various operations (like translation or rotation) on them. It is easier to investigate the movement of points in the plane. And so, Preparata FP and Muller DE in [PM] and Brown KQ in [BQ] first proposed the powerful means of duality which is, general spoken, a translation of a line in E^2 into a point in the dual space D^2.

definition 17: Let l be a line in E^2 with $y_l = a_l x + b_l$. Then let DUAL(l) be the function which maps line l one-to-one into the point A_l in the dual plane, such that DUAL(l) = A_l = (a_l, b_l).

observation 22: Consequently a point $B = (a_B, b_B) \in E^2$ is transformed into line l_B: $y = -a_B x + b_B \in D^2$.

It is easy to verify that parallel lines in E^2 imply points in D^2 which lie on the same vertical line. Furthermore we can imply, that the point of intersection of two lines in E^2 corresponds to a line through the two corresponding points in D^2. And if the point of intersection, say L_{12} of two lines l_1 and l_2 lies below a third line l_3, the line through the points DUAL(l_1) and DUAL(l_2) lies below point DUAL(l_3). More formally we can say:

$$a_{13}x_{L12} + b_{13} > y_{L12} \iff$$
$$\iff y_{DUAL(l3)} > a_{DUAL(L12)} x_{DUAL(l3)} + b_{DUAL(L12)}$$

for $l_1: y = a_{11}x + b_{11}$, $l_2: y = a_{12}x + b_{12}$, $l_3: y = a_{13}x + b_{13}$, and $L_{12} = (x_{L12}, y_{L12})$ the point of intersection of l_1 and l_2.

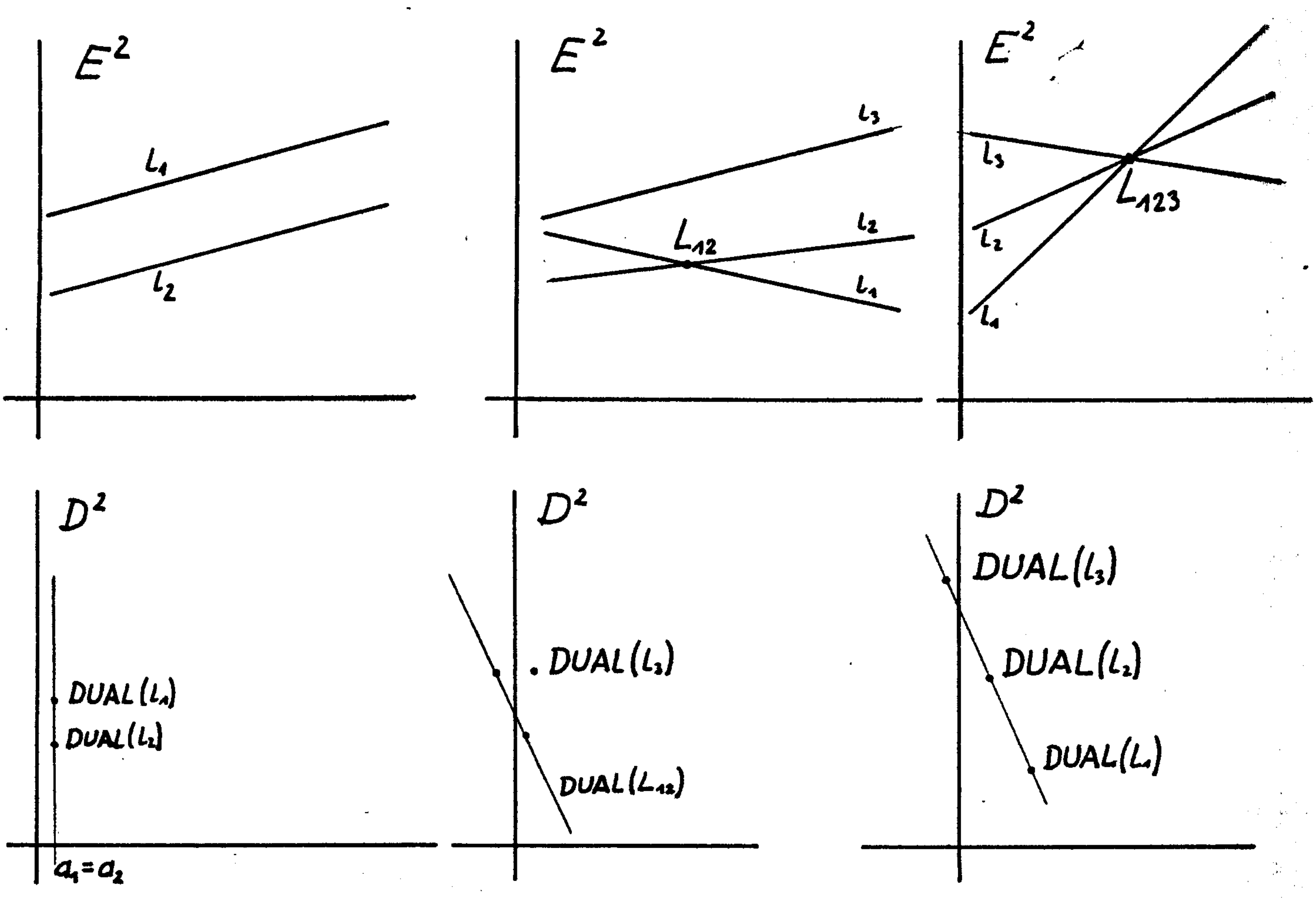


figure 16

It immediately follows that three or more lines in E^2 intersect in one point if and only if the dual mapped points are collinear.

We will now describe how the translation-restricted containment problem can be solved.

problem-definition: Decide if a T-transformation-image Q^* of polygon Q^0 exists, such that Q^* is in allowed position, and if, give all possible T-images.

definition 18: Assume that any vertex Q_j ($j \in \{1,...,k\}$) is in contact with edge p_i ($i \in \{1,...,n\}$), such that polygon Q is completely contained in halfplane h_i. We define $t_i(\Theta)$ as the line through vertex Q_1 parallel to edge p_i. Additionally, $ht_i(\Theta)$ is the halfplane which is bounded by $t_i(\Theta)$ and does not contain edge p_i. $I(\Theta)$ is the intersection of all halfplanes $ht_i(\Theta)$ for $i = 1,...,n$ ($I(\Theta) = ht_1(\Theta) \cap ht_2(\Theta) \cap ... \cap ht_n(\Theta)$).

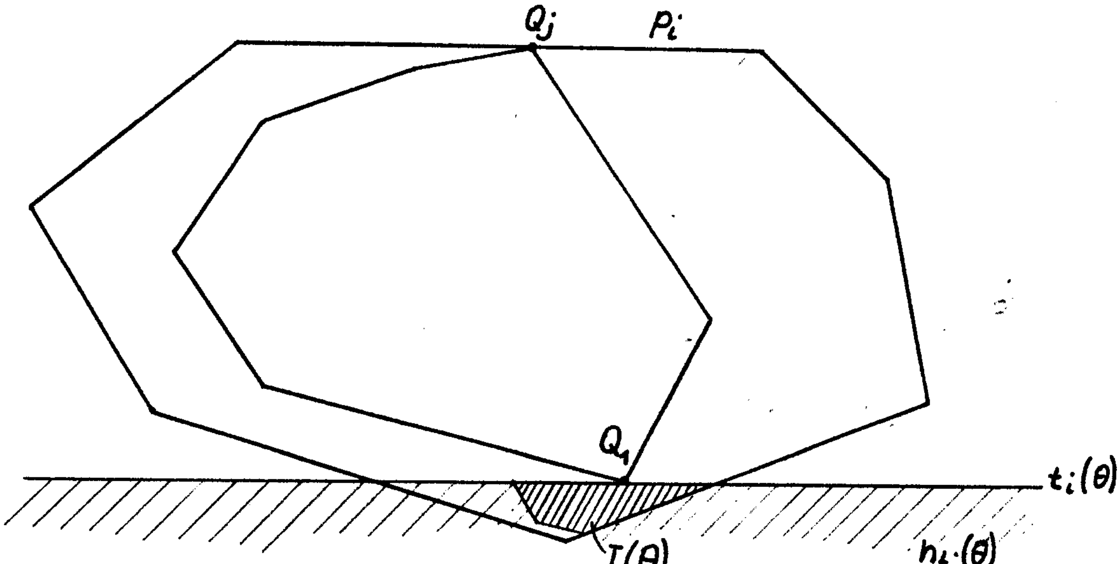


figure 17

Therefore, polygon Q can lie completely in h_i, if vertex p_i lies in $ht_i(\Theta)$. And furthermore an allowed position of Q exists just if $I(\Theta)$ is not empty, and in this case $I(\Theta)$ is obviously a convex polygon. And if such a polygon $I(\Theta)$ exists every T-image Q* is in allowed position if vertex Q_1 lies inside $I(\Theta)$.
And so, to determine whether a T-image Q* in allowed position exists, we need only calculate $I(\Theta)$.
As the first step we divide the edges of polygon P into two subsets: Assuming that P_1 is the vertex with maximal x-coordinate and P_l the vertex with minimal x-coordinate we get the two sets of edges of P $\{p_1,\ldots,p_{l-1}\}$ and $\{p_l,\ldots,p_n\}$ which define the two planes

$$UP(\Theta) = ht_1(\Theta) \cap \ldots \cap ht_{l-1}(\Theta)$$

and

$$LO(\Theta) = ht_l(\Theta) \cap \ldots \cap ht_n(\Theta),$$

such that $I(\Theta) = UP(\Theta) \cap LO(\Theta)$.

For further considerations Chazelle turns to a dual description of $I(\Theta)$. To this end line $t_i(\Theta)$: $y = a_i(\Theta)x + b_i(\Theta)$ is transformed into vertex $U_i(\Theta) = (a_i(\Theta), b_i(\Theta))$. And since the slope $a_i(\Theta)$ $(a_i(\Theta) = (y_{Qi} - y_{Qi+1}) / (x_{Qi} - x_{Qi+1}))$ of $t_i(\Theta)$ does not change during an operation on Q, we can use the short form a_i. It is easy to verify that a redundant line $t_i(\Theta)$ – i.e. a line which does not contribute an edge to $UP(\Theta)$ – has a slope which is greater than the slope of all lines $t_e(\Theta)$ with $1 \leq e < i$ and a slope smaller than all lines $t_f(\Theta)$ with $i < f \leq n$ (we assume that t_e and t_f are nonredundant) and that the intersection of line $t_e(\Theta)$ and $t_f(\Theta)$ lies below $t_i(\Theta)$. As we have seen in our discussion of duality the latter fact implies that vertex U_i lies above the line through U_e and U_f, and since

$a_- < a_i < a_+$ we can immediately imply that UP(Θ) is represented in the dual space by the bottom part of the convex hull of $\{U_1(\Theta),\dots,U_{l-1}(\Theta)\}$, which we denote with CU(Θ). The first vertex of CU(Θ) is $U_1(\Theta)$ and the last is $U_{l-1}(\Theta)$.
With CL(Θ) we denote the dual representation of LO(Θ), $U_l(\Theta)$ is the first vertex $U_n(\Theta)$ the last.

Chazelle has proved that LU(Θ) and LO(Θ) intersect in one point if and only if CL(Θ) and CU(Θ) intersect in one point. With this and the observation that CU(Θ) moves upwards (downwards resp.) when LU(Θ) moves upwards (downwards resp.) he implies the following lemma:

> <u>*lemma 23:*</u> LO(Θ) and LU(Θ) intersect in the interior if and only if CL(Θ) and CU(Θ) do not intersect. So, there is a T-image in allowed position with orientation Θ only if CU(Θ) and CL(Θ) do not intersect.
>
> □

In [CA] McCallum and Avis present an algorithm which computes the convex hull of a set of vertices in linear time, and since we need O(k+n) time to calculate the lines $t_i(\Theta)$ (vertices $U_i(\Theta)$ respectively) and additional O(n) time for the convex hull of I(Θ) (i.e. LU(Θ) and LO(Θ)) and since we need O(n) time to determine whether LU(Θ) and LO(Θ) intersect or not, the question if a T-image in allowed position exists can be answered in O(k+n) time.

During the R-transformation of polygon Q the distance h_i between the lines t_i ($i \in \{1,\dots,n\}$) and their appropriate edge p_i varies. Before we can describe the function with which h_i varies we have to define $y_i(\Theta)$:

$y_i(\Theta)$ is the function which evaluates the y-coordinate of Q_i when we assume that polygon is R-transformed in E^2 such that for every orientation Θ one vertex of Q is in contact with the x-axis and all vertices of Q lie above the x-axis:

$$y_i(\Theta) = d_{t+i} * \cos(\Theta + \gamma_i)$$

when d_{t+i} is the euclidian distance between Q_t and Q_i (Q_t is that vertex of Q which is in contact with the x-axis), and γ_i some constant.

h_i can be expressed as $y_i(\Theta+\beta_i)$ for some constant β_i. And therefore we can conclude the function which describes $b_i(\Theta)$ of line $t_i(\Theta)$:

$$b_i(\Theta) = \gamma_i y_i(\Theta+\beta_i) + \Delta_i$$

with $\gamma_i = -1 / \cos(\alpha_i)$, α_i the angle between the x-axis and vector (Q_{i+1}, Q_i) measured between 0 and 2π, and $\Delta_i = y_{Qi+1} - x_{Qi+1}(y_{Qi} - y_{Qi+1}) / (x_{Qi} - x_{Qi+1})$.

We are now ready to give the datastructure which describes the convex hull (as we did in the translation-restricted case we will only describe the datastructure for CU since that for CL is symmetric). L is a sequence of t cells which describe CU(Θ), i.e. L = CELL(CU_1) <-> ... <-> CELL(CU_t), where CU_i is the ith vertex of CU ($CU_1 = U_1$ and $CU_t = U_{l-1}$). Each CELL has enough space to contain the three parameters of y_i. Since, during the rotation of Q the parameters of the vertices U_i, $i \in \{1,...,n\}$ change we also introduce the linear list I each element of which can contain one of the following instructions:

[Θ',i,A,B,C]	at orientation Θ' vertex U_i get the new parameters A,B and C.
[Θ'',i,j,insert] ...	at orientation Θ'' vertex U_i is inserted next to U_j in counterclockwise order into L.
[Θ''',i,delete]	at orientation Θ''' vertex U_i is deleted from L.

Our goal is it now to compute I such that for every change of CU during the R-transformation of polygon Q from $\Theta = 0$ to $\Theta = 2\pi$ one entry exists. Chazelle does the computation of I with help of a "divide and conquer" algorithm:

"Divide and conquer" in this case means that we first divide the set of vertices U_i into two subsets of the same size (plus or minus one vertex) and calculate the two sequences L_1 and L_2, and I_1 and I_2 for the two subsets . The second step is to connect the two convex

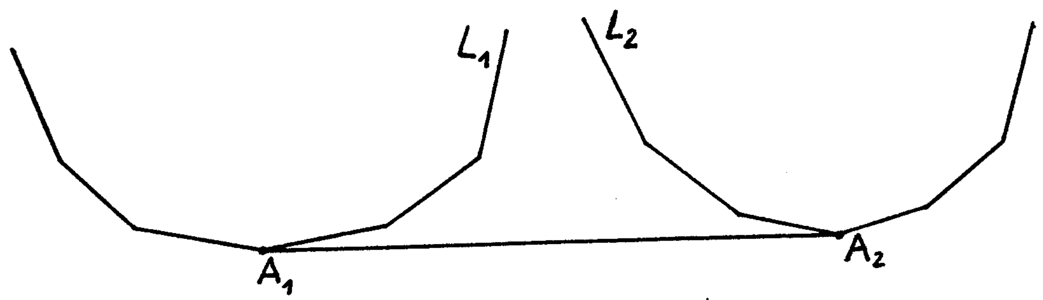


figure 18

hulls L_1 and L_2 (for orientation $\Theta = 0$) with an edge of support such that the resulting sequence L is a convex hull again (see figure 18). Then merge the sets I_1 and

I_2 such that the instructions in I are ordered with increasing orientation. Let A_1 and A_2 be the two edges of L_1, and L_2 respectively, adjacent to the edge of support. It can easily be verified that the edge (A_1, A_2) can change for increasing Θ: this may appear for example, in the moment when the predecessor of A_1 in L_1, A_1 itself and the successor of A_1 in L_1 become collinear. We need not merge instructions of I_1 and I_2 into I if they do not relate to vertices of $\Delta(A_1, A_2)$ and do not vary L. $\Delta(A_1, A_2)$ is the set of vertices U_i, such that $U_i \in \Delta(A_1, A_2)$ if $a_{A1} < a_{Ui} < a_{A2}$. But an instruction has to be inserted into I - additional to the instructions of I_1 and I_2 - if the edge of support changes for some Θ. It is left to mention that L_1 and I_1 (L_2 and I_2) are calculated recursively in the same way.

We can now estimate the time for the calculation of L and I, and also the space requirement of I (that of L is obviously O(n)) which is proportional to the time since insertion and deletion in I can be done in constant time:

$$T(N) = 2T(N/2) + R(N) + 2R(N/2) + O(N) \text{ and } T(1) = 0.$$

R(N) accounts for the instruction in I which can be split into two subsets: (i) the instruction of the kind (Θ',i,A,B,C); these can at most be kN since the function for h_i may change k times, and (ii) the insert and delete instructions which we evaluate with S(N). And therefore $R(N) \leq kN + S(N)$.

Let $f(\Theta) = \alpha b_a(\Theta) + \beta b_b(\Theta) + \gamma b_c(\Theta)$ be the function which is zero if the three vertices U_a, U_b and U_c are collinear. In the interval $[0,2\pi]$ this function $f(\Theta)$ is 3k+1 times a function of F (F is a set of functions of the kind $f(\Theta) = a*\sin(\Theta + b) + c$) , and since each interval is

less than π wide the function $f(\Theta)$ takes on any value at most twice over each interval and therefore the number of orientation at which $f(\Theta) = 0$ is less than $6k+2$. And because of the combinatorial number of triplets of vertices U_a, U_b and U_c we could assume an estimate $S(N) = O(kN^3)$.
But Chazelle proved that $R(N) = O(kN^2)$. It is not the purpose of this paper to duplicate Chazelle's proof, but we would like to explain the main idea:

> Assume that cu_{ab} is an edge of the convex hull CU at orientation Θ' and cu_{cb} is an edge of CU at orientation $\Theta'' > \Theta'$ (obviously cu_{ab} and cu_{cb} cannot be edges of CU simultaneously). If edge cu_{ab} is an edge of CU at orientation $\Theta''' > \Theta''$ again, edge cu_{cb} cannot be an edge of CU at any orientation $\Theta'^{v} \in [\Theta''', 2\pi]$. And furthermore Chazelle proves that every edge cu can be a part of CU at most 3 times. And since the combinatorial number of various edges cu is $O(kN^2)$ the resulting time for the calculation of I is $T(n) = 2T(n/2) + O(kn^2) = O(kn^2)$.

We will now show Chazelle's main algorithm and how it makes use of the datastructures L and I (L' and I' respectively, for CL). The first step is to decide if a containing placement exists at orientation $\Theta = 0$. And if it exists report it in $O(n \log(k+n))$ time. If not, compute the two pairs of edges of CU and CL which intersect, and apply I on L, and I' on L'. By varying L and L' for increasing orientation Θ the two pairs of intersecting edges also change. A containing placement is found in the moment when CU and CL are in contact in one point.
Chazelle proves that the time of this main algorithms is also proportional to kn^2 and concludes:

theorem: For any pair of simple polygons Q and P (P convex) with, respectively, n and k vertices, it is possible, in time $O(kn^2)$, to determine whether there exists a containing placement of Q reachable by translation and rotation, and if there is one, report its location.

For the sake of completeness we finally mention that in the case when neither P nor Q is convex, Chazelle gives an $O(k^3n^3(k+n)\log(k+n))$ solution.

7. Applications

Additional to the original problem we can solve the problem of finding the maximal expansion of polygon Q which fits into polygon P. Expansion μ°_{max} is said to be maximal if the expansion μ° for every orientation Θ' ($\Theta' <> \Theta$; Θ is the orientation at which the T-optimal RTE-image of polygon Q takes on expansion μ°_{max}). Before we describe the appropriate algorithm we proof:

lemma 24: Let $[\Theta',\Theta'']$ be a rotation interval, $M_{\Theta'\Theta''}$ the appropriate rotation center and C_{ab} one of the obstacle contacts.
We claim that the expansion of polygon Q can take on an extreme-value during the RTE-transformation from Θ' to Θ'' around $M_{\Theta'\Theta''}$ only at the orientations Θ', Θ'' and (if it exists) at orientation Θ''', for $\Theta' < \Theta''' < \Theta''$ and $S(M_{\Theta'\Theta''},Q_a,p_b) = \pi/2$ at orientation Θ'''.

proof: Assume that we divide the rotation interval into two parts: $[\Theta',\Theta''']$ and $[\Theta''',\Theta'']$. It is obvious that during the rotation from Θ' to Θ''' (Θ''' to Θ'' respectively) the distance between $M_{\Theta'\Theta''}$ and Q_a – and therefore also the expansion of polygon Q – steady decreases (increases resp.). This implies that polygon Q has its minimal expansion at orientation Θ''', and its maximum either at Θ' or Θ''. If Θ''' does not exist (this occurs if $S(M_{\Theta'\Theta''},Q_a,p_b) \geq \pi/2$ at Θ', or if $S(M_{\Theta'\Theta''},Q_a,p_b) < \pi/2$ at Θ' and $S(M_{\Theta'\Theta''},Q_a,p_b) < \pi/2$ at Θ'') polygon Q takes on the maximal (minimal resp.) expansion at one of the orientations Θ' and Θ''. □

From lemma 24 we can conclude, that we must just compare the expansion of polygon Q at the end of each rotation interval with the maximal expansion which polygon Q has taken on until this point. The expansion of polygon Q at the end of each rotation interval, say $\mu^{\circ}_{\theta''}$ is calculated by multiplying the expansion at the beginning of the rotation interval, say $\mu^{\circ}_{\theta'}$ by the expansion $\mu_{\theta'\theta''}$ by which polygon Q is expanded during the rotation ($\mu^{\circ}_{\theta''} = \mu^{\circ}_{\theta'} * \mu_{\theta'\theta''}$).

As a subproblem of this, the question if a convex polygon fits into another convex polygon (Chazelle's approach) can be answered(i.e. $\mu_{max} > 1$).

An other application is to find the maximal expansion of Q such that Q can be completely rotated in P. We can get this expansion in a similar way as μ°_{max}.

If the rotation intervals are stored in an array, with help of binary search the maximal expansion of Q that fits into P for a given interval of orientation can be answered in O(log kn) time. The size of this array is $O(kn^2)$.

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