Semester Report WS05/06 of Florian Zickfeld

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Field of Research:	Discrete Mathematics
Topic:	Schnyder Woods and Geometric Representations of Graphs
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Current Research

Schnyder Woods and Orthgonal Surfaces

Schnyder Woods are very useful combinatorial structures on 3-connected planar graphs. They consist of an orientation and coloring of the edges. Schnyder Woods are in close relation with 3-dimensional orthogonal surfaces and my work in this field is aimed at improving the understanding of this relation. I gave some background on the topic in the report for the summer term 2005 ([4]). Therefore, I omit to repeat this here and refer to [2] for a comprehensive treatment of the material. The work on this topic is joint with Stefan Felsner.

I spent much of my time on the "height problem". It has a similar flavor as the representation of orthogonal surfaces by a Schnyder Wood and face weights, which I represented in the report for the summer term 2005. Let $\mathcal{O} \subset \mathbb{R}^3$ be an orthogonal surface, and $p \in \mathcal{O}$ a point on that surface. We define the *height* of p as

$$h(p) = p_1 + p_2 + p_3.$$

Let $\mathcal{V}$ be the set of minima of $\mathcal{O}$, which correspond to the vertices of S and let $\mathcal{F}$ be the maxima, which correspond to the bounded faces of S.

Problem 1 Let $\mathcal{O} \subset \mathbb{R}^3$ be an orthogonal surface, and S a Schnyder Wood induced by $\mathcal{O}$. Does the Schnyder Wood S together with the set $\{h(x)|x \in \mathcal{V} \cup \mathcal{F}\}$ uniquely determine $\mathcal{O}$?

I was not able to solve the height problem in full generality nor restricted to orthogonal surfaces which induce a triangulation. The following facts make me confident that nevertheless there should be an affirmative answer to Problem 1.

Stacked Triangulations We can show that every stacked triangulation is uniquely determined by its Schnyder Wood and the height vector $(h(x))_{x \in \mathcal{V} \cup \mathcal{F}}$.

Enumeration I have enumerated all Schnyder Woods on triangulations with up to twelve vertices with a computer program. All of them turned out to uniquely determine the orthogonal surface inducing them together with the corresponding height vector.

On my webpage notes on the techniques I tried to deploy to solve the problem are available [5].

Split-Merge Transition Graph

Bonichon, Felsner and Mosbah presented operations on Schnyder Woods called *merge* and *split* in [1]. We hope that the study of the transition graph of these operations on the set of Schnyder Woods with n vertices will give further insight into the structure of this set. I was able to show that the minimal degree of this graphs is $4/3 \cdot n$. The maximal degree is quadratic in n when we allow all kinds of splits and $6 \cdot n$ in the limit when restricting the set of allowed splits to the so called *short splits*. The goal of the work on this topic is to give a good upper bound for following question.

Problem 2 What is the diameter of the transition graph?

Lifting Stacked Triangulations

Together with Günter Ziegler I am trying to answer the following question.

Problem 3 Is it possible to lift every stacked triangulation T on n vertices to a 3-polytope P with the following properties:

- all vertices (0-dimensional faces) of P have integer coordinates
- the size of the embedding is polynomial in n?

We have two partial results on Problem 3. Using Tutte's approach of proving Steinitz's Theorem (see [3]) we can show that "dense" stacked triangulations can be embedded in $O(n^4)$. I will briefly describe the approach. With the embedding of a stacked triangulation we associate a rooted tree in the following way. The bounded triangle defined by the three outer vertices is the root. The subtree of every triangle (facial or separating) is the set of bounded faces contained in that triangle. Thus, every inner vertex of the tree has three children and the leaves of the tree are the bounded faces of the triangle. The depths of the triangulation is the depth of the associated tree and the depth of a vertex not on the outer face is the minimum depth of triangle separating it from the outer face. We say that a stacked triangulation of depth D is complete, if the associated tree is a complete ternary tree of depth D.

It is quite easy to see that a complete stacked triangulation of depth D has a spring embedding on the $3^D \times 3^D$ grid. To see this start with a triangle with vertices $(0,0), (1,0), (0,1)$ and define the embedding of the first $k+1$ levels inductively be scaling the embedding of the first k levels by a factor three and placing a new vertex at the barycenter of every bounded triangle. Of course, in this embedding all inner edges have spring weight one. The complete stacked triangulations of depth D has $1/2(3^D + 5)$ vertices, therefore this embedding is linear in n. Using the approach by J. Richter-Gebert ([3]) this give an integral lifting of size $O(n^4)$.

We are also able to lift "sparse" stacked triangulations, i.e. stacked triangualtions for which the associated tree is a caterpillar on a grid of size $O(n^4)$.

I am working on the generaliztion of the approaches for the extremal examples to find a satisfying answer to Problem 3.

In the context of stacked triangulations I am also interested in Problem 4, on which I am working with Stefan Felsner.

Problem 4 Find an algorithm that embeds every stacked triangulation on the $c \cdot n \times c \cdot n$ grid, $2/3 \leq c < 1$.

This problem is interesting, because the examples that give the currently best known lower bound of $2/3 \cdot n \times 2/3 \cdot n$ for the size of grid drawings of planar graphs with n vetices are sparse stacked triangulations. We can embed the complete stacked triangulation of depth $2D$ on a grid of size $(2/3)^D 2n \times (2/3)^D 2n$. Again, as this construction relies on the depth rather than the number of vertices of the triangulation, the generalization to all stacked triangulations remains an interesting open problem.

Activities

Since July 2005, I

- attended the Summer School on Geometric Combinatorics, Vienna, 18.-29. July ’05
- attended a Summer Academy organized by the Studienstiftung des deutschen Volkes, Rot an der Rot, 6.-20. August ’05
- attended the Eurocomb 05, Berlin, 5.-9. September ’05 and presented a poster. Furthermore, I have helped with the organization during the conference week and with the preparation of the conference booklet.
- attended the 13th Symposium on Graph Drawing, Limerick, 12.-14. September ’05
- attended CGC-Workshop, Hiddensee, 25.-28. September ’05 and gave a talk
- attended the Monday lectures and colloquia of the CGC and gave a talk
- attended the lecture ”Topologie” at TU Berlin
- organized and attended the Noon Seminar of the workgroup ”Diskrete Mathematik” at TU Berlin
- attended the seminar ”Partielle Ordnungen” of the workgroup ”Diskrete Mathematik”
- attended a PhD student conference of the Studienstiftung des deutschen Volkes, 4.-6. December 2005

Preview

I am planning to

- take three weeks of French classes in La Rochelle, France, from 12. February ’06 - 05. March ’06
- attend the block course ”Embeddings of planar graphs”, Berlin, 20. - 31. March ’06

- attend a PhD student conference of the Studienstiftung des deutschen Volkes, 11. - 14. May '06

References

[1] N. Bonichon, S. Felsner, and M. Mosbah. Convex drawings of 3-connected planar graphs. In J. Pach, editor, *Proc. of Graphs Drawing 2004*, LNCS 3383, pages 60–70, 2004.

[2] S. Felsner. *Geometric Graphs and Arrangements*. Vieweg, 2004.

[3] J. Richter-Gebert. *Realization Spaces of Polytopes*. Springer, 1996.

[4] F. Zickfeld. Semester report SS05 of Florian Zickfeld. `http://www.math.tu-berlin.de/~zickfeld/CGC-Summer05.pdf`, 2005.

[5] F. Zickfeld. Orthogonal surfaces and the height problem. `http://www.math.tu-berlin.de/~zickfeld/HeightProblem.pdf`, 2006.