Visualizing Harmonic Analysis Transforms

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Abstract

The Laplace transform is the more general harmonic transform. The $z$- and the Fourier-transform are specializations of the former. A visualization of the connections between the different continuous and discrete transforms is presented in this short note. The visualization can be of educational value.

Enter Laplace

The Laplace transform of a real function $f$ is defined as

$$\mathcal{L}(s) = \int_{0}^{\infty} e^{-st} f(t) dt \quad s \in \mathbb{C}$$

The two-sided Laplace transform is defined as

$$\mathcal{L}(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt \quad s \in \mathbb{C}$$

In each case we have a product of the function $e^{-st}$ with the real function $f$.

The Fourier transform is just a specialization of the Laplace transform, when the complex number $s$ is of the form $s = i\omega$, that is purely imaginary. Thus the Fourier transform $F$ of $f$ is given by

$$F(\omega/2\pi) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$
The function $e^{-i\omega t}$, for $-\infty < \omega < \infty$ and $-\infty < t < \infty$, can be represented by a matrix $E$ with continuous indices of the form:

$$E = \begin{pmatrix} \vdots & e^{-i\omega t} & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \end{pmatrix}$$

where the entries in the matrix are all values of $e^{-i\omega t}$ arranged like points in a plane. The infinite dimensional vector $f$ can contain all values of $f(t)$ for $-\infty < t < \infty$ and we then write

$$\int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt = \begin{pmatrix} \vdots & e^{-i\omega t} & \vdots \\ \vdots & & \vdots \end{pmatrix} \begin{pmatrix} f(t) \\ \vdots \\ \vdots \end{pmatrix} = Ef$$

The continuous multiplication operation is an integration. This notation is just a way of visualizing the Fourier transform as a linear operator acting on a function (as is done in quantum mechanics with the Dirac notation).

**Visualizing $\mathcal{L}$ and $\mathcal{F}$**

We can generalize the visualization shown above to three variables, and represent the Laplace operator by a three-dimensional arrangement of the values $e^{-(a+i\omega)t}$, as shown in Fig.1.

![Figure 1: The volume of values $e^{-(a+i\omega)t}$](image)

The cube in Fig.1 contains at each position the value of $e^{-(a+i\omega)t}$, for $a$, $\omega$, and $t$ running from $-\infty$ to $\infty$. 
The Laplace transform of $f$ is the product of the cube with $f$. We can think of this product as the weighted sum of all vertical cuts through the cube, as shown in Fig.2. The product with the function $f$ “collapses” the cube onto a square. We can think of a product of an operator with a function as dimensionality reduction in the direction of the product.

![Figure 2: The Laplace transform as dimensionality reduction](image)

Fig.3 shows the specialization of the Laplace transform for the case $a = 0$, that is, for the Fourier transform. This represents the product of the vector with a cut of the cube at $a = 0$. In the case of the Fourier transform we have to scale the argument $\omega$ by $2\pi$, but that is just a detail. The Laplace transform $L(s)$ is thus a function of a complex argument, while the Fourier transform is a function of a real argument $\omega$ (Fig.4).
The *z*-Transform

The bilateral *z*-transform is a specialization of the Laplace transform for discrete values of *t*, i.e. *t* = ..., −2, −1, −0, 1, 2, .... Therefore

\[ F(z) = \sum_{k=-\infty}^{\infty} z^{-k} f(k) \]

In this case, the function being transformed is discrete. Since any complex number *z* can be written as *z* = *e*\(^{a+\omega}i\), we have the specialization shown in Fig. ?? where the vector of boxes represents a discrete vector.

Fourier series

In the case of continuous \( F \), but a set of discrete frequencies \( \omega_1, \omega_2, \ldots, \omega_N \), we can compute a Fourier series as

\[ F(\omega_k/2\pi) = \int e^{-i\omega_k t} f(t) dt, \quad \text{for } k = 1, \ldots, N \]

This is the specialization shown in Fig.6.
We can also handle the case of a discrete function $f$, by specializing the $t$’s, as shown in Fig.7. This is called the discrete frequency Fourier transform, the dual of the Fourier series. This is just a special case of the $z$-transform.

Specializing the Fourier transform for discrete frequencies and discrete values of $t$, leaves us with a conventional matrix-vector multiplication for the computation of the discrete Fourier transform.

**The complete visualization**

Finally, we are left with the following complete visualization of the transformations and their connections.
Figure 8: Visualization of the discrete Fourier transform
Figure 9: Complete visualization of the harmonic transforms