Mnemonics for the Poisson distribution

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May 2010

Abstract

This short note relates the Poisson distribution to the Taylor expansion of $\exp(x)$, making it easier to remember its definition.

The Poisson distribution

The Poisson distribution describes the probability of ocurrence of seldom events (telephone calls, goals in a soccer match, etc.). It is defined as

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

for $x = 0, 1, 2, \ldots$ The parameter λ is the expected number of ocurrences of the event in the interval being considered. The Poisson distribution approximates the binomial distribution when the probability p of a Bernoulli trial is low and $\lambda = np$, where n is the number of trials. The constant λ must be that, a constant, it should not change as time advances.

If you are like I am, chances are that you will forget the definition after first having seen it, or that you will have to deduce the expression again from the binomial distribution.

However, if you look more closely to the definition of $p(x)$, you will notice immediately that the terms $\lambda^x/x!$ are just the monomials of the Taylor expansion of e^λ, that is,

$$e^\lambda = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \tag{1}$$

Since we require the sum of all terms in the Poisson distribution to be one this is easily achieved multiplying e^λ by $e^{-\lambda}$, that is,

$$1 = \frac{\lambda^0}{0!}e^{-\lambda} + \frac{\lambda^1}{1!}e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \cdots$$

The individual terms are just the individual elements in the definition of the Poisson distribution for $x = 0, 1, 2, \ldots$ The series is easy to remember.

Another way of arriving to the Poisson distribution is by considering the binomial expression

$$(1 + \lambda/n)^n = \binom{n}{0}\left(\frac{\lambda}{n}\right)^0 + \binom{n}{1}\left(\frac{\lambda}{n}\right)^1 + \cdots + \binom{n}{k}\left(\frac{\lambda}{n}\right)^k + \cdots + \binom{n}{n}\left(\frac{\lambda}{n}\right)^n$$

which can be rewritten as

$$(1 + \lambda/n)^n = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \cdots + (1 - 1/n)(1 - 2/n)\cdots(1 - k/n)\frac{\lambda^k}{k!} + \cdots$$

In the limit, when $n \to \infty$, we obtain the expression (1) again.