Chapter 10

Scheduling of dependent threads
10.1 Introduction

• In the previous chapters we assumed that the threads to be executed were either independent on each other or described by a TIG.
• Now we consider the dependencies between the threads in that way that an order of the thread execution must be observed.
• On the set of threads a partial order is defined that specifies which thread has to be executed before which other thread.
• The threads are then modeled as a Task Precedence Graph (TPG), i.e. as directed acyclic graph (DAG). The thread weights indicate the execution times of the threads.

Example of a DAG
The upper numbers are the thread numbers, the lower ones are the execution times.
Scheduling of dependent threads

- Threads as parts of a parallel program that is to be executed on a cluster computer.

**Goal**
- For a given number of processors minimize the schedule length (makespan) under the restriction of the order dependencies.

**Variants:**
- specific forms of DAGs (e.g. tree, forest,...)
- Additional consideration of communication times (delays along the edges)
- Additional consideration of deadlines
- Additional consideration of the communication network’s topology
Example

How many processors should we use?

Precedence graph

1
1

2
3

5
2

3
4

6
3

4
2

7
2

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Example

Gantt-Chart

for 3 processors

0 5 10
1 1 3 4
2 2
3 5 6 7

for 2 processors

0 5 10
1 1 5 3 4
2 2 6 7

Precedence graph

1 2 5
3 4 6 7

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For the sake of simplicity we assume that

- the processors are homogeneous (same speed) and
- the latencies between each two processors are the same.

Realistic?
Model

For the sake of simplicity we assume that

- the processors are homogeneous (same speed) and
- the latencies between each two processors are the same.

That means the execution and message latencies depend only on the threads, not on the processors.

In addition, intermediate results are to be transferred at synchronization points.

That leads to the following model:

- $P$: Set of processors
- $T$: Set of threads (vertices)
- $n = |T|$: number of threads
- $E^T \subseteq T \times T$: precedence relation (edge set)
- $\alpha_{ij}$: amount of information exchanged (edge weight)
- $\beta_i$: execution time of thread $i$ (vertex weight)
Other quantities

\( ms \) schedule length (makespan)
\( f_i \) finishing time of thread \( T_i \)
\( s_i \) starting time of thread \( T_i \)

where the following holds:

\[ ms := \max \{ f_i \mid T_i \in T \} \]

\( \pi : T \to P \) mapping of the threads to the processors
Considering communication cost

Let $\pi(T_i)$ be the processor to which $T_i$ is assigned.

**Model A**

Total cost = $ms + cc$

where $cc = \mid \{ (T_i, T_j) \in E^T \mid \pi(T_i) \neq \pi(T_j) \} \mid \cdot \alpha$

The message latencies $\alpha_{ij}$ are assumed to be constant: $\alpha_{ij} = \alpha$.

**Model B**

Total cost = $ms + cc$

where $cc = \mid \{ (P_k, T_i) \mid \exists k : P_k = \pi(T_j) \land (T_i, T_j) \in E^T \} \mid \cdot \alpha$

If two successors of $T_i$ are executed on the same, but different from $\pi(T_i)$ processor, then the result of $T_i$ needs to be shipped only once to the processor on which the successors reside. This is considered in model B (in contrast to model A).
Considering communication cost

**Model C**

Model C explicitly integrates the communication overhead in the calculation of the schedule.

Communication costs between threads on the same processor are zero.

Let \((T_i, T_j) \in E^T\), \(\pi(T_i) = P_k\). Then the following must hold for \(T_j\):

\[
\begin{align*}
    s_j &\geq f_i, & \text{if } \pi(T_j) = P_k \text{ (} T_j \text{ starts after } T_i \text{ finished)} \\
    s_j &\geq f_i + \alpha_{ij}, & \text{otherwise (} T_j \text{ starts after } T_i \text{ finished and transmitted the results)}
\end{align*}
\]

Total cost = \(ms\)
NP-completeness

The NP-Completeness of the scheduling problem is given for the following problem instances:

<table>
<thead>
<tr>
<th>Graph</th>
<th># Processors</th>
<th>Execution time</th>
<th>Communication time</th>
<th>Communication model</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>$m$</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>forest</td>
<td>$m$</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>tree</td>
<td>$m$</td>
<td>1</td>
<td>1</td>
<td>Model A</td>
</tr>
<tr>
<td>tree</td>
<td>$m$</td>
<td>1</td>
<td>1</td>
<td>Model B</td>
</tr>
<tr>
<td>arbitrary</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>Model B</td>
</tr>
<tr>
<td>arbitrary</td>
<td>unlimited</td>
<td>1</td>
<td>$&gt;1$</td>
<td>Model C</td>
</tr>
</tbody>
</table>
Optimal Algorithms

For the following problem instances optimal efficient algorithms are known (unit execution time, communication model C):

<table>
<thead>
<tr>
<th>Graph</th>
<th># Processors</th>
<th>comm.-time</th>
<th>Authors</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>2</td>
<td>0</td>
<td>Coffman&amp;Graham, 1972</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>tree</td>
<td>$m$</td>
<td>0</td>
<td>Hu, 1961</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>interval order</td>
<td>$m$</td>
<td>0</td>
<td>Papadimitriou &amp; Yanakakis, 1979</td>
<td>$O(n+e)$</td>
</tr>
<tr>
<td>arbitrary</td>
<td>$m$</td>
<td>$c$</td>
<td>Jung et al.</td>
<td>$O(n^{c+1})$</td>
</tr>
<tr>
<td>interval order</td>
<td>$m$</td>
<td>1</td>
<td>Ali &amp; El-Rewini, 1993</td>
<td>$O(e + np)$</td>
</tr>
<tr>
<td>tree</td>
<td>2</td>
<td>1</td>
<td>El-Rewini &amp; Ali, 1994</td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>
10.2 List Scheduling

- For general scheduling problems of dependent threads, usually the List Scheduling is used.
- It is a heuristic off-line-algorithm that not necessarily produces optimal schedules (such with minimal length).
- Only for trees it is optimal.

Algorithm schema:

Given: Precedence graph as DAG, vertices weighted with priorities
Goal: makespan of minimal length for \( p \) processors
Initialization: insert all source vertices (vertices without predecessor) into the list.

Loop:

\[ \text{while list not empty do} \]

(i) take vertex with highest priority from the list.
(ii) select an idle processor to execute this vertex.
(iii) check all vertices that are not yet assigned to a processor and not yet in the list, whether all direct predecessors are executed. If yes, insert that vertex into the list.
We use the graph of slide 4 and give all threads the same priority.

The list can then organized as a FIFO queue.

Which plans does the list algorithm produce?

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For three processors

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For four processors
Example

We use the graph of slide 4 and give all threads the same priority.

The list can then organized as a FIFO queue.

The list algorithm produces the following plans:

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
<td>-</td>
</tr>
</tbody>
</table>

For three processors

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>4</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

For four processors
Is the schedule for 4 processors optimal?

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
<td>-</td>
</tr>
</tbody>
</table>

For three processors

<table>
<thead>
<tr>
<th>time</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>3</td>
<td>-</td>
<td>-</td>
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<tr>
<td>3</td>
<td>10</td>
<td>4</td>
<td>6</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

For four processors
Variants of List Scheduling

- If more than one thread is ready to execute, i.e. it is in the list, then the priority decides which to take next. If not externally given, the priority can be set according to different strategies targeting different goals.

- **Def.** The **path length** in a dependency graph is defined as the sum of all vertex weights (execution times) along a path including the first and the last vertex.

- **Def.** The **Level** (static b-Level) of a vertex $x$ is the length of the longest path from $x$ to a sink, i.e. to a vertex without successor (bottom).

- **Def.** The **Co-level** (static t-Level) of a vertex $x$ is the length of the longest path from $x$ to a source, i.e. to a vertex without predecessor (top).

- **Def.** The **critical path** (cp) is the length of the longest path from a source to a sink.
**Example**

**Def.** The **Level** (static b-Level) of vertex x is the length of the longest path from x to the sink.

**Def.** The **Colevel** (static t-Level) of vertex x is the length of the longest path from x to a source.

<table>
<thead>
<tr>
<th>No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Colevel</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Def. The **Level** (static b-Level) of vertex $x$ is the length of the longest path from $x$ to the sink.

Def. The **Colevel** (static t-Level) of vertex $x$ is the length of the longest path from $x$ to a source.
Variants of List Scheduling

**HLF  Highest Level First:**

- The vertex with the longest chain of successors receives the highest priority. The strategy is sometimes also called CP (critical path), since the chosen vertex lies on the critical path.
- Because the critical path determines the whole schedule length, this strategy is promising.
- HLF is optimal for tree-like graphs with unit execution times and generally good in all practical cases. (with random graphs at most 5% worse than the optimal solution in 90% of cases.)
- HLF takes the height of the tree that depends on vertex x as a priority criterion.
- Other variants choose the breadth or cardinality of the subtree as priority instead of the height.
- The aim is to deblock (put in the list) as many threads as possible.
10.3 Algorithms considering communication

- The general approach of List Scheduling can be extended to graphs with edge weights.
- The calculation of levels as priorities is somewhat more complicated since, in contrast to the static levels of HLF, the values now depend on the placement of the predecessors.

- WHY?
10.3 Algorithms considering communication

- The general approach of List Scheduling can be extended to graphs with edge weights.
- The calculation of levels as priorities is somewhat more complicated since, in contrast to the static levels of HLF, the values now depend on the placement of the predecessors.
- We distinguish (for each vertex $x$)
  - $b$-level: length of longest path from vertex $x$ to some leaf vertex or sink, respectively. ($b$ like bottom)
  - $t$-Level: length of longest path from a source to vertex $x$ ($t$ like top)
- The path lengths contain not only the execution times (vertex weights) but also the communication times (edge weights).
- Reference point for the calculation of the levels is the starting point of a vertex.
Calculation of vertex attributes

**Reminder:** \( \alpha_{ij} = \) edge weight, \( B_i = \) vertex weight

**t_level:**
construct list of nodes in topological order (TopList)

```plaintext
for each node \( T_i \in \text{TopList} \) do
    max \leftarrow 0
    for each parent \( T_x \) of \( T_i \) do
        if (t_level\((T_x) + \beta_x + \alpha_{xi} > \max)\)
            max \leftarrow t_level\((T_x) + \beta_x + \alpha_{xi} \)
    t_level\((T_i) \leftarrow \max
```

**b_level:**
construct list of nodes in reverse topological order (RevTopList)

```plaintext
for each node \( T_i \in \text{RevTopList} \) do
    max \leftarrow 0
    for each child \( T_y \) of \( T_i \) do
        if (b_level\((T_y) + \alpha_{iy} > \max)\)
            max \leftarrow b_level\((T_y) + \alpha_{iy} \)
    b_level\((T_i) \leftarrow \beta_i + \max
```
Calculation of vertex attributes

- Some scheduling algorithms use the attribute „as late as possible“ (ALAP). The ALAP value indicates the latest start time of the thread, that doesn't lead to an in increased schedule length (makespan) ms.

**Calculation ALAP:**

construct list of nodes in reverse topological order (RevTopList)

```plaintext
for each node \( T_i \in \text{RevTopList} \) do
  \( \text{minft} \leftarrow \text{length of critical path} \)
  for each child \( T_y \) of \( T_i \) do
    if \((\text{alap}(T_y) - \alpha_{iy} < \text{minft})\)
      \( \text{minft} \leftarrow \text{alap}(T_y) - \alpha_{iy} \)
    \( \text{alap}(T_i) \leftarrow \text{minft} - \beta_i \)
```
Example

<table>
<thead>
<tr>
<th>thread</th>
<th>level</th>
<th>b-level</th>
<th>t-level</th>
<th>ALAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_8$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example

<table>
<thead>
<tr>
<th>thread</th>
<th>level</th>
<th>b-level</th>
<th>t-level</th>
<th>ALAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>T_1</td>
<td>11</td>
<td>23</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>T_2</td>
<td>8</td>
<td>15</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>T_3</td>
<td>8</td>
<td>14</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>T_4</td>
<td>9</td>
<td>15</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>T_5</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>T_6</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>T_7</td>
<td>5</td>
<td>11</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>T_8</td>
<td>5</td>
<td>10</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>T_9</td>
<td>1</td>
<td>1</td>
<td>22</td>
<td>22</td>
</tr>
</tbody>
</table>
Algorithms for fast networks

Assumptions

- The communication times between arbitrary processors are constant, i.e. the network topology does not matter.
- No contention for bandwidth, i.e. many messages can be sent over the same link without additional delay.
- The processor elements are able to execute a thread and simultaneously send messages.
Insertion Scheduling Heuristic (ISH)

- Calculate Level (static b-Level) for each vertex
- Build the ready list in descending level-order from root vertices (entry nodes)

**While** (ready-list not empty) **do**
- Place the first thread of the ready list to the processor that allows the earliest execution.
- If this placement induces idle times, find as many threads from ready list as possible that can be scheduled into this idle time, unless they can start on another processor earlier.
- update ready list.

- Complexity: \(O(n^2)\)
ISH Example

![Graphical representation of an ISH example]

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ISH Example

Level

P1 | P2 | P3 | P4
---|----|----|----
0  | 1  | 0  | 0  
2  | 2  | 0  | 0  
4  | 4  | 0  | 0  
6  | 6  | 0  | 0  
8  | 8  | 0  | 0  
10 | 10 | 0  | 0  
12 | 12 | 0  | 0  
14 | 14 | 0  | 0  
16 | 16 | 0  | 0  
18 | 18 | 0  | 0  
20 | 20 | 0  | 0  

Scheduled order

1  2  4  5  9  8  7  4  3  5

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Modified Critical Path Method (MCP)

Initialization:

Calculate ALAP times of each vertex.
Build for each vertex a list that contains the ALAP times of itself and its successors in descending order.
Sort these lists in ascending lexicographic order.
Create a vertex list in that order.

Loop:

while vertex list not empty do

Place the first vertex of the list to that processor that allows the earliest execution.
Remove that vertex from list.

Complexity $O(n^2 \log n)$
MCP Example

ALAP Schedule order

P1 P2 P3 P4

0 2 4 6 8 10 12 14 16 18 20

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MCP Example
MCP Example

ALAP Schedule order
Earliest Time First Method (ETF)

**Initialization:**
Calculate the static b-Level of each vertex.
Insert source vertices into ready list.

**Loop:**

```plaintext
while vertex list not empty do
  Calculate earliest starting time on each processor for each vertex in the ready list.
  Select the vertex-processor-pair with the smallest start time.
  Place the vertex accordingly.
  Insert new ready vertices into ready list.
```

Complexity $O(n^2)$
ETF Example

Level Schedule order

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ETF Example

Level
Schedule order

P1   P2   P3   P4

0
2
4
6
8
10
12
14
16
18
20

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ETF Example

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10.4 Scheduling with Clustering

- Approaches using clustering decompose the allocation process into two phases:
  - Phase 1: Clustering, i.e. combining all threads that are allocated to the same processor.
  - Phase 2: Ordering; i.e. determining at which times the threads start execution.

- There are as many clusters formed as processors are available.
- The clustering process starts with single element clusters and stepwise merges clusters to larger ones.
- Threads of the same cluster run on the same processor. Communication between those threads is free of cost.
Linear Clustering

A clustering is called non-linear, if two independent threads are assigned to the same cluster, otherwise it is called linear.
Clustering

- The clustering modifies the DAG by zeroing some edge weights
- By doing so, the length of the critical path may also be affected.
- We make the distinction:
  - The (original) DAG
  - the DAG after clustering (clustered DAG, CDAG)
  - the DAG after ordering (scheduled DAG, SDAG)
- The critical path of the SDAG is called **dominant sequence** (DS) of the CDAG.
Critical Path and Dominant Sequence

CDAG

CP: $T_1$, $T_2$, $T_7$
CP length: 9

Gantt-Chart

Critical Path: $T_1$, $T_2$, $T_7$
CP length: 9

SDAG

DS: $T_1$, $T_3$, $T_4$, $T_5$, $T_6$, $T_7$
DS length: 10
Sarkar's Algorithm

- Initialization:
  - All vertices are forming single element clusters
  - All edges are unmarked
  - All edges are being sorted according to descending communication cost.

- Repeat
  - Set unmarked edge with highest weight to 0 (merge clusters), if the makespan is not increased by that.
  - Mark that edge
  - If two clusters are merged, the edges are ordered according to HLF-rule (highest b-level first.).

  **until** all edges marked

---

<table>
<thead>
<tr>
<th>Step</th>
<th>edge</th>
<th>Zeroing</th>
<th>makespan</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
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</tr>
<tr>
<td>1</td>
<td>(T₁, T₂)</td>
<td>(T₁, T₂)</td>
<td>13,5</td>
</tr>
<tr>
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<td>(T₃, T₄)</td>
<td>(T₃, T₄)</td>
<td>12,5</td>
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<tr>
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<td>(T₃, T₅)</td>
<td>(T₃, T₅)</td>
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<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>(T₄, T₆)</td>
<td>(T₄, T₆)</td>
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</tr>
<tr>
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<td>(T₅, T₆)</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>(T₁, T₃)</td>
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<tr>
<td>8</td>
<td>(T₆, T₇)</td>
<td>∅</td>
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</tr>
</tbody>
</table>
Sarkar's Algorithm

Complexity: $O(e(n+e))$
Dominant Sequence Clustering (DSC)

- **Initialization:**
  - All vertices are single element clusters
  - All edges are unmarked
  - \( r=0 \)
  - Calculate \( D_{S_0} \)
  - Initialize Ready List with source vertices

- **While not all edges marked**
  - Let \((T_i, T_j)\) be the topmost unmarked edge in \( D_{S_r} \)
  - Mark this edge
  - Delay zeroing of edge until \( T_j \) becomes ready.
  - Select ready vertex \( T_k \) as the one which runs to the longest path through ready vertices of the SDAG
  - Zero those incoming edges of \( T_k \), that minimize the \( t \)-level of \( T_k \).
  - Schedule \( T_k \) after the last already scheduled vertex of its cluster.
  - Insert successors of \( T_k \) into ready list that just became ready, if available
  - If all edges in \( D_{S_r} \) are marked: increment \( r \) and find new \( D_{S_r} \)

end while
### DSC-Algorithm

<table>
<thead>
<tr>
<th>Step</th>
<th>Edge</th>
<th>ready</th>
<th>Zeroing</th>
<th>makespan</th>
</tr>
</thead>
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<td>$T_1$</td>
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<tr>
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<tr>
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<tr>
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<td>9</td>
</tr>
</tbody>
</table>
DSC-Algorithm

Complexity: $O((n+e) \log n)$
Further references