A ~~note~~ TUTORIAL on some empirical properties of the Mandelbrot Set

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TUTORIAL

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Abstract

In this note we inspect the kind of patterns produced by the quadratic iteration used to generate the Mandelbrot set. A very simple relation involving the Fibonacci numbers can be found between the relative positions of the disks which form the Mandelbrot set and the number of attractors of the iteration. Using this insight the concept of escape time can be generalized to points inside and not just ouside the Mandelbrot set. The pictures obtained with this method produce the illusion of perspective and resemble somewhat the classical ray-tracing images of crystal balls.

1. Introduction

The Mandelbrot set is a never ending source of surprise and inspiration for computer scientists. Its well known cardioid shape and its exuberant boundary have become a kind of trademark for all those investigations concerning chaos and nonlinear dynamics that are being pursued around the world. There is hardly a computer science student who has not yet heard of the set or who has not seen a picture of it.

The standard view of the Mandelbrot set (we call it M in what follows) depicts the points in the complex plane which belong to M with a black pixel. Other points of the plane are colored according to the so called escape time of the complex quadratic iteration given by the transformation rule $z \rightarrow z^2 + c$, where c is the complex number which represents the point being investigated (the iteration is started with z=(0,0)). As long as the succesive points of this iteration remain in the region around the origin bounded by the circle of radius 2, it is said that the iteration does not escape to infinity. If the iteration skips on step k over the borderline of the circle, then it is guaranteed that the iteration diverges to infinity and k is

called its escape time at point c. This point is colored according to a table associating succesive natural numbers to different colors (up to an upper bound, of course).

The most interesting graphics of the Mandelbrot set which have been made show not the set proper but its outer border. There are around M extremely chaotic regions which when colored in the appropriate way produce very nice pictures. Peitgen et al [1986, 1988] have shown just how wonderful these pictures can be.

In most of these computer graphics there is a great contrast between the color richness of the boundary of M and the sober black used to represent M itself. It is as if a big misterious mysterious region laid ~~just~~ exists on at the other side of the colorful boundary, an unknown and unknowable land. As a matter of fact the internal structure of M can be visualized in a very simple way. Nevertheless this is seldom done, because it is implicitly assumed that in M itself only uninteresting details can be found. Some programmers, for example, have tried to color the points in M according to the distance to the origin of the last number calculated in the complex iteration. Unfortunately this yields not very interesting graphics: the only difference to the standard view of M is that each bud or bubble of the set contains concentric rings centered on each ~~one of them~~ Bud [Becker et al 1989]. Other researchers have adopted alternative methods to inspect the interior of M which are slightly more attractive [Entwistle 1989].

However the most natural way of visualizing the interior of M is simply to extend the criteria used to graph the complement of the set. Points outside of M are colored according to their escape times to infinity. Points inside M can be colored according to their escape times to certain convergence points or attractors. This is a generalization of the first rule because we can think of the points outside of M as points whose iteration converge to infinity, and we just test in each iteration step ~~if~~ whether our succesion has already entered a ball centered at infinity (whose complement is a ball centered at the origin). Figure 1 was produced making use of this idea. It shows the Mandelbrot set in its standard position, but now the interior of the set is not simply black, it displays a structure given by the escape times of the points which form it. The misterious black zone has disappeared and we get a pattern which produces the illusion of a three dimensional body protruding out of the two dimensional page.

spelling

2. What happens in the interior of the Mandelbrot set?

In order to understand better how the structured pattern of Figure 1 was obtained it is ~~instructing~~ INSTRUCTIVE to look at the succesion of complex numbers obtained with the iteration $z \rightarrow z^2 + c$ with different c's. One gets ~~a surprising~~ an INTERESTING connection between the position of M's buds and the Fibonacci numbers. This "embedding" of the Fibonacci numbers in the Mandelbrot set was pointed out in another context by Peitgen and Douady [1986, p. 61].

Figure 2 shows the points obtained in the quadratic iteration when c is the complex number (–0.11,0.6425). The first points of the iteration are the ones located in the tips of the spiral

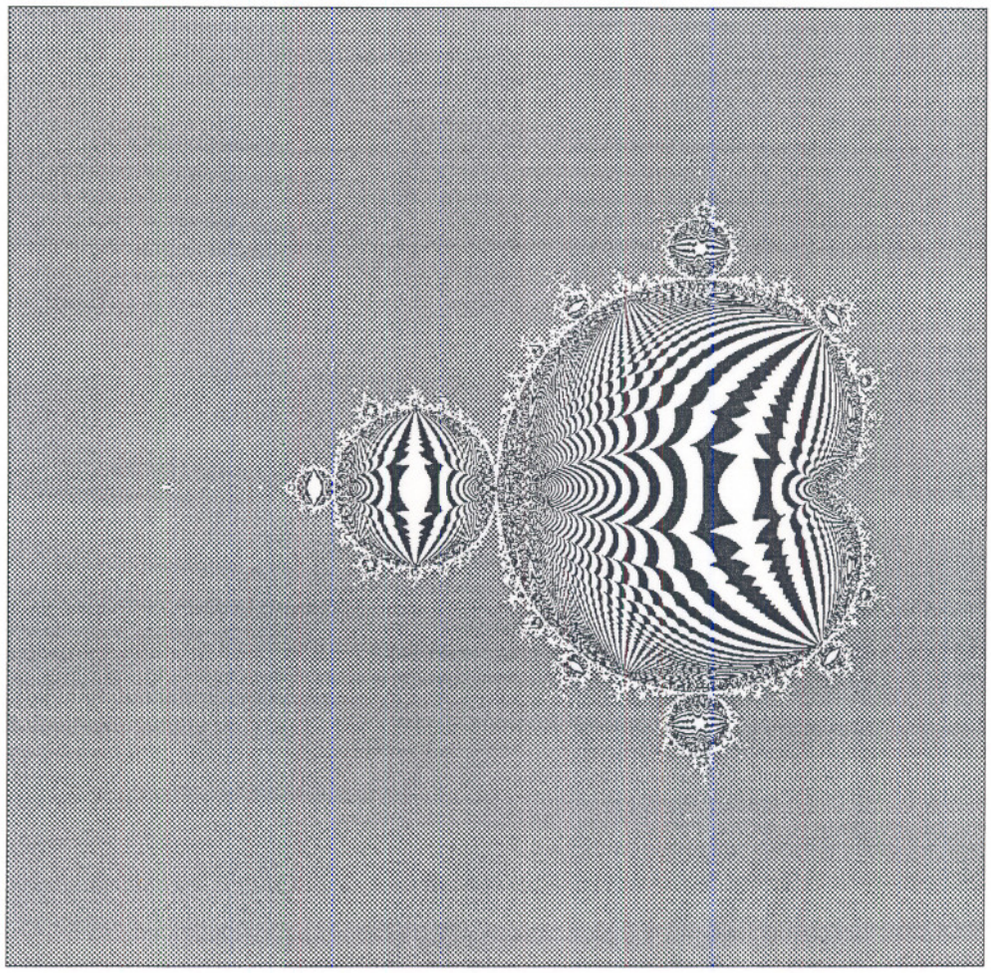

Figure 1: The inner structure of the Mandelbrot set

Figure 2: Pattern of 1000 iterations with c=(-0.11,0.6425)

arms. The iteration skips from one spiral arm to the next, but always approaching a limit point in the middle of the arms. Note the symmetry of the produced pattern. The point which we used as our c for the iteration is one located within the main cardioid of M. If we make the same experiment using other values of c taken from the interior of the cardioid we get similar results: the succesion converges to a single point through several spiral arms. Figure 3 for example looks very much like the tail of a peackcock and was produced using the value c=(0.27,0.006).

A c in the interior of the circle located to the left of the cardioid can be used to launch the iteration and we get the result shown in Figure 4. This time there are two different concentration points for the succesion. Each cluster of points shows the same kind of spiral structure found before. Even more interesting is the way in which the succesion jumps from one cluster to the other. In each step the succesion approaches one of the concentration points and in the next step jumps in the neighborhood of the other. When the iteration is visualized on a computer screen then a point can be seen jumping between one cluster and the other. After a big enough number of iterations the succesion alternates exclusively between the two concentration points. We can call this two points the two attractors of the cycle of period 2 of the succesion. Douady calls succesions which fall into a cycle after a finite number of steps preperiodic [1986, p. 164].

large

Similar experiments with other c's located in the interior of the second biggest bud of M (the disk centered at (-1,0)) are enough to sugest that in this region the complex succesion has two attractors (whereas in most of the cardioid the succesion has one attractor).

What about other buds of M? Further experimentation show that if a c from the next bud to the left is taken, then the succesion has four attractors whose neighborhoods are visited one after the other by the quadratic iteration. If again a c in the next bud to the left is used eight attractors are found, etc. That is the number of attractors in the succesive buds of M to the left exhibit the period doubling typical of many chaotic processes [Norton 1989].

commas

The bud on top of the main cardioid can now be examined. Iterations which use a c in this region posses three different attractors. We call this bud the 3-disk. The next bigger bud to the left of this one and over the cardioid generates iterations with five attractors (this we call the 5-disk). The biggest bud between the 3-disk and the 5-disk generates iterations with 8 attractors. As can be seen it is possible to find out how many attractors the iterations in each bud generate just by adding the attractor number of the two nearest bigger buds at each side of the one examined. In this way we get the Fibonacci numbers. Figure 5 shows in which way the number of attractors is distributed in the Mandelbrot set.

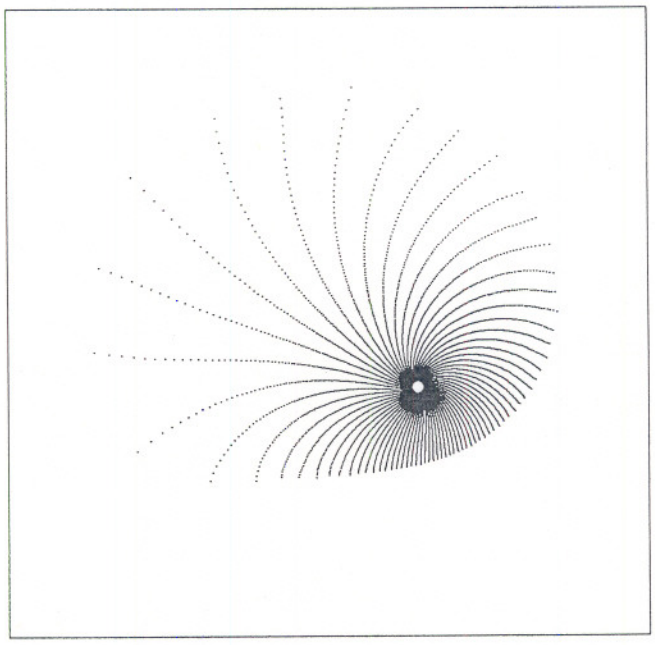

Figure 3: Pattern of 15000 iterations with c=(0.27,0.006)

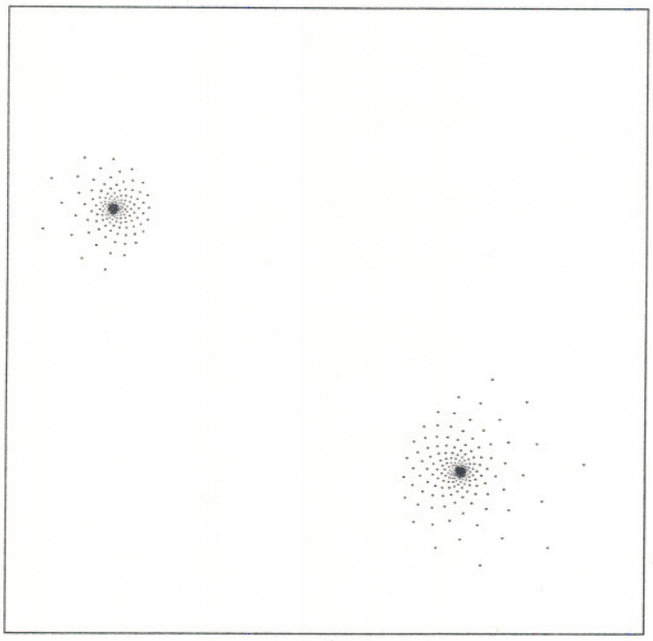

Figure 4: Pattern of 1000 iterations with c=(-0.78,0.10)

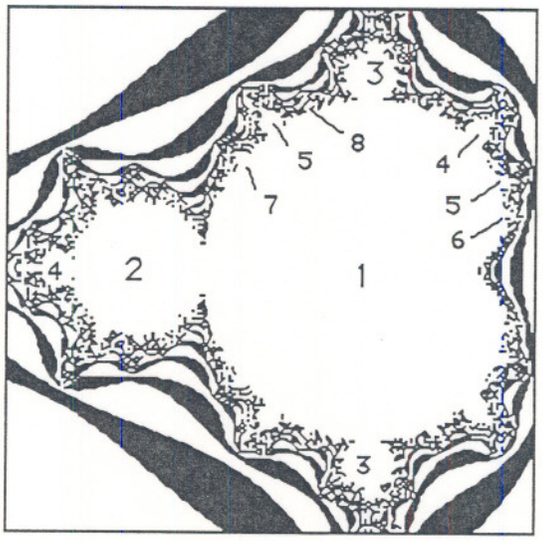

Figure 5

The above discussion holds true for c's not too close to the boundary of each bud. If a c close to the boundary of the main cardioid is used then we get more chaotic looking iterations like the one shown in Figure 6. Here the spiral arms are not so evident as before, and after 20000 iterations the limit point has been approached, but as one can see, very slowly. Yet more interesting is to look at the iteration pattern as one moves from the interior to the exterior of the Mandelbrot set. Figure 7 shows just that. Here the first iteration pattern at the top left shows the succesion approaching one attractor through spiral arms. A slightly different c transforms the pattern into a more complicated structure (top right). The next pattern (middle left) is astonishing: this time the iteration appears not to converge to a single point, but rather seems to remain "stationary" over a very narrow twisted band. The next three iteration patterns show how the iteration "explodes" on crossing the boundary of M. In the last pattern the iteration jumps to "infinity" (very far from the origin) after just a few steps.

Many things can be learned from these simple experiments. After a few tries it is possible to generate iteration patterns with an spiral structure and the desired number of spiral arms. All that has to be done is to select a point in the cardioid near to the bud which exhibits the periodicity which is equal to the number of arms that we want in our spiral. That is why in Figure 2 the number of spiral arms is three. The c which was used is a point in the cardioid near to the 3-disk region. If an spiral pattern with 5 arms is wanted, all that is needed is a c in the cardioid but near to the 5-disk. It is not difficult to see that in this way spiral patterns with any number of arms can be generated (because there are an infinite number of buds between any two buds on the cardioid).

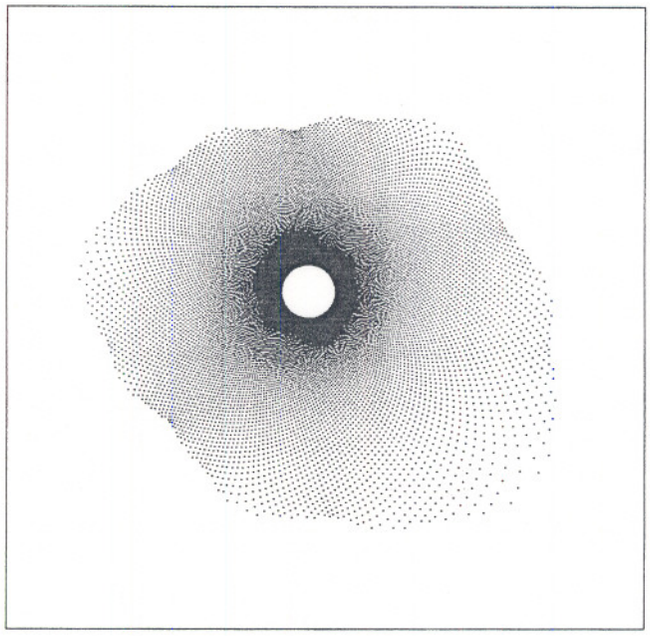

Figure 6: Chaotic pattern obtained with c=(-0.3905,0.5867)

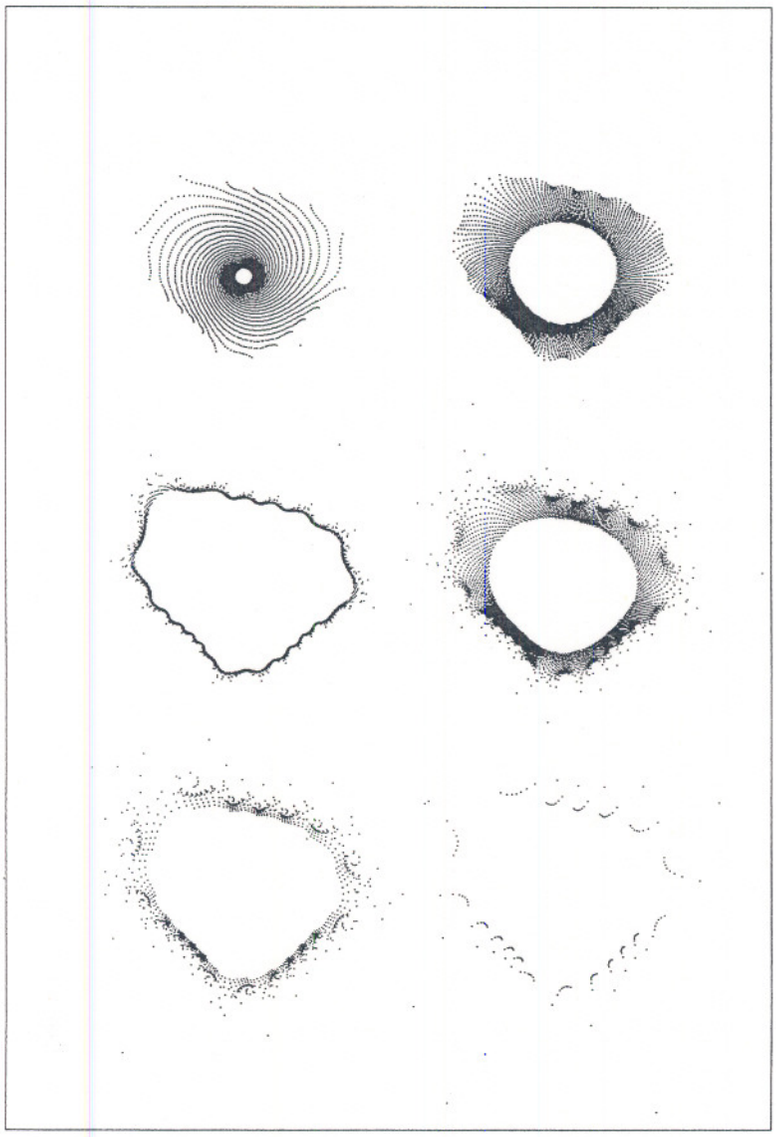

Figure 7: Six iteration patterns with slightly different c's

In summary: what we learn when we look at the iteration pattern produced by different c's is that the complexity of M is not restricted to points outside of it and close to its boundary. The interior of M is also very interesting and deserves more that just the dark coloring used in the standard view of the set.

3. Visualizing the interior of the Mandelbrot set.

We start this section with a new picture of the Mandelbrot set obtained using the ideas explained above. Figure 8 shows M in its "classic" position but now black and white stripes are present inside and outside of M.

We have already pointed out that all we need in order to visualize the interior of M is to generalize the escape time concept for the complex iteration $z \rightarrow z^2 + c$. We can do it in this way: For each pixel in the explored region the corresponding value of c is computed and a certains number of iterations is performed (say 1000). Points outside of M escape far from the origin at some step. Points inside M remain bounded by the circle of radius 2 and centered at the origin. It is not difficult to test if the iteration converges to a single point or if it forms clusters around two, three or more attractors. We define the escape time of the iteration as the number of steps in which the succesion of points skip inside a circle centered around one of the attractors and of radius ε, where ε is assumed to be sufficiently small. If each pixel is colored according to its escape time (in our examples odd escape times are black, even escape times are white) then very nice pictures of M are generated.

Figure 9 and 10 show the 3-disk and the 5-disk respectively. The colored escape regions of M give now the illusion of spheres floating above a bigger sphere. Some colleagues who saw these pictures were automatically reminded of the classical ray-tracing images of crystal balls. It is possible to look now at the Mandelbrot set in a different way: try to visualize the smaller buds of M as spheres floating farther away above another sphere and receding toward the horizon. It is not difficult to perceive this illusion.

PERHAPS

Incidentally: maybe there is another connection between the number of attractors generated by a bud and the number of ramifications at the top of each bud. Figure 10 shows this clearly. Five ramifications correspond to the 5-disk which lies below. Further inspection of M reveals that as far as the eye can see this rule remains unchanged. That means that the number of bifurcations above the buds is ruled also by the Fibonacci sequence.

Figure 11 is also very impressive. The grey background provides a good contrast to the whole army of "spheres on fire" floating above the 2-disk of the Mandelbrot set.

OK

I WONDER WHY YOUR INTERIOR CONTOUR BANDS ARE JAGGED / NOT SMOOTH?

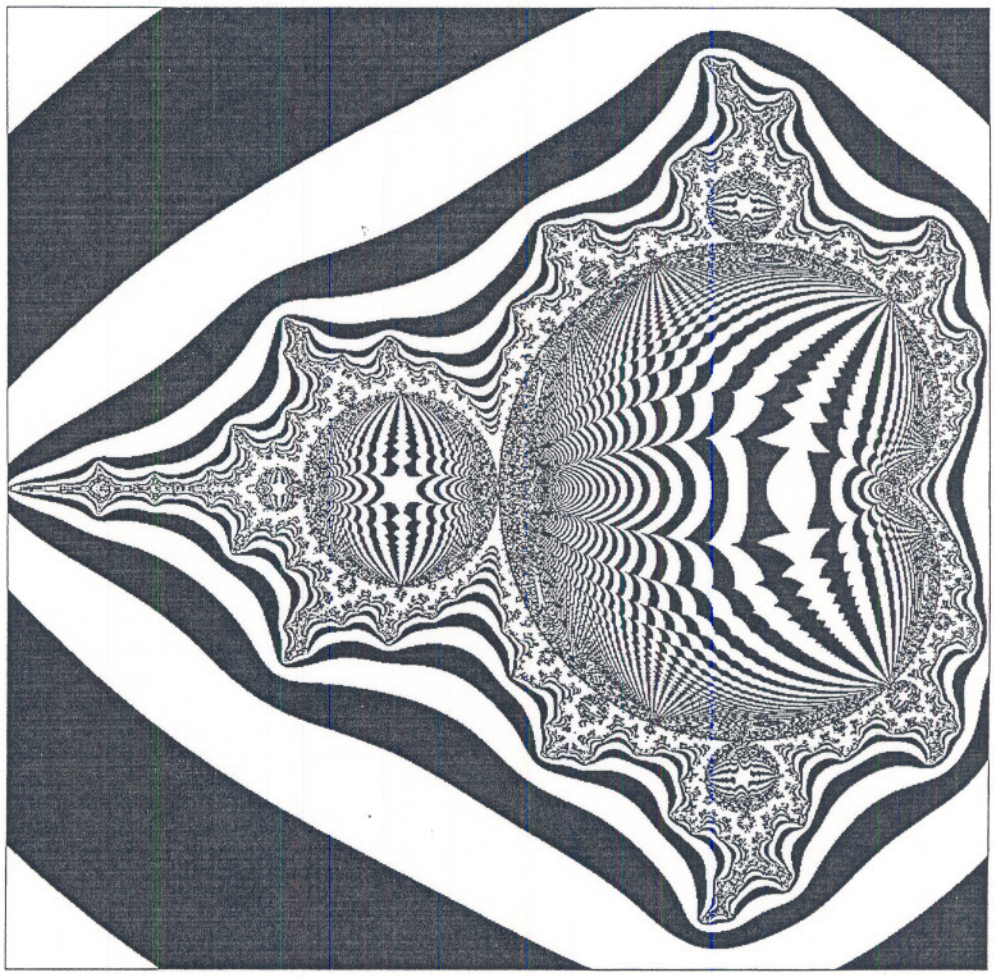

Figure 8: Escape zones inside and outside of M

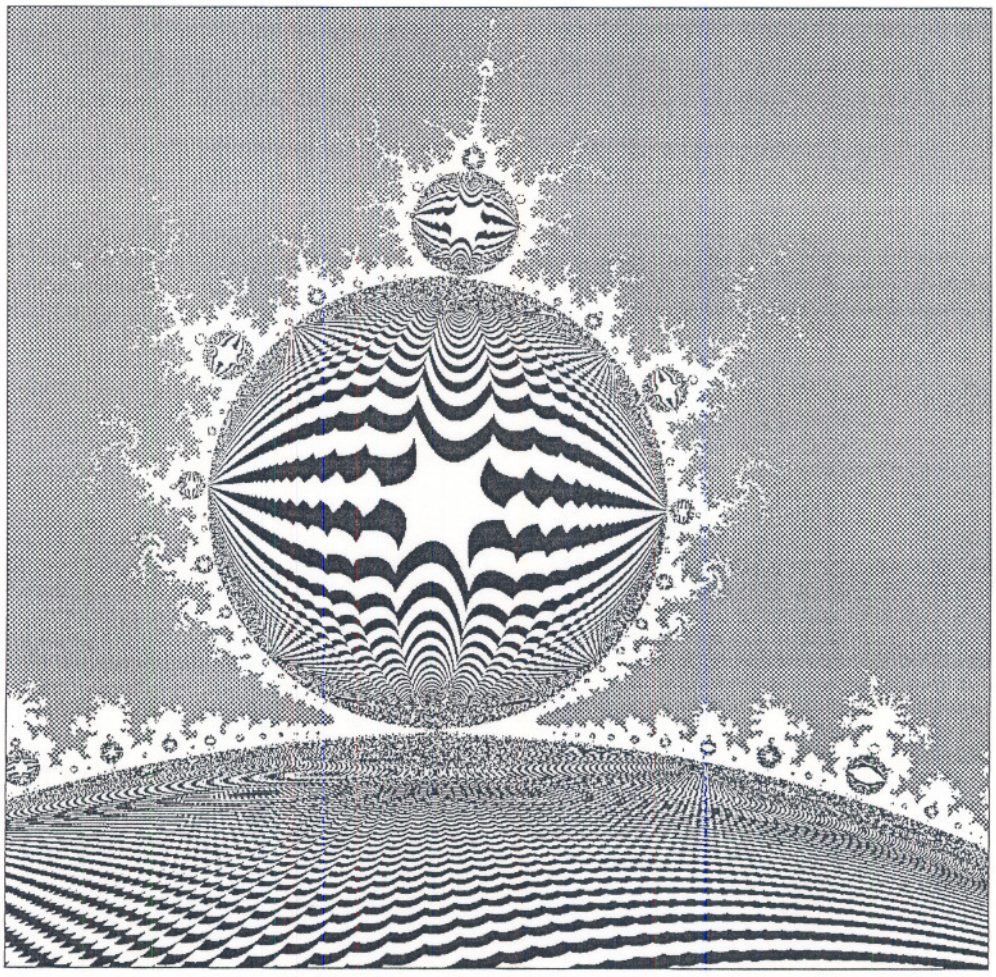

Figure 9: The upper 3-disk inner structure

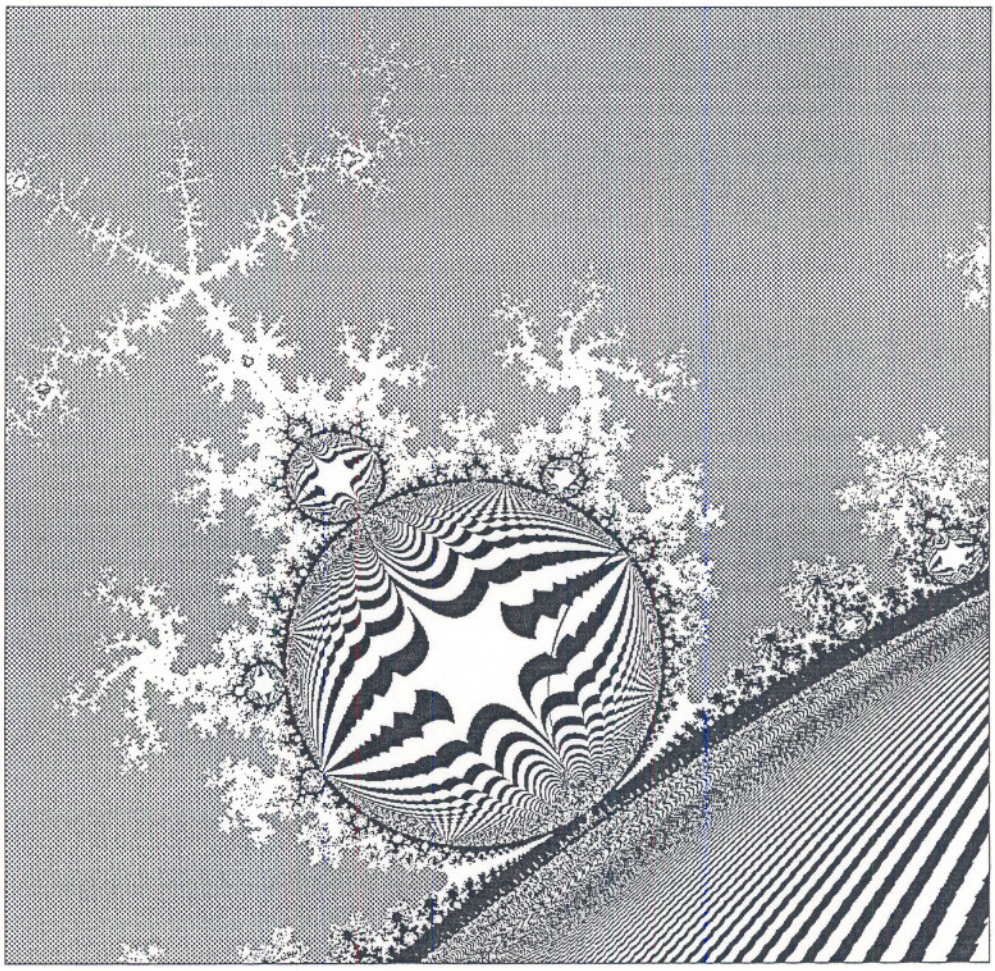

Figure 10: The upper 5-disk escape zones

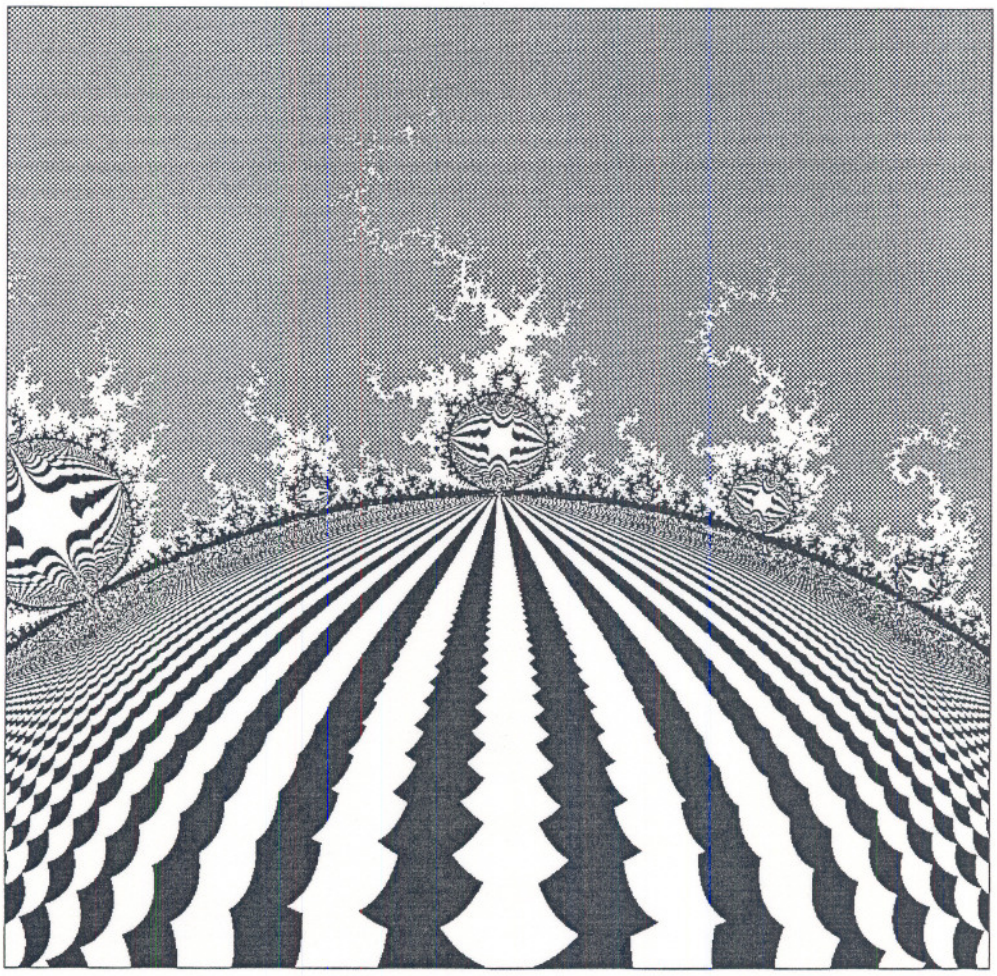

Figure 11: A zoom of the 2-disk showing some smaller buds

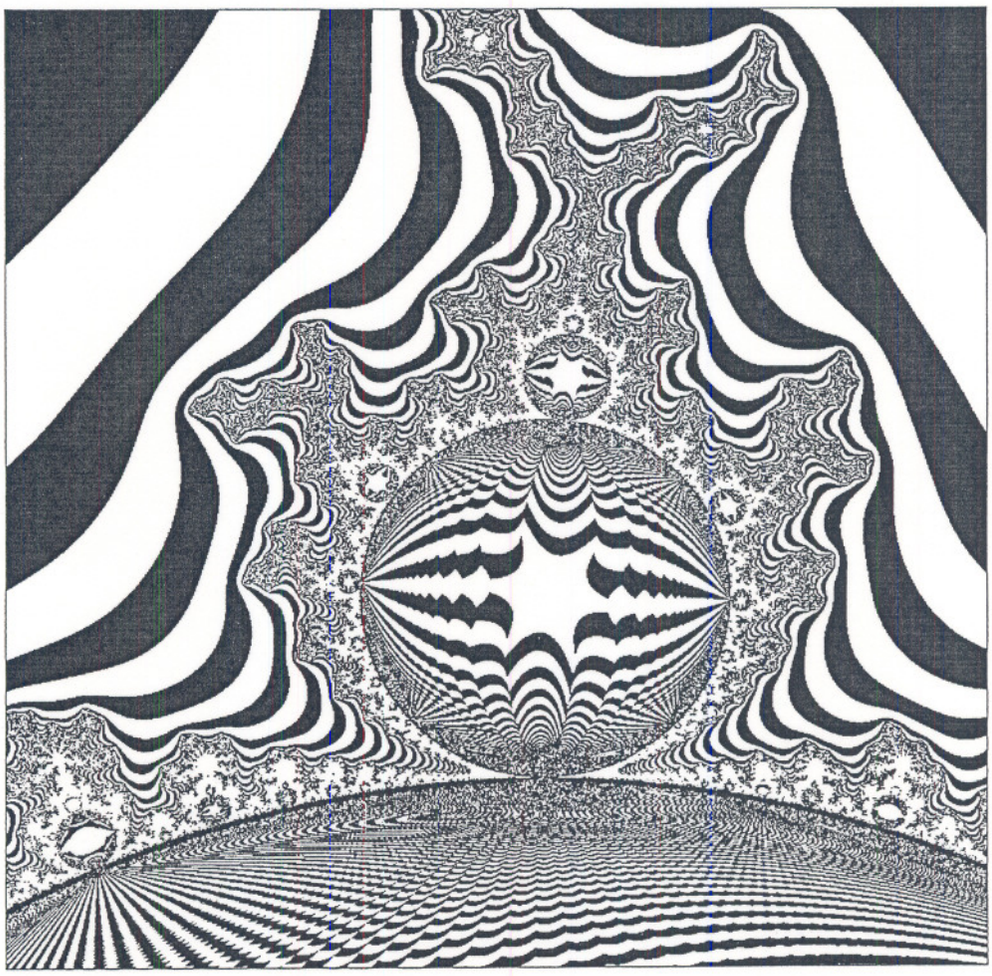

Figure 12: Escape zones outside and inside the upper 3-disk

BE A LITTLE MORE FORMAL

4. Conclusions

NOT Needed

Somebody may say: So what? That the Mandelbrot set is a very complicated beast was already known. That its interior is also very complicated is also a known fact.
To this objections it can be answered that although M is complex, just how complex it is and in which form is something that we are just beginning to realize. An empirical exploration of the set may pave the way to some new theorems about the internal structure of M. The experiments that we described in this note are just one of the many kinds of empirical explorations which can be made in order to elucidate the internal estructure of M. At the very least these explorations produce new views of M which are very intriguing, unexpectedly "three dimensional" and surely more interesting than a sea of blackness.
Another possible application of the ideas developed in this note could be to simplify the calculation of M. Instead of iterating 1000 times for each point in M and waiting to see if the iteration does not skip to infinity, it would be much more economical to inspect the succesive points in each iteration step. If one or more attractors have been reached, then it does not pay to continue iterating because we know that in succesive steps the iteration will be just jumping from one attractor to the other. A large percentage of the pixels in the interior of the main cardioid of M converge to a single point just after 10 or 20 iterations. In this case it is safe to stop iterating and go to the next point. This idea should give a significant speed-up when in a picture big "black" regions of M have to be calculated.
But there is something more about M which fills one with wonder. A careful examination of the iteration patterns produced with different values of c lets one understand how, by varying the single parameter c in the iteration, it is possible to produce all kinds of spiral forms. These spiral patterns resemble the ones found in fluid dynamics and in the heavens (there are spiral galaxies with two, three or more arms). For the form of these spiral patterns there is no universally accepted theory. Galactic patterns are sometimes explained by shock waves in the galactic material, sometimes by other means. The iteration patterns of M show that just one parameter can yield many different forms. If we conceptualize the movement inside a galaxy as an "iteration", whereby the state in the next step is determined by the system state in the previous one, could it be possible to think of the spiral form of the galaxy just as the attractor for the especial iteration being performed? The complexity of the patterns produced by the N-body problem represented by the galaxy could then be reduced to a simpler problem involving attractors of complex non-linear iterations. The last pictures in this note try to suggest to the reader the new possibilities offered by this approach to a very old problem. One of the more intriguing things that I found in my experiments with the iteration pattern was that the attractor always lies close to the center of mass of the iterated points (even at an

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THIS TYPE OF SPECULATION SHOULD BE MADE BY AN EXPERT ASTRONOMER

early stage). Could it be that the Mandelbrot set contains the answer to some of these questions?

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INSIDE M M-SET

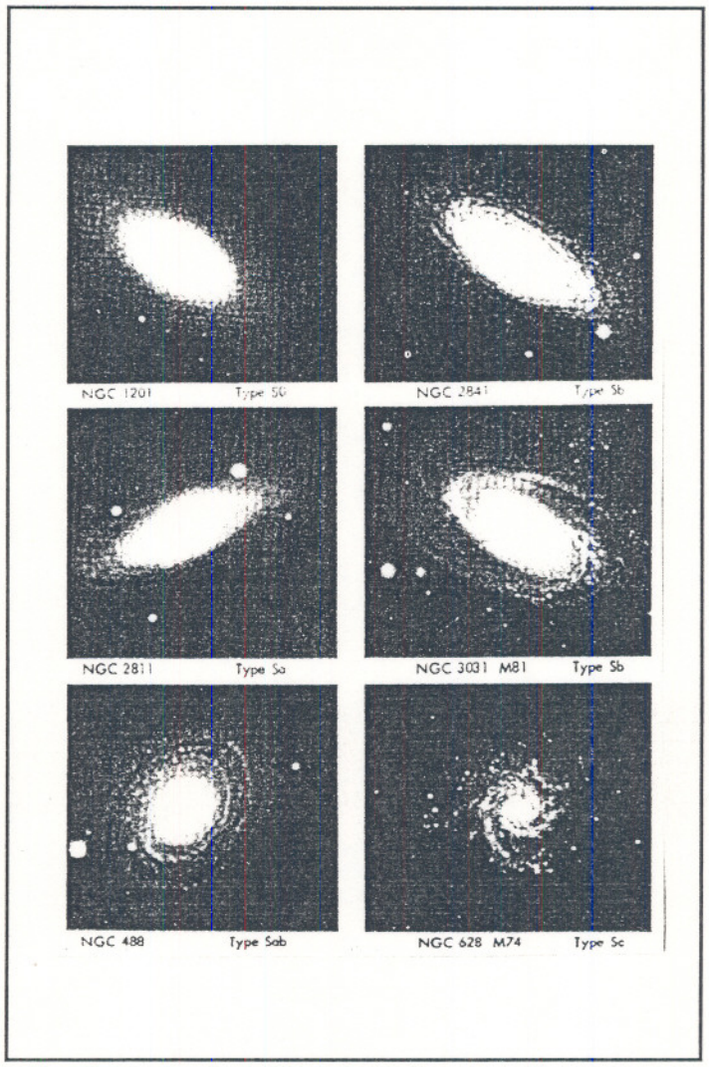

Figure 13: Some spiral and elliptic galaxies

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