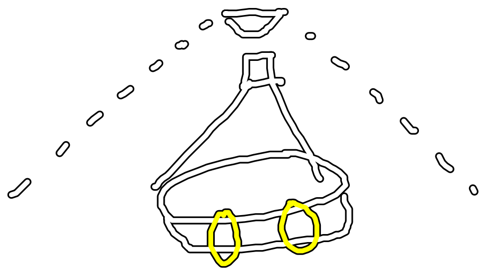
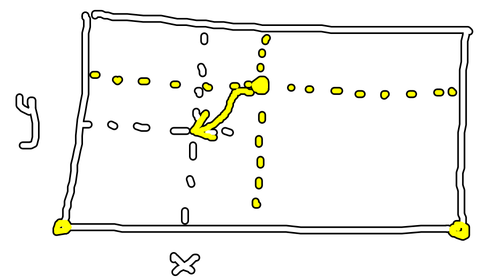
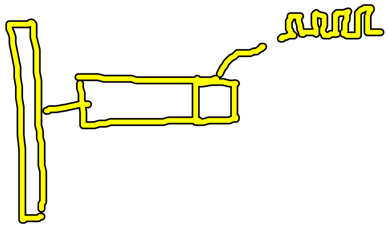
Kalman Filter



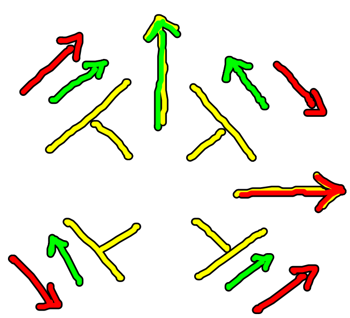




(x,y) Messung (Vision)

(x',y') " (Odometrie)

⋮



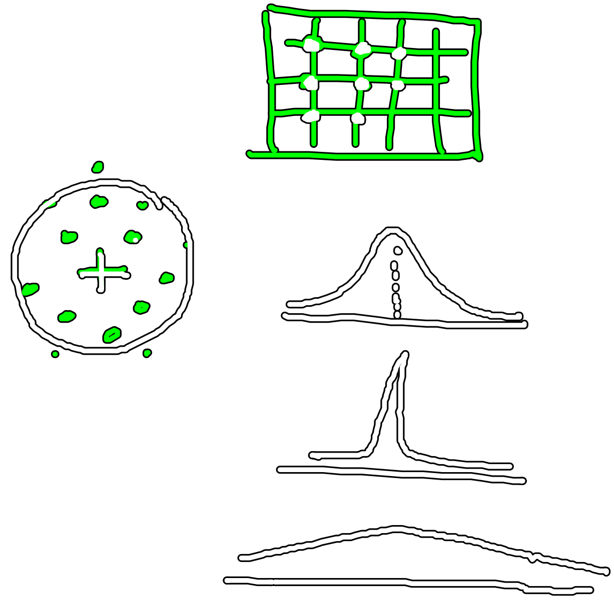
$$\vec{x}_1 = (x,y)$$

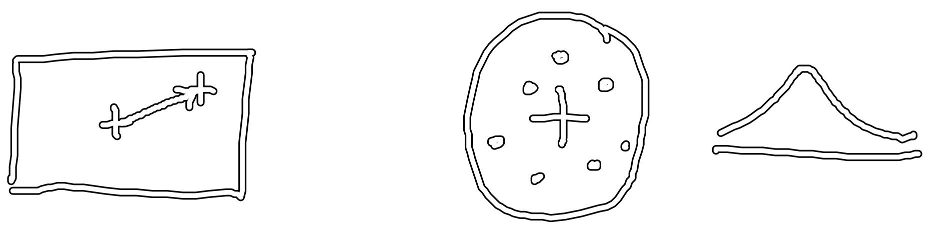
$$\vec{x}_2 = (x',y')$$

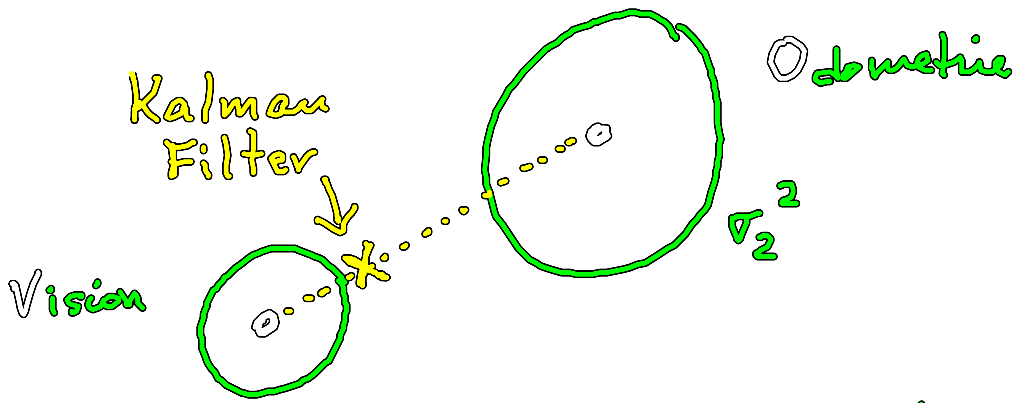
(x',y')

(x,y)

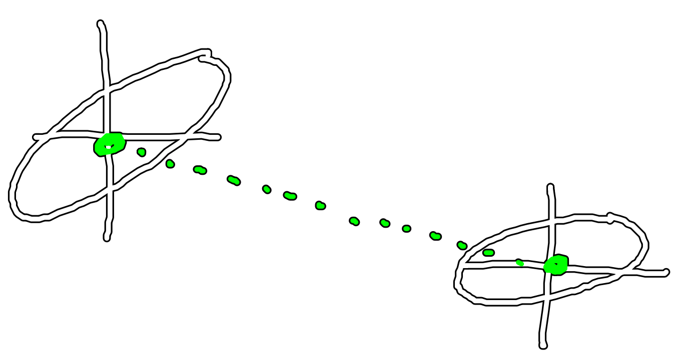
Fehler







σ_1^2 = mittlere quad. Fehler



Motivation

$$x_1, x_2, \dots, x_n$$

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

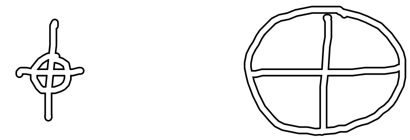
Neue Messung x_{n+1}

$$\mu_{n+1} \; ?$$

$$\mu_{n+1} = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i$$

$$\mu_{n+1} = f(\mu_n, x_{n+1})$$

1. Mess. 2. Mess



$$\mu_{n+1} = \frac{n}{n+1} \left(\underbrace{\frac{1}{n} \sum_{i=1}^{n} x_i}_{} + \frac{1}{n} x_{n+1} \right)$$

$$= \frac{n}{n+1} \cdot \mu_n + \frac{1}{n+1} x_{n+1}$$

$$= \mu_n + K \left(x_{n+1} - \mu_n \right)$$

$$K = \frac{1}{n+1}$$

Kalman Filter

$$\mu_{n+1} = \mu_n + \underbrace{K}_{\text{Kalman gain}} \big(\underbrace{x_{n+1} - \mu_n}_{\text{Diff}} \big)$$

Neu ....

1. Messung ....

$$\sigma_n^2 = \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \mu_n \right)^2$$

....

σ_{n+1}^2 ?

$$\sigma_{n+1}^2 = g\left(\sigma_{n+1}^2 , x_{n+1} \right)$$

$$\sigma_{n+1}^2 = \frac{1}{n+1} \sum_{i=1}^{n+1} (x_i - \mu_{n+1})^2$$

$$= \frac{1}{n+1} \sum_{i=1}^{n+1} \left(\underbrace{x_i - \mu_n} - k(x_{n+1} - \mu_n) \right)^2$$

$$= \frac{1}{n+1} \Big[\overbrace{\sum_{i=1}^{n} (x_i - \mu_n)^2} + (x_{n+1} - \mu_n)^2$$

$$\cancel{\sum_{i=1}^{n} -2k(x_i - \mu_n)}(x_{n+1} - \mu_n)$$

$$-2k(x_{n+1} - \mu_n)(x_{n+1} - \mu_n)$$

$$+ \sum_{i=1}^{n} k^2 (x_{n+1} - \mu_n)^2 + k^2 (x_{n+1} - \mu_n)^2 \Big]$$

$$= \frac{1}{n+1} \Big[n\sigma_n^2 + (x_{n+1} - \mu_n)^2 (1 - 2k + nk^2 + k^2) \Big]$$

$$= \frac{n}{n+1} \left(\sigma_n^2 + k (x_{n+1} - \mu_n)^2 \right)$$

$$k = \frac{1}{n+1}$$

$$(1-K)^2 + nK^2 = nK$$

$$= \left(1 - \frac{1}{n+1}\right)^2 + n\left(\frac{1}{n+1}\right)^2$$

$$= \left(\frac{n}{n+1}\right)^2 + \frac{n}{(n+1)^2} = \frac{n^2+n}{(n+1)^2}$$

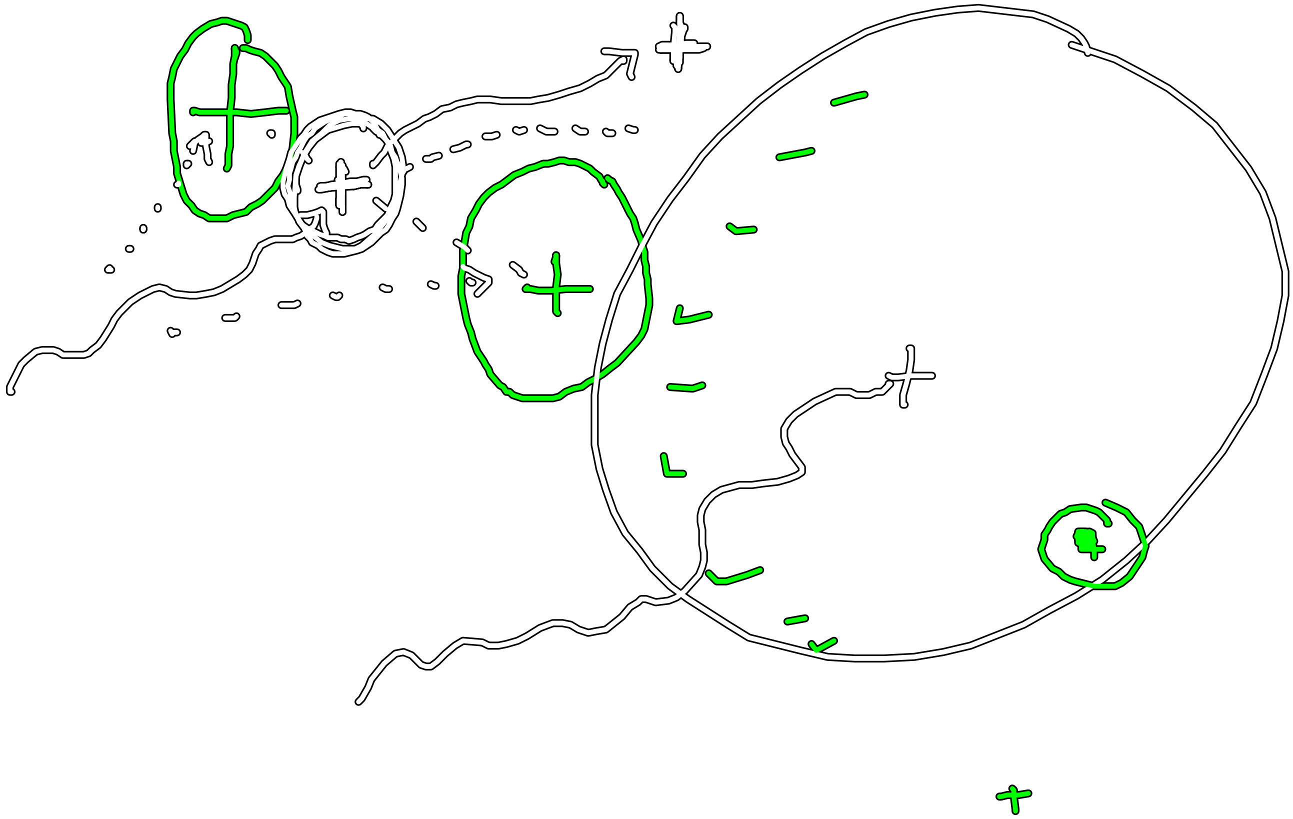
$$= \frac{n(n+1)}{(n+1)^2} = \frac{n}{n+1}$$

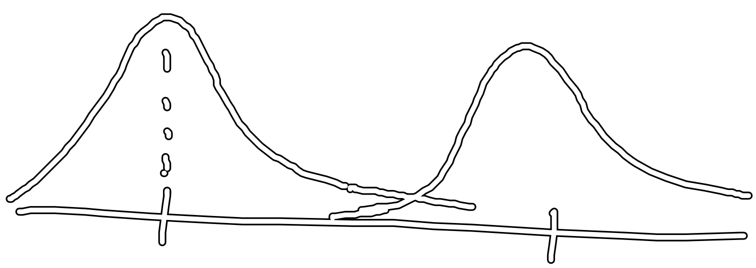
$$\boxed{\sigma_{n+1}^2 = (1-K)\left(\underbrace{\sigma_n^2 + K(x_{n+1}-\mu)^2}\right)}$$

$$1-K = 1 - \frac{1}{n+1} = \frac{n}{n+1} < 1$$

Kalman Filter

$$\begin{cases} \mu_{n+1} = \mu_n + K(x_{n+1} - \mu) \\ \sigma'^2_{n+1} = \sigma_n^2 + K(x_{n+1} - \mu)^2 \\ \sigma_{n+1}^2 = (1-K)\,\sigma'^2_{n+1} \end{cases}$$





$$\hat{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2}$$

Falls $\sigma_1 = \sigma_2$

$$\hat{x} = \frac{1}{2} x_1 + \frac{1}{2} x_2$$

$$\hat{x} = \frac{\frac{1}{\sigma_1^2} x_1 + \frac{1}{\sigma_2^2} x_2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

$$\rightarrow \boxed{\hat{x} = \frac{\sigma_2^2 x_1 + \sigma_1^2 x_2}{\sigma_1^2 + \sigma_2^2}}$$

$$\rightarrow \hat{x} = x_1 + K (x_2 - x_1)$$

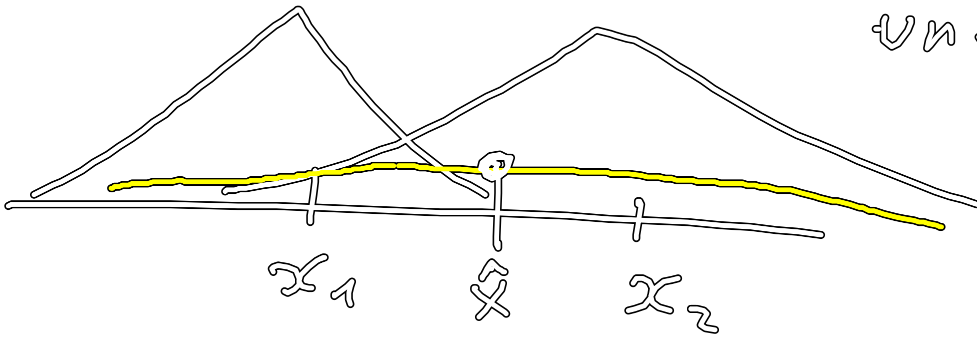
$$K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

$$= x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (x_2 - x_1)$$

$$= \frac{x_1 (\cancel{\sigma_1^2} + \sigma_2^2 \cancel{- \sigma_1^2}) + x_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

$$\underline{\qquad} \cdot \underline{\qquad}$$

unabhängig



$$p(x) = \underbrace{p(x|x_1)}_{}\,\underbrace{p(x|x_2)}_{}$$

$$p_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-x_1)^2}$$

$$p_2(x) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2\sigma_2^2}(x-x_2)^2}$$

$$p_1(x)\,p_2(x) = C\, e^{-\frac{1}{2\sigma_1^2}(x-x_1)^2-\frac{1}{2\sigma_2^2}(x-x_2)^2}$$

Exponent

$$-\frac{1}{2\sigma_1^2}\left(x^2 - 2x x_1 + x_1^2\right)$$

$$-\frac{1}{2\sigma_2^2}\left(x^2 - 2x x_2 + x_2^2\right)$$

$$= -\frac{1}{2}\left\{ x^2\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right) - 2x\left(\overbrace{\frac{x_1}{\sigma_1^2}+\frac{x_2}{\sigma_2^2}}\right) + Cts \right\}$$

$$e^{\overbrace{A}^{var}+\overbrace{B}^{ct}} = e^A \cdot \underbrace{e^B} = c' e^A$$

$$= -\frac{1}{2}\,\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left(x - \frac{x_1\sigma_2^2+x_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2$$

$$+ \; Cts$$

$$\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}$$

$$-2x\,\frac{(x_1\sigma_2^2+x_2\sigma_1^2)}{\sigma_1^2\sigma_2^2}$$

Exponent

$$= -\frac{1}{2\,\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}\left(x - \underbrace{\frac{x_1\sigma_2^2+x_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}}\right)^2$$

$$\begin{cases} \hat{x} = \dfrac{x_1 \sigma_2^2 + x_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} \\[2ex] \hat{\sigma}^2 = \dfrac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \end{cases}$$

$$K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

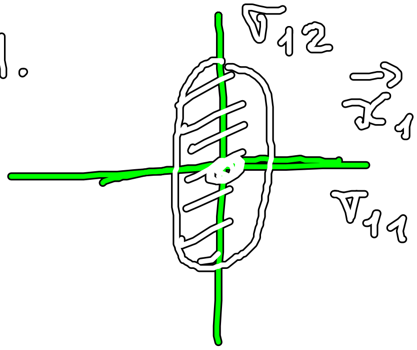
$$\boxed{\begin{cases} \hat{x} = x_1 + K(x_2 - x_1) \\ \hat{\sigma}^2 = (1-K)\sigma_1^2 \end{cases}}$$

$\sigma_2^2 \ll \sigma_1^2 \qquad K \approx 1$

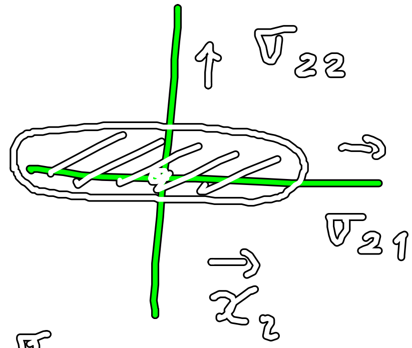
$\sigma_1^2 \ll \sigma_2^2 \qquad K \approx 0$

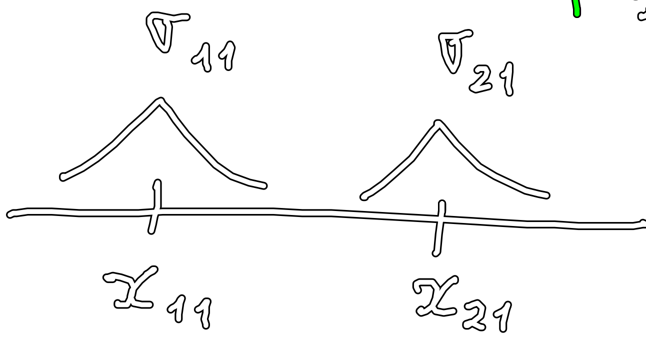
n-Dimensionen (unkorreliert)

1.



2.





<u>n-Dim</u>

$$\Sigma_1 = \begin{pmatrix} \sigma_{11}^2 & 0 \\ 0 & \sigma_{12}^2 \end{pmatrix}$$

Kovarianz

$$\Sigma_2 = \begin{pmatrix} \sigma_{21}^2 & 0 \\ 0 & \sigma_{22}^2 \end{pmatrix}$$

$$\Sigma_1^{-1} = \begin{pmatrix} \frac{1}{\sigma_{11}^2} & \\ 0 & \frac{1}{\sigma_{12}^2} \end{pmatrix}$$

$$\Sigma_2^{-1} = \begin{pmatrix} \frac{1}{\sigma_{21}^2} & 0 \\ 0 & \frac{1}{\sigma_{22}^2} \end{pmatrix}$$

$$\hat{\vec{x}} = \frac{\Sigma_2 \vec{x}_1 + \Sigma_1 \vec{x}_2}{\Sigma_1 + \Sigma_2}$$

$$= (\Sigma_2 \vec{x}_1 + \Sigma_1 \vec{x}_2)(\Sigma_1 + \Sigma_2)^{-1}$$

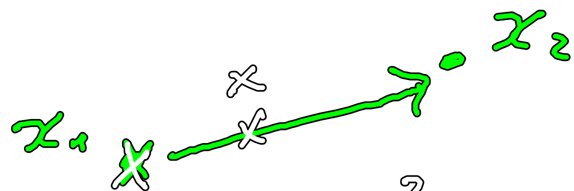
$$\hat{\vec{x}} = \left\{ \begin{pmatrix} \sigma_{21}^2 & 0 \\ 0 & \sigma_{22}^2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + \begin{pmatrix} \sigma_{11}^2 & 0 \\ 0 & \sigma_{12}^2 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} \right\}$$

$$\cdot \begin{pmatrix} \frac{1}{\sigma_{11}^2 + \sigma_{21}^2} & 0 \\ 0 & \frac{1}{\sigma_{21}^2 + \sigma_{22}^2} \end{pmatrix}$$

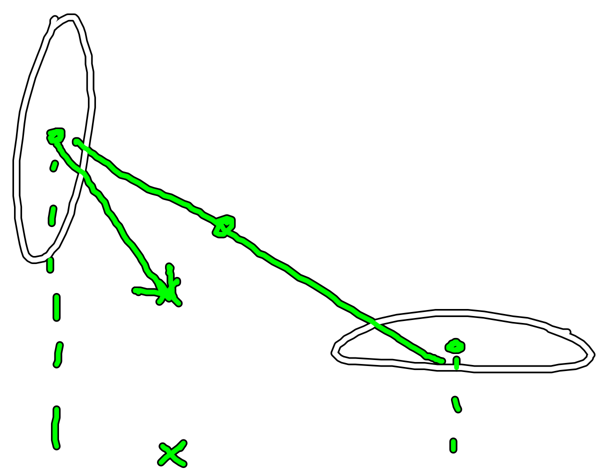
$$\hat{x}_1 = \frac{x_{11}\sigma_{21}^2 + x_{21}\sigma_{11}^2}{\sigma_{11}^2 + \sigma_{21}^2}$$

Kalman Filter n-dim

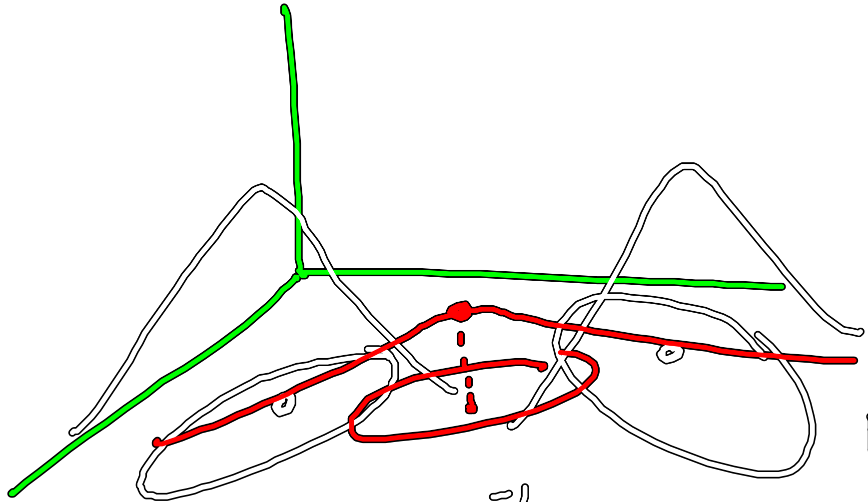
$$
\begin{cases}
\hat{x} = \underline{\vec{x}_1} + K \underbrace{(\vec{x}_2 - \vec{x}_1)} \\
\hat{\Sigma} = (I - K)\Sigma_1 \\
K = \Sigma_1 (\Sigma_1 + \Sigma_2)^{-1}
\end{cases}
$$



$$
K = \begin{pmatrix} \dfrac{\sigma_{11}^2}{\sigma_{11}^2 + \sigma_{21}^2} & 0 \\ 0 & \dfrac{\sigma_{21}^2}{\sigma_{11}^2 + \sigma_{21}^2} \end{pmatrix}
$$



n-dim

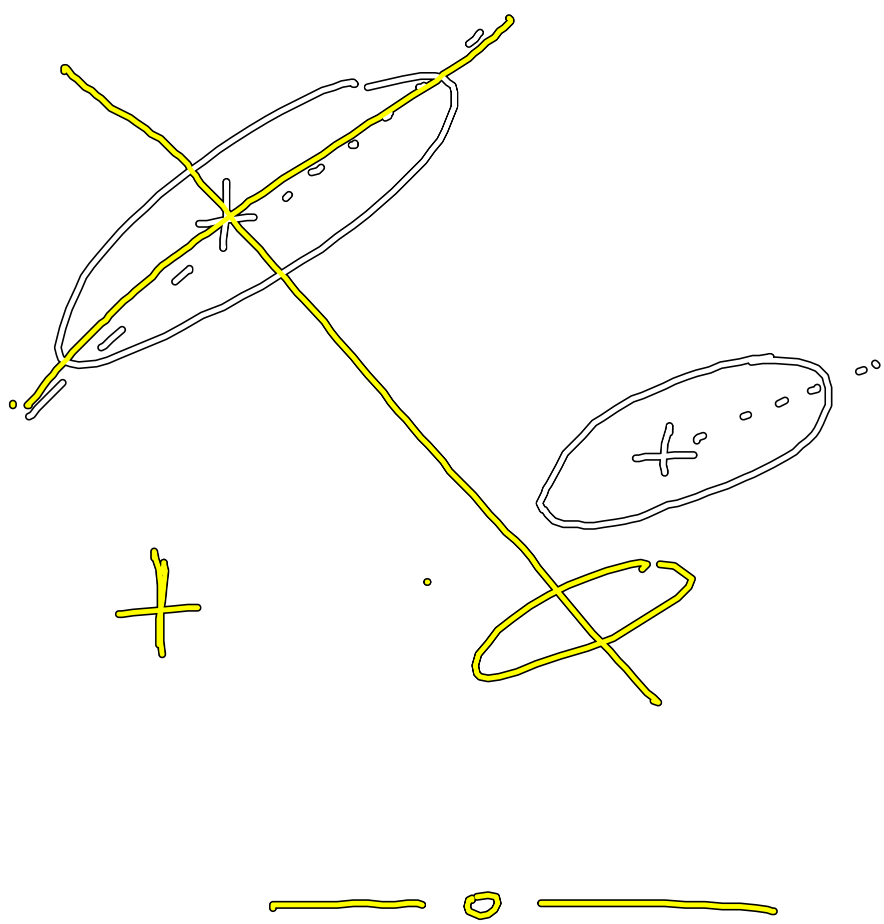


$$e^{-\frac{1}{2}\underbrace{\cdots\,\Sigma_1^{-1}\cdots}_{QF}} \times e^{-\frac{1}{2}\underbrace{\cdots\,\Sigma_2^{-1}\cdots}_{QF}}$$

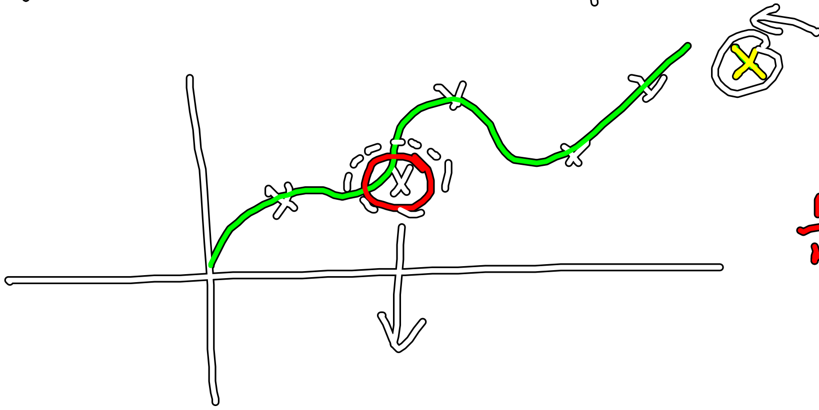
$$QF + QF$$

$$\downarrow$$

$$QF$$



Rekursive Min. Sq. $\longrightarrow$ Kalman

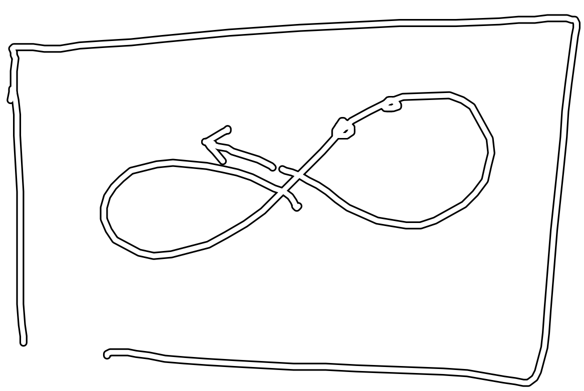


$\frac{1}{n}\sum x_i$

$a_0 + a_1 x^1 + a_2 x^2 + \ldots$

K

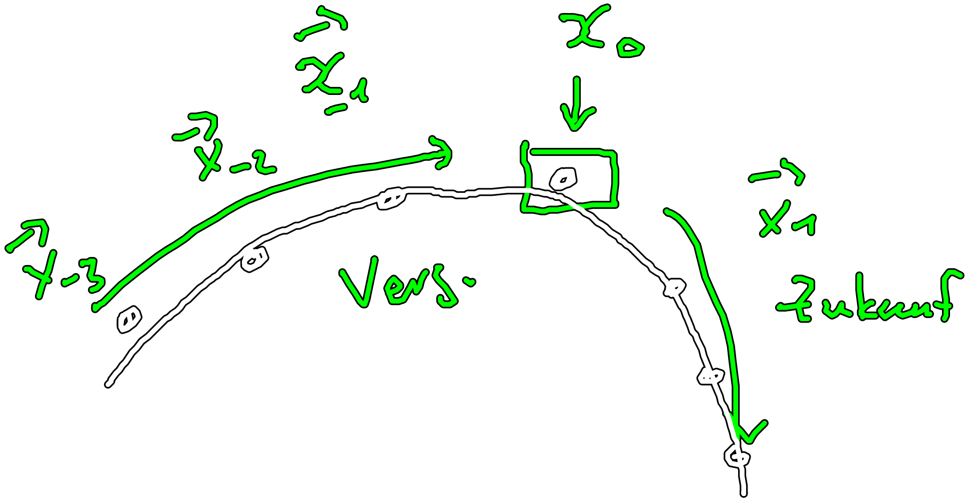
Übung



Abschätzen

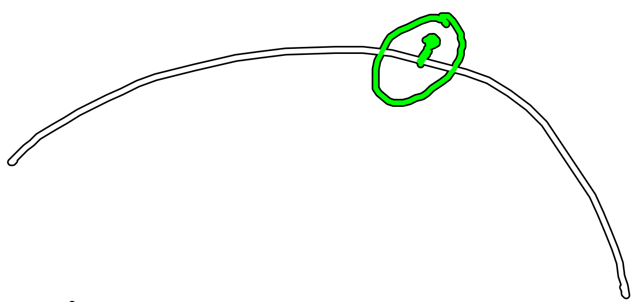
Verteilung des Fehlers

$x, y \quad - \quad x_r, y_r$
x', y'
x'', y''
$\vdots$



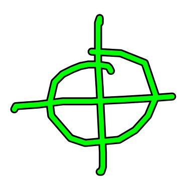
$$\begin{cases} \vec{x}_0 = \underline{a_{-3}}\,\vec{x}_{-3} + \dots + \underline{a_{-1}}\,\vec{x}_{-1} \\ \quad + \underline{\underline{a_1}}\,\vec{x}_1 + \dots \underline{a_3}\,\vec{x}_3 \end{cases}$$

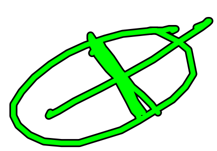
Min. Q.



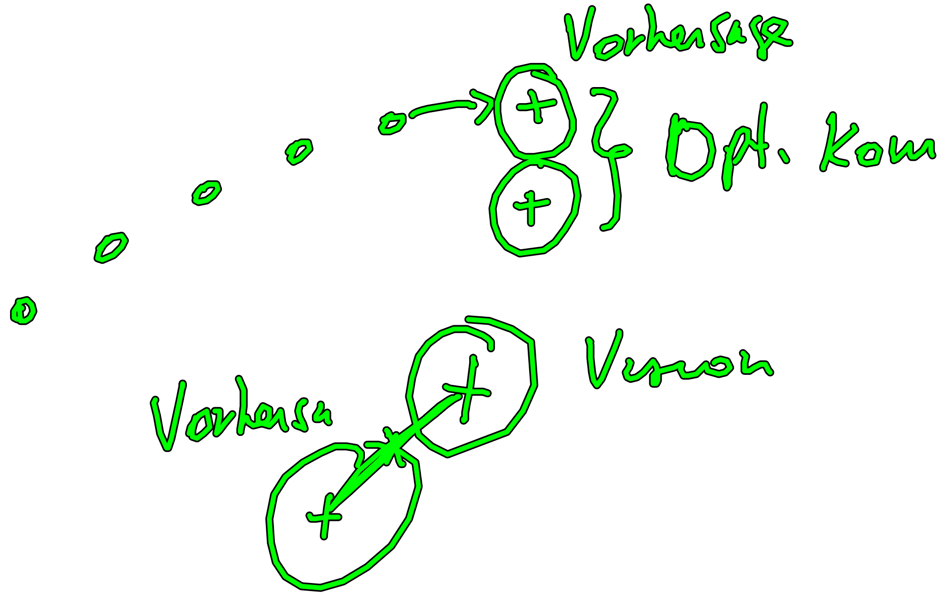
Schätzung des Versuchfehlers

- Verteilung $\underline{x,y}$
- " Roboter





<u>Kalman</u> <u>Filter</u>


Vorhersage
Opt. Kom
Vorhersa
Version