Semester Report WS05/06 of Carsten Schultz

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Field of Research:	Topological Combinatorics
Topic:	Homomorphism complexes of graphs
Postdoc	at the program from June to December 2005

Field of Research

In the proof of Kneser's Conjecture, Lovász introduced the neighbourhood complex of a graph and showed that a graph whose neighbourhood complex is $(k-1)$-connected for some integer $k \geq 0$ has chromatic number at least $k+2$ [Lov78], connecting combinatorics and topology in a surprising way. Subsequently, other graph complexes have been studied. In particular, for graphs G and H, the complex $\mathrm{Hom}(H, G)$ has been introduced, whose vertices are the graph homomorphisms from H to G. Since $\mathrm{Hom}(K_2, G)$ is homotopy equivalent to the neighbourhood complex of G, it is natural to ask if similar results may be obtained by replacing K_2 with other graphs. As a starting point in this direction, Lovász conjectured that the chromatic number of G is at least $k+3$ if $\mathrm{Hom}(C_{2r+1}, G)$ is $(k-1)$-connected. Here C_{2r+1} denotes a circuit of odd length.

These questions gained new impetus by the work of Babson & Kozlov [BK06a, BK06b]. The complex $\mathrm{Hom}(C_{2r+1}, G)$ carries a natural free $\mathbb{Z}_2$-operation induced by an automorphism of C_{2r+1} that flips an edge. Babson & Kozlov proposed to study the cohomological index of this action. Because of the functoriality of Hom, Lovász' conjecture would follow from

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(C_{2r+1}, K_n) \leq n-3 \qquad \text{for all } n \geq 3 \text{ and } r \geq 1. \quad (1)$$

They proved this for odd n and proved Lovász' conjecture in full generality by doing a similar calculation for even n.

Plan of Research

In my application to the CGC program, I had stated that the calculations in the proof by Babson & Kozlov are involved enough to make it worthwhile to attempt a simplification of their proof as a starting point for studying Hom-complexes, and that some Hom-complexes are interesting to combinatrial geometers in their own right.

Results

In [Sch05a] I gave a proof a of (1) for all n that is also considerably simpler than the previous proof for odd n. For this it had been useful that I had learned from Frank Lutz, who is an associate member of Günter Ziegler's Discrete Geometry Group, about a conjecture by Péter Csorba, which states that $\mathrm{Hom}(C_5, K_n)$ is homeomorphic to a Stiefel manifold, the unit tangent space of the $(n-2)$-sphere [Cso05], and which had led to their work on Hom-complexes which are manifolds [CL05]. I proved this conjecture in [Sch05b]. In December I greatly profited from discussions with Rade Živaljević, who spent a week in Berlin partly on invitation of the CGC program. This led me to generalise his elegant argument which he had used in [Živ05] to prove a special case of Lovász' conjecture. I was able to obtain the following result.

Theorem ([Sch06]). *Let G, G' be graphs with involutions, the involution on G flipping an edge, and $k \geq 1$. If*

- ▷ $\mathrm{coind}_{\mathbb{Z}_2} \mathrm{Hom}(G, G'^{\mathbb{Z}_2}) \geq k-1$,
- ▷ *there is a graph homomorphism from G to G' that commutes with the involutions, and*
- ▷ $\mathrm{Hom}(G, G')$ *is* $(k-1)$*-connected,*

then

$$\text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(G', H) + k \leq \text{cohom-ind}_{\mathbb{Z}_2} \mathrm{Hom}(G, H)$$

for all graphs H with $\mathrm{Hom}(G', H) \neq \emptyset$.

Here, $G'^{\mathbb{Z}_2}$ is a graph whose vertex set is the set of all orbits of the involution on G'. Its edge set is the largest one such that $\{o_0, o_1\} \in E(G'^{\mathbb{Z}_2})$ and $u_i \in o_i$ together imply $\{u_0, u_1\} \in E(G')$.

This yields an even simpler proof of (1) as the special case $G = K_2$, $G' = C_{2r+1}$, $k = 1$. It also yields new graphs T for which $\mathrm{Hom}(T, G)$ gives a lower bound on the chromatic number of G and results on the relative strengths of these bounds.

Activities

I attended the workshop of the CGC program on Hiddensee where I presented [Sch05b]. I presented [Sch05a] at the program's colloquium.

Preview

I presented the results of [Sch06] immediately after the end of the program at the combinatorics workshop in Oberwolfach. I attend an Algebraic Topology program at the Institut Mittag-Leffler for four weeks in January and February and will continue to be a member of Günter Ziegler's group in 2006. There I will further study applications of Algebraic Topology to Combinatorics. I also plan to take part in the fall program at the MSRI dedicated to this field.

References

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